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Authentic SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS for JEE MAIN, ADVANCED & KVPY

Er. Vaibhav Singh

Strategic Book for
Class 11/ 12 & Engineering Exams

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FOREWORD

The competitive exams like JEE test an aspirant's conceptual knowledge & how fast he/ she solve the problems with accuracy. So it becomes necessary that the students should know the short-cut methods in addition to the traditional methods of analysis. Keeping this in mind DISHA Publication brings a unique & innovative book *Authentic* **SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS** for JEE Main, Advanced & KVPY to enable aspirants for advanced abilities to Solve KVPY, JEE Main & Advanced level Questions well within the stipulated time.

An earnest effort has been made to bring the book *Authentic* **SHORTCUTS, TIPS, TRICKS & TECHNIQUES in MATHEMATICS**. We have really worked hard researching for the best possible Tips, Tricks, Techniques and Shortcut Solutions which students must know and can utilize in the examination hall.

- Shortcuts to help you in providing a different perspective to a concept/ problem thus strengthening your conceptual understanding.
- Tips provide you the Most Important Points to remember that aids in Conceptual Understanding & Problem Solving.
- Tricks empower you with magical tools that help you develop unique approaches to solve a problem.
- Shortcut Solutions provides alternate faster methods that save you a lot of time during examination.

The book encompasses 26 Chapters, which start with Review of Key Notes and Formulae, followed by Shortcuts, Tips, Tricks and Techniques which are further followed by Illustrations demonstrating Shortcut Solutions. The book in all contains:

1. 250+ Chapter-wise Shortcuts, Tips & Tricks to solve JEE Level Problems.
2. 400+ Illustrations with Shortcut Solutions of JEE Level Questions including JEE Past Years Questions.
3. 25+ International Techniques to crack JEE Advanced Level Questions.
4. 500+ Chapter-wise JEE Level Questions Exercise with Accurate & Shortest Possible Solutions.
5. Chapter-wise 350+ Important Key Notes Formulae.

This book provides you with hundreds of short-cut methods for the most conceptual and relevant problems. This book can also be used as a REVISION BOOK for various competitive exams. I hope that the book will fulfill the needs of the students for which it has been designed. We have made our best efforts to keep the book error-free but some errors might have crept in by mistake. We request our readers to highlight these errors and their fruitful suggestions so that we can keep on improving this book.

Author: Er. Vaibhav Singh

No Matter where You Prepare from, keep this book as your companion.
It would definitely improve your score by 25-30%.

Contents

Unit–I: Calculus

Unit–II: Algebra

Unit–III: Trigonometry

Unit–IV: Coordinate Geometry

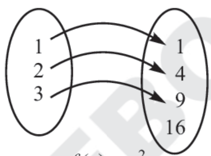
UNIT-I : CALCULUS

1 Functions

Review of Key Notes and Formulae

Definition: If A and B are two non-empty sets, then the rule that, for each and every element of set A is uniquely associate with set B.

$f : A \rightarrow B$



$f(x) = x^2$

Domain: All elements of set A

$$D_f = \{1, 2, 3\}$$

Co-domain: All elements of Set B

$$Co\text{-}D_f = \{1, 4, 9, 16\}$$

Range: Elements of set B which are involved in mapping.

$$R_f = \{1, 4, 9\}$$

Different Types of Functions

1. Polynomial function: Function in the form of:

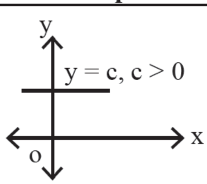
$$f(x) = a_0 x^n + a_1 x^{n-1} + \text{...........} + a_n \,;\, a_0 \neq 0 \,;\, \text{Degree} = n$$

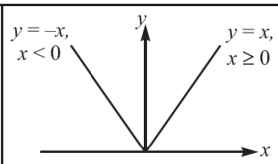
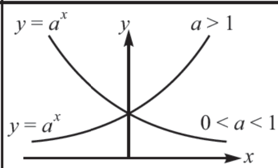
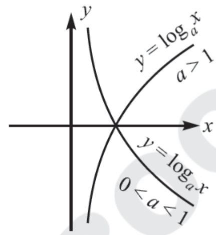
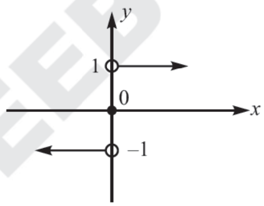
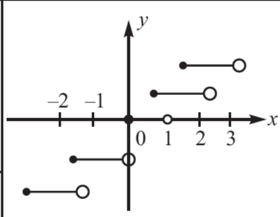
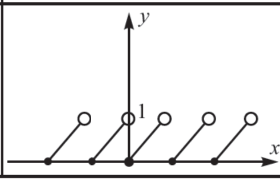
where, $n, n-1, n-2$ .... are non-negative integers. Domain of $f(x) = R$

2. Rational function: Functions in form of

$$f(x) = \frac{p(x)}{q(x)} ; q(x) \neq 0$$

where, $P(x)$ and $q(x)$ are polynomial in x. Domain of $f(x) = R - \{x : q(x) = 0\}$

	Function	Graph	Domain & Range
3.	**Constant function:** $y = f(x) = c \quad \forall x \in R$, where c is a constant		Dom : $x \in R$ Range : $y = \{c\}$

4.	**Modulus function:** $y = f(x) = \lvert x \rvert$		Dom: $x \in R$ Range: $y \in [0, \infty]$
5.	**Exponential function:** $y = f(x) = a^x$, where $a > 0, a \neq 1$		Dom: $x \in R$ Range: $y \in (0, \infty)$
6.	**Logarithmic function:** $y = f(x) = \log_a x$ where $a > 0, a \neq 1$		Dom: $x \in (0, \infty)$ Range: $y \in R$
7.	**Signum function:** $y = f(x) = \text{Sgn}(x)$ $\Rightarrow \quad f(x) = \begin{cases} \dfrac{\lvert x \rvert}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$		Dom: $x \in R$ Range: $y \in \{-1, 0, 1\}$
8.	**Greatest integer function:** $y = f(x) = [x]$ x if x is integer; greatest integer less than x		Dom: $x \in R$ Range = $\{z\}$
9.	**Fractional part function:** $y = f(x) = \{x\}$ $\{x\} = x - [x]$		Dom: $x \in R$ Range: $y \in [0, 1)$
10.	**Trigonometric function:**		
	Functions	**Domain**	**Range**
	$y = \sin x$	$x \in R$	$y \in [-1, 1]$
	$y = \cos x$	$x \in R$	$y \in [-1, 1]$

$y = \tan x$	$x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$	$y \in R$
$y = \cot x$	$x \in R - \{n\,\pi\}$	$y \in R$
$y = \operatorname{cosec} x$	$x \in R - \{n\,\pi\}$	$y \in (-\infty, -1] \cup [1, \infty)$
$y = \sec x$	$x \in R - \{(2n+1)\frac{\pi}{2}\}$	$y \in (-\infty, -1] \cup [1, \infty)$

Equal or Identical Functions:

Two functions $f(x)$ and $g(x)$ are said to be identical if.

(i) Domain of $f(x)$ = Domain of $g(x)$

(ii) Co-domain of $f(x)$ = Co-domain of $g(x)$

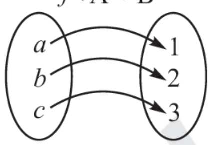
(iii) $f(x) = g(x)$ for every x belonging to their domain.

Classification of Functions:

1. **One-one function:** The mapping $f : A \to B$

   (Injective Function) is one-one function if different elements in A have different images in B.

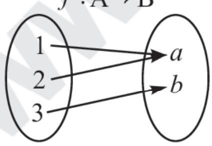
   $f : \mathrm{A} \to \mathrm{B}$

   $$\boxed{\begin{array}{l} x_1, x_2 \in A \\ x_1 \neq x_2 \;\Rightarrow\; f(x_1) \neq f(x_2) \end{array}}$$

2. **Many-one function:** The mapping $f : A \to B$ is many-one two or more than two different elements in A have the same image in B.

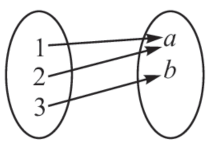
   $f : \mathrm{A} \to \mathrm{B}$

   $$\boxed{\begin{array}{l} x_1, x_2 \in A \\ x_1 \neq x_2 \;\Rightarrow\; f(x_1) = f(x_2) \end{array}}$$

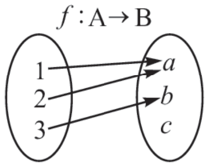
3. **Onto (surjective) function:** A function is said to be onto if

   $\delta : \mathrm{A} \to \mathrm{B}$

   $$\boxed{\text{Range} = \text{Co-domain}}$$

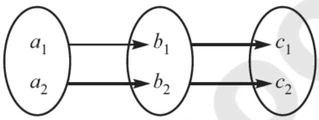
4. Into function: A function is said to be into if $\boxed{\text{Range} \neq \text{Co-domain}}$



★**Note** $\Rightarrow$ $\boxed{\text{Bijective function} \Leftrightarrow \text{One-one} + \text{Onto}}$

Composition of Function:

If $f : A \to B$ and $g : B \to C$ are two functions, then composite function of f and g are $go'f : A \to C$ will be defined as $g(f(x)) = gof(x)\ \forall\ x \in A$



Even and Odd Functions

(i) If $f(x) + f(-x) = 0\ \forall\ x \in$ Domain of $f(x)$

$\Rightarrow$ Odd function $\Rightarrow$ Symmetric about origin

(ii) If $f(x) = f(-x)\ \forall\ x \in$ Domain of $f(x)$

$\Rightarrow$ Even function $\Rightarrow$ Symmetric about y-axis

★**Note** $f(x) = 0$ is even as well as odd function.

Homogeneous Function:

Functions consists of variables both x & y such that $f(x, y)$ is homogeneous if:

$$f(tx, ty) = t^n f(x, y)$$

$\downarrow$

Homogeneous of degree 'n'

Periodic Function:

A function is periodic if its each value is repeated after a definite interval. So a function is periodic if there exists a positive real number 'T' such that

$$f(x + T) = f(x)\ \forall\ x \in D_f$$

★ Period $= nT$; $n \in I$

$\therefore$ $\boxed{f(x + nt) = f(x)}$; $n \in I$

★**Note:** Constant function has no fundamental period.

Inverse of a Function:

If $f : A \rightarrow B$ is a one-one and onto fn. both then we can define the inverse of the function as $g : B \rightarrow A$, such that $f(x)\ y \Rightarrow \underbrace{g(y) = x}_{\text{Inverse of } f(x).}$

Properties of Invertible Function:

(i) Inverse of bijective function is unique and bijective.

(ii) $(f^{-1})^{-1} = f$

(iii) $(gof)^{-1} = f^{-1}og^{-1}$

(iv) Inverse of a function is a mirror image about $y = x$ line.

TIPS AND TRICKS: (T-1)

There are only two polynomial functions exists, which satisfies the condition

$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$

are $\boxed{f(x) = 1 \pm x^n}$ $; n \in I^+ : x \in R$

Illustration 1

Find the polynomial function which satisfies the condition $f(x) + \left(\frac{1}{x}\right) = f(x). \left(\frac{1}{x}\right)$ of degree 3 and is always increasing function.

Short-cut solution :

$\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^3$ ($\because$ degree is 3)

Now, for function is always $\uparrow$sing

So, $f(x) = 1 + x^3$ or $f(x) = 1 - x^3$

Differentiate, $f'(x) = 3x^2 > 0$ & $f'(x) = -3x^2 < 0$

$\Rightarrow$ $\uparrow$sing $\Rightarrow$ $\uparrow$sing *fn*

Hence, we conclude that the required function is

$f(x) = 1 + x^3$

TIPS AND TRICKS: (T-2)

If $y = f(x) = [x]$ is a greatest integer function then,

$$[x] + [-x] = \begin{cases} 0 \ ; & x \in I \\ -1 ; & x \notin I \end{cases}$$

Illustration 2

Find the value of $\int_a^{3\pi} [2\cos x]\, dx$, where [ ] is the greatest integer function.

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\int_a^{3\pi} [2 \cos x]\, dx$, where [ ] is greatest integer function.

Let $\quad I = \int_a^{3\pi} [2 \cos x]\, dx \quad$ ... (1)

$\boxed{\text{Apply, King Property} : \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx}$

$\Rightarrow \quad I = \int_0^{3\pi} [-2 \cos x]\, dx \quad$ ... (2)

Now, add (1) and (2)

$\Rightarrow \quad 2I = \int_0^{3\pi} [2 \cos x] + [-2 \cos x]\, dx$

Since 2 cos x is not always '0' or Integer

$\Rightarrow \quad 2I = \int_0^{3\pi} (-1)\, dx$

$\therefore I = \dfrac{-3\pi}{2}$

TIPS AND TRICKS: (T-3)

If $y = f(x) = \{x\}$ is a fractional part function.

then, $\{x\} + \{-x\} = \begin{cases} 0 \; ; & x \in I \\ 1 \; ; & x \notin I \end{cases}$

TIPS AND TRICKS: (T-4)

${}^nC_r$ and ${}^rC_n$, simultaneously possible only when $n = r$

Illustration 3

Let $f(x) = {}^{x+1}C_{2x-8}$, $g(x) = {}^{2x-8}C_{x+1}$ if $h(x) = f(x) \cdot g(x)$.

Then find domain and range of $h(x)$.

Short-cut solution :

$\boxed{\text{Using T-4}}$ $x + 1 = 2x - 8$

$\Rightarrow \quad x = 9$

Hence domain is $x \in \{9\}$ and $R_f = 1$.

TIPS AND TRICKS: (T-5)

Range of the function $f(x) = \dfrac{ax+b}{cx+d}$; $x \neq \dfrac{-d}{c}$ is $R - \left(\dfrac{d}{c}\right)$

Illustration 4

Find the range of the function $f(x) = \dfrac{2x+1}{5x-2}$; $x \neq \dfrac{2}{5}$

Short-cut solution :

Using T-5 $\quad y \in R - \left\{\dfrac{2}{5}\right\}$

TIPS AND TRICKS: (T-6)

To find Range of $f(x) = \cos(K \sin x)$

where $K \in R^+$ then,

If $K \in [0, \pi)$ $\Rightarrow$ Range is $y \in [\cos K, 1]$

If $K \in [\pi, \infty)$ $\Rightarrow$ Range $\rightarrow y \in [-1, 1]$

Illustration 5

Find range of the function $y = \cos(2 \sin x)$

Short-cut solution :

Using T-6 $\quad \because \; K = 2$ (case IInd) and $K < \pi$

$\Rightarrow$ Range $\rightarrow y \in [\cos 2, 1]$

Illustration 6

Find range of the function $y = \cos(3 \sin x)$

Short-cut solution :

Using T-6 $\quad \because \; K = 3 \Rightarrow K < \pi$ (case IInd)

$\Rightarrow$ Range $\rightarrow y \in [\cos 3, 1]$

Illustration 7

Find the range of the function $f(x) = \cos(4 \sin x)$

Short-cut solution :

Using T-6 $\quad \because \; K = 4 \Rightarrow K > \pi$ (case Ist)

Hence, $\quad$ Range $\rightarrow y \in [\cos -1, 1]$

TIPS AND TRICKS: (T-7)

Identification of function using graph:

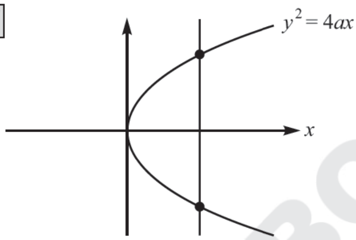
If it is possible to draw lines parallel to y-axis which cuts the curve more than one point then the given relation is not a function and when the line cuts the curve at only one point than it is a function.

Illustration 8

Check whether it is a function or not $y^2 = 4\,ax$

Short-cut solution :

Using T-7



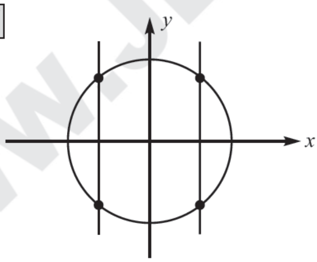
Vertical line cuts the graph more than one time then it is not a function.

Illustration 9

Check whether $x^2 + y^2 = a^2$ is a function or not

Short-cut solution :

Using T-7



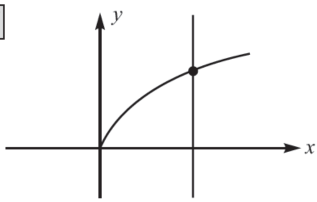
$\Rightarrow$ It is not a function.

Illustration 10

Check whether $y = \sqrt{x}$ is a function or not.

Short-cut solution :

Using T-7



$\Rightarrow$ It is a function since it cuts the curve once.

TIPS AND TRICKS: (T-8(i))

Graphical approach to check one-one or many-one function

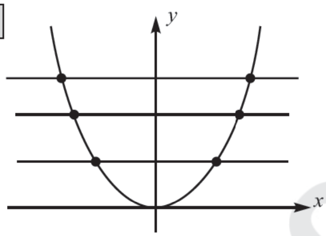
Construct the graph and draw lines parallel to x-axis, if it cuts the graph one time then it is a function and if it cuts more than one time then it is a many-one function.

Illustration 11

Check whether $y = x^2$ is a one-one or many-one function.

Short-cut solution :

$\boxed{\text{Using T-8(i)}}$



It is a many-one function.

TIPS AND TRICKS: (T-8(ii))

Calculus approach to check one-one or many-one function

Differentiate the function. $y = f(x)$ $\{f(x)$ must be differentiable$\}$

★ If $\dfrac{dy}{dx} > 0 \Rightarrow$ Monotonically Increases $\Rightarrow$ one-one function

★ If $\dfrac{dy}{dx} < 0 \Rightarrow$ Monotonically Decreases $\Rightarrow$ one-one function

★ If $\dfrac{dy}{dx} > 0 \Rightarrow$ for some 'x' and $\dfrac{dy}{dx} > 0$ for some x then function is many-one function.

Illustration 12

Check whether one-one or many-one function.

(i) $f(x) = x^3$ (ii) $f(x) = x^2$

Short-cut solution :

$\boxed{\text{Using T-8(ii)}}$

(i) $f(x) = x^3 \Rightarrow f'(x) = 3x^2 > 0 \Rightarrow$ one-one function

(ii) $f(x) = x^2 \Rightarrow f'(x) = 2x \begin{cases} x > 0 \Rightarrow \uparrow\text{ses} \\ x < 0 \Rightarrow \downarrow\text{ses} \end{cases}$ Many- one function

★**Note:** (1) All trigonometric functions are many-one function.

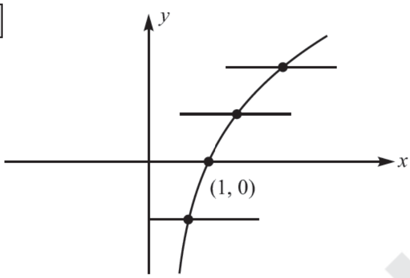
(2) All inverse trigonometric function are one-one function.

Illustration 13

Check for one-one or many-one function for $f(x) = \log_e x$

Short-cut solution :

Using T-8(ii)



It is one-one function.

Illustration 14

Check whether the following functions $f(x)$ are either one-one or many-one function.

(i) $f(x) = e^x$

(ii) $f(x) = \sin x$

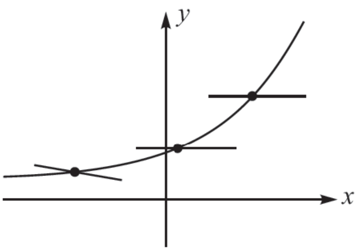
(iii) $f(x) = |x|$

(iv) $f(x) = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Short-cut solution :

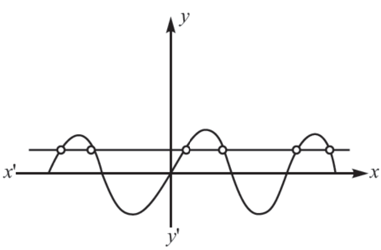
Using T-8(ii)

(i) $f(x) = e^x$

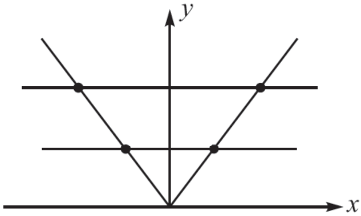


One-one function.

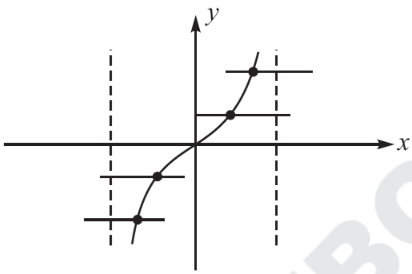
(ii) $f(x) = \sin x$



$\Rightarrow$ Many-one function.

(iii) $f(x) = |x|$  $\Rightarrow$ Many-one function.

(iv) $f(x) = \tan x$ in $x \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right)$



$\Rightarrow$ One-one function.

Illustration 15

Check whether one-one or many-one function for

$f(x) = f : R \to R$

$f(x) = x^3 + x^2 + 7x + \sin x$

Short-cut solution :

$\boxed{\text{Using T-8(ii)}}$

Differentiate $\quad f'(x) = 3x^2 + 2x + 7 + \cos x$

$$= (x^2 + 2x + 1) + (2x^2 + 6\ ) + \cos x$$

$$f'(x) = \underset{\geq 0}{(x+1)^2} + \underset{\geq 6}{(2x^2 + 6)} + \underbrace{\cos x}_{[-1,1]}$$

$\Rightarrow \qquad f'(x) > 0 \quad \Rightarrow \quad$ one-one function

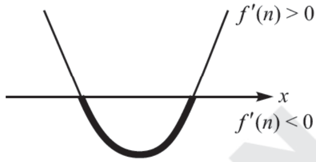
Illustration 16

Check whether one-one or many-one function for

$f(x) = x^3 + 6x^2 + 11x$

Short-cut solution :

Using T-8(ii)



Differentiate $f'(x) = 3x^2 + 12x + 11$

This is a parabola open upwards

Now, we find discriminant.

$$D = 144 - 132$$

$\Rightarrow \quad D > 0$ (two distinct roots)

So this implies $f'(x) > 0$ and $f'(x) < 0$ both.

$\Rightarrow$ Many-one function.

TIPS AND TRICKS: (T-9)

Short trick to check whether onto or into function for polynomial functions. For $x \in R$

Put, $x \to \infty$ If $\Rightarrow f(x) \to \infty$

Put, $x \to -\infty$ If $\Rightarrow f(x) \to -\infty$

Hence, onto function.

Illustration 17

Check whether onto or into function.

(i) $f(x) = x^3$

(ii) $f(x) = x^2$

Short-cut solution :

Using T-9

(i) Since $f(x)$ is odd function.

$$\left.\begin{aligned} \Rightarrow x \to +\infty, &\quad f(x) \to +\infty \\ x \to -\infty, &\quad f(x) \to -\infty \end{aligned}\right\} \Rightarrow \text{One-one function and onto function.}$$

(ii) $f(x) = a_0 x^{2n} + a_1 x^{2n-2} + \ldots\ldots\ldots$

$$\left.\begin{aligned} \text{Put, } x \to +\infty, &\quad f(x) \to \infty \\ x \to -\infty, &\quad f(x) \to \infty \end{aligned}\right\} \Rightarrow \text{Many-one function and into function.}$$

Illustration 18

Check whether the given function is bijective or not

$f(x) = x^3 + 5x + 1$ **[AIEEE 2009]**

Short-cut solution :

Using T-8(ii)

On differentiating $\to f(x) = 3x^2 + 5 > 0$s

$\Rightarrow$ one-one function.

Using T-9

$$\left.\begin{array}{l} x \to +\infty, \quad f(x) \to \infty \\ x \to -\infty, \quad f(x) \to -\infty \end{array}\right\} \Rightarrow \text{Onto function}$$

Hence, above function is bijective.

TIPS AND TRICKS: (T-10)

If set A contains 'm' elements and another set B contains 'n' elements, then the

(i) Total number of functions = $(n)^m$
For f : A $\to$ B

(ii) Number of one-one function is $\boxed{{}^nP_m}$ for $n \geq m$ and 0 (zero) for $n < m$

(iii) Number of many-one function = Total number of functions – One-one function

(iv) Number of onto functions are
- (a) If $n < m \Rightarrow n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots$
- (b) If $n = m \Rightarrow n\,!$
- (c) If $n > m \Rightarrow 0$

(v) Number of into function are
- (a) If $n \leq m \Rightarrow$ Total number of functions – Onto functions
- (b) If $n > m \Rightarrow (n)^m$ [Total – Onto]

(vi) Number of constant functions = n.

(vii) If A and B are two sets having n-elements and '2' elements respectively.
Then number of onto functions from A to B
is $2^n - 2$ if $n \geq 2$ and '0' if $n > 2$

Illustration 19

If $A = \{1, 5, 9, 7, 14, 22\}$ and $B = \{2, 3, 5, 6\}$, then number of:

(a) Total functions
(b) One-one functions
(c) Many-one functions
(d) Onto function
(e) Into functions
(f) Constant functions

Short-cut solution :

Since, $m = 6$ and $n = 4$

(a) Using T-10(i) $\Rightarrow (n)^m = (4)^6 = 4096$

(b) Using T-10(ii) $\Rightarrow$ Since $n < m \Rightarrow$ One-one function = 0

(c) Using T-10(iii) $\Rightarrow$ Many-one function = Total – (One-one functions)
$= 4096 - 0 = 4096$

(d) Using T-10(iv) Since $n < m$

$= (n^m) - {}^nC_1\,(n-1)^m + {}^nC_2\,(n-2)^m + \ldots\ldots.$

$= 4^6 - {}^4C_1 \cdot 3^6 + {}^4C_2 \cdot (2)^6 - {}^4C_3 \cdot (1)^6 + {}^4C_4 \cdot (0)^6$

$= 4096 - 4 \times 729 + 6 \times 64 - 4 + 0$

$= 1560$

(e) Using T-10(v) Total – Onto

$4096 - 1560 = 2536$

(f) Using T-10(vi) Constant function $= n = 4$

TIPS AND TRICKS: (T-11)

If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have same number of the elements. If A has n elements, then number of the bijections from A to B is $\boxed{n!}$

Illustration 20

If $A = \{2, 3, 4, 5, 6\}$, then the total number of bijection function in $f : A \to A$ is

(a) 110 (b) 115 (c) 120 (d) 125

Short-cut solution :

Using T-11 Since $n = 5$

$\Rightarrow 5\,! = 120$ **Ans. (c)**

TIPS AND TRICKS: (T-12)

In order to find fundamental period of

(i) $\sin\,(2\,n\pi\,\{x\}) \longrightarrow \boxed{\frac{1}{n}}$; where $\{x\}$ is fractional part function and $n \in I^+$

(ii) $\sin\,(2n+1)\,\pi\,\{x\} \longrightarrow \boxed{1}$

(iii) If $f\,(x)$ is periodic function with fundamental period 'T' then $\frac{}{f(x)}$ and $\sqrt{f(x)}$ will also be periodic with fundamental period 'T'.

Illustration 21

Find the fundamental period of

(i) $\sin(2\pi(\{x\}))$ (ii) $\sin(4\pi\{x\})$

Short-cut solution :

Using T-12(i) $\frac{1}{n} = \frac{1}{1} \Rightarrow \boxed{T = 1}$

(ii) sin (4π {x})

Using T-12(ii) $\frac{1}{n} = \frac{1}{2} \Rightarrow \boxed{T = \frac{1}{2}}$

Illustration 22

Find the fundamental period of $f(x) = \sec x$

Short-cut solution :

Using T-12(i) $f(x) = \frac{1}{\cos x}$

Since period of $\cos x$ is 2π

Hence period of $f(x) = \sec x$ is also 2π.

Illustration 23

Find the fundamental period of $f(x) = \frac{1}{\sqrt{\tan x}}$

Short-cut solution :

Using T-12(iii) $f(x) = \sqrt{\cot x}$

Since fundamental period of $\cot x$ = is 'π'

Then, fundamental period of $\sqrt{\cot x} = \frac{1}{\sqrt{\tan x}}$ is also 'π'

TIPS & TRICKS

TIPS AND TRICKS: (T-13)

If $f(x)$ is periodic with fundamental period 'T' than $f(ax + b)$ is also periodic with fundamental period $\frac{T}{|a|}$

Illustration 24

Find the fundamental period of $f(x) = \sin(2x + 3)$

Short-cut solution :

Using T-13 Since fundamental period of $\sin x$ is 2π

$\Rightarrow T = \frac{2\pi}{|2|} \Rightarrow T = \pi$

Illustration 25

Find the period of $f(x) = \{-3x + 5\}$ where {*} is fractional part function.

Short-cut solution :

Using T-13 Since period of $y = \{x\}$ is 1

Hence, $T = \frac{1}{|-3|} \Rightarrow T = \frac{1}{3}$

TIPS AND TRICKS: (T-14)

Let $f(x)$ and $g(x)$ be the two functions which are periodic then period of $h(x) = f(x) + g(x)$ is $\boxed{\text{LCM of } T_1 \text{ and } T_2}$ as one its periods.

where, T_1, T_2 are period of $f(x)$ and $g(x)$ respectively.

★**Note:** LCM of irrational number $= \dfrac{\text{LCM of numerator}}{\text{HCF of denominator}}$

Illustration 26

Find period of $f(x) = \cos(\sin x) + \cos(\cos x)$

Short-cut solution :

$\boxed{\text{Using T-14}}$ Since, period of $\cos(\sin x)$ is π and priod of $\cos(\cos x)$ is π
Hence the priod of $f(x) = \text{LCM}(\pi, \pi) = \pi$.

Illustration 27

Find period of $f(x) = -\sin\dfrac{2x}{7} + \cos\dfrac{3x}{5}$.

Short-cut solution :

$\boxed{\text{Using T-13}}$ Since period of $-\dfrac{\sin 2x}{7}$ is 7π

and period of $\dfrac{\cos 2x}{5}$ is $\dfrac{10\pi}{3}$

$\boxed{\text{Using T-14}}$

Hence, the period of $f(x) = \text{LCM}\left(7\pi, \dfrac{10\pi}{3}\right)$

$$\Rightarrow \frac{\text{LCM of } N^r}{\text{HCF of } D^r} = \frac{\text{LCM}(7\pi, 10\pi)}{\text{HCF}(1, 3)} = \frac{70\pi}{1}$$

TIPS AND TRICKS: (T-15)

Let $y = f(x)$ is a function and it satisfies the relation $f(n + a) + f(n + b) =$ constant then period of this junction is $2|b - a|$.

Illustration 28

If $f(x) + f(x + 5) = 12$, the period of $f(x)$ is:

Short-cut solution :

Using T-15 $2\,|5-0| = 10.$

Illustration 29

If $f(x+2) + f(x+9) = 30$, then period of $f(x)$ is:

Short-cut solution :

Using T-15 $2\,|9-2| = 14.$

TIPS AND TRICKS: (T-16)

$$[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$$

where, $n \in N$ and [*] is greatest integer function.

Illustration 30

Find the value of

$$\left[\frac{1}{5}\right] + \left[\frac{1}{5} + \frac{1}{100}\right] + \left[\frac{1}{5} + \frac{2}{100}\right] + \ldots\ldots + \left[\frac{1}{5} + \frac{99}{100}\right]$$

Short-cut solution :

Using T-16 $[nx] = \left[100 \times \frac{1}{5}\right] = 20$

SHORTCUTS: (SC-1)

To find number of solution of $f(x) = g(x)$ where $f(x)$ and $g(x)$ are functions. Then draw graph for both LHS and RHS on same Cartesian plane and find number of point of intersection.

Number of point of intersection = Number of solutions/ Roots

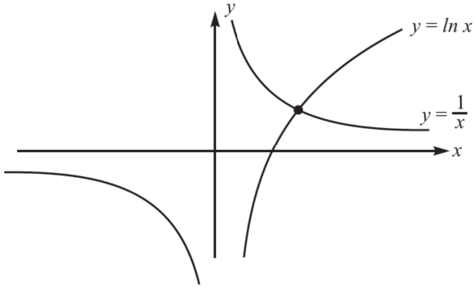
Illustration 31

Find number of solution of $x \cdot ln\, x = 1$

Short-cut solution :

Using SC-1 $\because \quad ln\, x = \frac{1}{x}$; $x > 0$ and $x \neq 0$

Now, we draw graph for $y = ln\, x$ and $y = \frac{1}{x}$; $x > 0$



Number of intersection = 1 = Number of solutions

Illustration 32

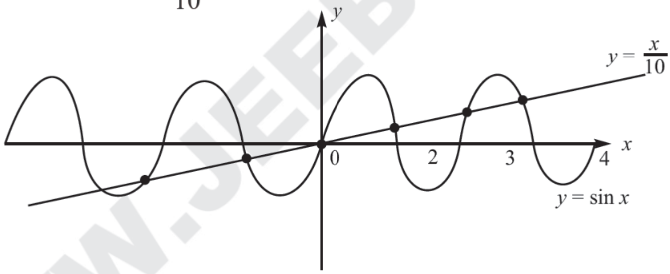
Find number of solutions of $\sin x = \frac{x}{10}$.

Short-cut solution :

Using SC-1 Now, we will draw graphs for $y = \sin x$ and $y = \frac{x}{10}$

$\because$ $\sin x \in [-1, 1]$

$\Rightarrow$ $-1 \le \frac{x}{10} \le 1 \Rightarrow -10 \le x \le 10$



We have to sketch the curve when $x \in [-10, 10]$

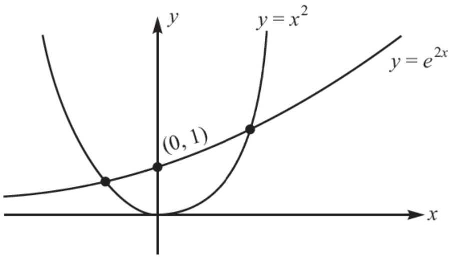
Hence, number of intersection = 7 = Number of solutions.

Illustration 33

Find the number of solutions of $e^{2x} = x^2$.

Short-cut solution :

Using SC-1 Now, we will draw graphs for $y = e^{2x}$, $y = x^2$



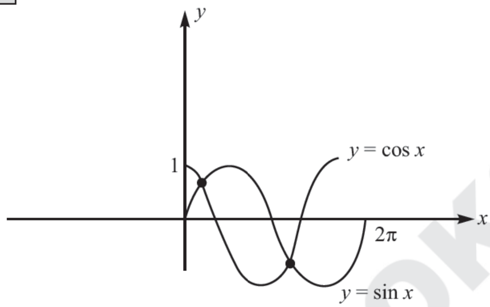
Number of intersection = Number of solutions = 2.

Illustration 34

Find the number of solutions of $\sin x = \cos x$ in $x \in [0, 2\pi]$.

Short-cut solution :

Using SC-1 Draw graphs of $y = \sin x$ and $y = \cos x$



Number of intersection in $x \in [0, 2\pi] = 2 =$ Number of solutions

SHORTCUTS: (SC-2)

In order to find domain and range of function, the shortest way is to draw graph. Graphs are also useful for one-one, many-one, onto and into.

Domain = Existence of graph along x - axis
Range = Existence of graph along y - axis

Illustration 35

Find the domain and range also check function is one-one or not in its domain. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 2x - 3}$

Short-cut solution :

Using SC-2 $f(x) = \dfrac{x^2 - 5x + 4}{x^2 + 2x - 3} = \dfrac{(x-4)(x-1)}{(x+3)(x-1)}$; $x \neq 1, -3$

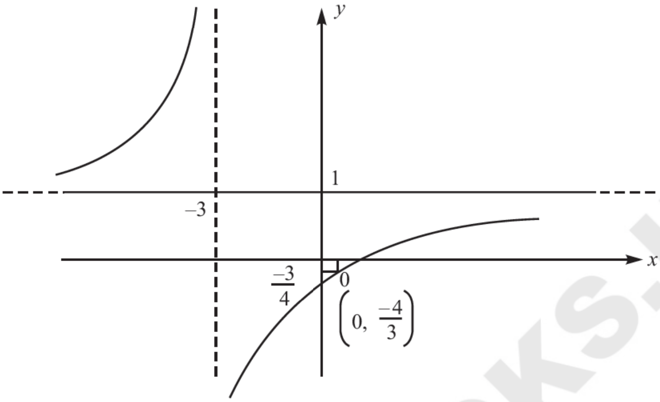
First of all we will check monotonicity.

Differentiating $f(x)$ w.r.t. x, we get

$$\Rightarrow \quad f'(x) = \frac{d}{dx}\left(\frac{x-4}{x+3}\right) = \frac{(x+3)-(x-4)}{(x+3)^2} = \frac{7}{(x+3)^2}$$

$$\Rightarrow \quad f'(x) > 0 \Rightarrow \text{Increasing function}$$

Graph



$\Rightarrow$ Domain : $x \in R - \{-3, 1\}$

Range : $x \in R - \left\{1, \frac{-3}{4}\right\}$

As shown function is **one-one**.

Illustration 36

$f : [2, \infty] \to Y$

$f(x) = x^2 - 4x + 5$ is both one-one and onto if

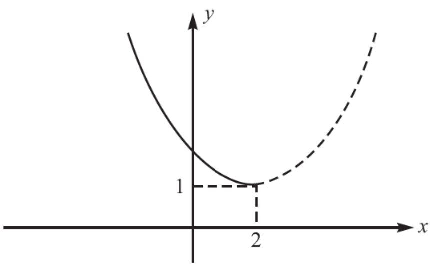
(a) $Y = R$ (b) $Y = [1, \infty)$

(c) $Y = [4, \infty)$ (d) $Y = [5, \infty)$

Short-cut solution :

Using SC-2 Rewrite, $f(x) = (x - 2)^2 + 1 \Rightarrow$ Parabola with vertex (2, 1)

Graph



$$y_{min} = 1 \; ; y_{max} = \infty$$

Hence, $Y = [1, \infty)$.

Illustration 37

$f: R \to R$

$$f(x) = \begin{cases} x^2 + 2mx - 1 ; x \le 0 \\ mx - 1 ; x > 0 \end{cases} \text{ then find the value of m}$$

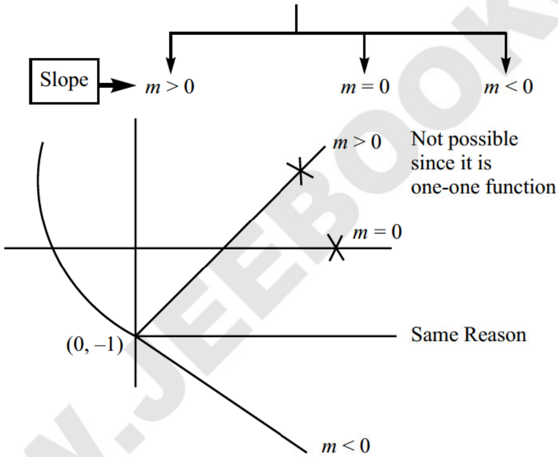
Short-cut solution :

Using SC-2

Graph

For $x < 0$, it is parabola open upwards

For $x > 0$, there are three possibilities (line)



Hence, $m < 0 \Rightarrow m \in (-\infty, 0)$.

SHORTCUTS: (SC-3)

Inverse of many-one function does not exist, only one-one onto function are invertible.

$$f'(x) \ge 0 \quad \text{or} \quad f(x) \le 0$$

Illustration 38

If $f: R \to R$, $f(x) = x^3 + (a+2)x^2 + 3ax + 5$ is invertible mapping. Find 'a'.

Short-cut solution :

Using SC-3 Invertible $\Rightarrow$ One-one + Onto

Hence $f'(x) \ge 0$ or $f'(x) \le 0$

$\Rightarrow$ $f'(x) = 3x^2 + 2x(a+2) + 3a \ge 0$ or $\le 0 \ \forall\ n \in R$

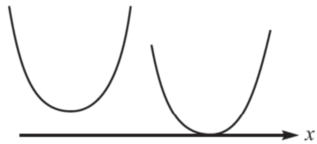
$\Rightarrow \qquad \boxed{D \le 0}$

Hence, $4(a+2) - 4.9a \le 0$

$$a^2 - 5a + 4 \le 0$$

$$(a-1)(a-4) \le 0$$

$\Rightarrow \qquad a \in [1, 4]$



SHORTCUTS: (SC-4)

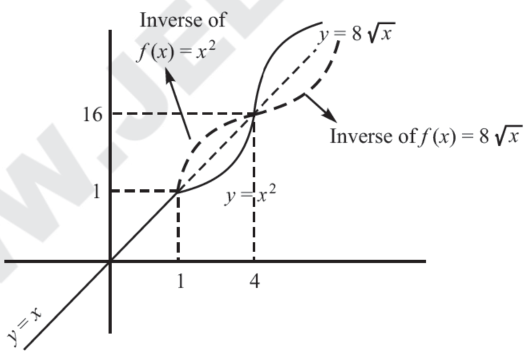
Inverse of a function is a mirror image about $y = x$ line. So copy the graph along other side of $y = x$ line.

Illustration 39

If $f(x) = \begin{cases} x & ; \quad x < 1 \\ x^2 & ; \quad 1 \le x \le 4 \\ 8\sqrt{x} & ; \quad x > 4 \end{cases}$. Then find $f^{-1}(x)$

Short-cut solution :

$\boxed{\text{Using SC-4}}$ **Draw Graph**



Hence, $f^{-1}(x) = \begin{cases} x & ; \quad x < 1 \\ \sqrt{x} & ; \quad 1 \le x \le 16 \\ \dfrac{x^2}{64} & ; \quad x > 16 \end{cases}$

SHORTCUTS: (SC-5)

Even functions are symmetric about y-axis odd functions are symmetric about origin.

Illustration 40

If $f(x) = \begin{cases} x^2 & ; \ x \in (0,1] \\ 2-x ; & x > 1 \end{cases}$. Define $f(x)$ for $x < 0$ if $f(x)$ is

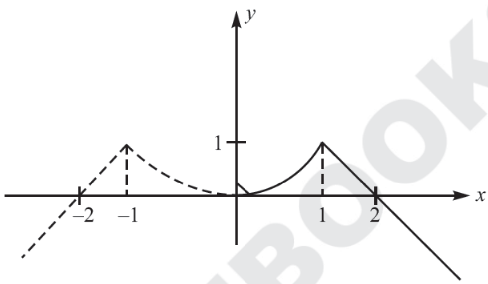
(i) Even function (ii) Odd function

Short-cut solution :

$\boxed{\text{Using SC-5}}$

(i) **Even Function**

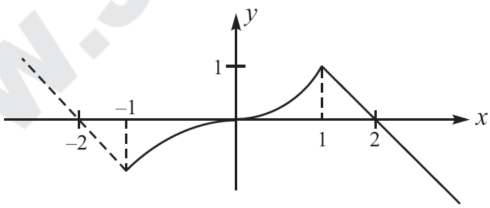
Draw graph of $f(x)$ using short cut $\Rightarrow$ Symmetry about y-axis



Hence, $f(x) = \begin{cases} x+2 & ; \ x \in (-\infty, -1) \\ x^2 & ; \ x \in (-1, 0) \end{cases}$

(ii) **Odd Function**

Draw graph of $f(x)$ using shortcut $\Rightarrow$ Symmetry about origin



Hence, $f(x) = \begin{cases} -x-2 & ; \ x \in (-\infty, -1) \\ -x^2 & ; \ x \in (-1, 0) \end{cases}$

TECHNIQUE

If f^{-1} be the inverse of bijective function $f(x)$ then $f(f^{-1}(x)) = x$.

Apply the formula of f on $f^{-1}(x)$ and use of the identity $f(f^{-1}(x)) = x$ to solve for $f^{-1}(x)$

Illustration 41

Find the inverse of the function $f(x) = \log_a\left(x + \sqrt{x^2+1}\right); a > 1$

Short-cut solution :

Using Tech.

$$f\left(f^{-1}(x)\right)=x$$

$$\Rightarrow \log_a\left(f^{-1}(x)+\sqrt{\left(f^{-1}(x)\right)^2+1}\right)=x$$

$$\Rightarrow f^{-1}(x)+\sqrt{\left(f^{-1}(x)\right)^2+1}=a^x \quad \text{...(i)}$$

and $-f^{-1}(x)+\sqrt{\left(f^{-1}(x)\right)^2+1}=a^{-x}$ ..(ii)

From (i) and (ii), $f^{-1}(x)=\left(\dfrac{a^x-a^{-x}}{2}\right)$

Concept Booster Exercise

1. Find a polynomial of degree '5' which satisfies the relation $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ which is always decreasing function.

2. Find a polynomial which satisfies $f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$ $\forall\, x \in R - \{0\}$ and the condition $f(3) = -26$, then determine $f'(1)$.

3. Find: $I = \int_0^{3\pi} \{2\cos x\}\, dx$; where {} is fractional part function.

4. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is **[AIEEE 2004]**
(a) $\{1, 2, 3\}$ (b) $\{1, 2, 3, 4, 5, 6\}$ (c) $\{1, 2, 3, 4\}$ (d) $\{1,2,3,4,5\}$

5. If $f : R \to S$, defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$, is onto, then the interval of 'S' is: **[AIEEE 2004]**
(a) $[0, 3]$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) $[-1, 3]$

6. The range of the function $f(x) = \dfrac{2+x}{2-x}, x \neq 2$ is **[AIEEE 2002]**
(a) R (b) $R - \{-1\}$ (c) $R - \{1\}$ (d) $R - \{2\}$

7. The range of $f(x) = \dfrac{3x-1}{2x+1}$; $x \neq \dfrac{-1}{2}$
(a) $R - \left\{\frac{2}{3}\right\}$ (b) $R - \left\{\frac{3}{2}\right\}$ (c) R (d) $R - \left\{\frac{2}{5}\right\}$

8. The range of the function $f(x) = \cos\left(\frac{1}{2}\sin x\right)$
(a) $[\cos 2, 1]$ (b) $\left[\cos\frac{1}{2}, 1\right]$
(c) $[-1, 1]$ (d) None of these

9. The range of the function $f(x) = \cos(5 \sin x)$
(a) $[\cos 5, 1]$ (b) $[-1, 1]$ (c) $\left[\cos\frac{1}{5}, 1\right]$ (d) $[-1, 1)$

10. Which of the following is/are the functions **[AIEEE 2002]**
(a) $y^2 = 4x$ (b) $x^2 = 8y$ (c) $x^2 + y^2 = 4$ (d) $\left[\cos\frac{1}{5}, 1\right]$

11. The function $f : R \to R$ defined by $f(x) = \sin x$ is:
(a) Into (b) Onto (c) One-one (d) Many-one

12. The period of the function $f(x) =$ if $f(x+10)+f(x+20)=50$

(a) 30 (b) 40 (c) 60 (d) 20

13. The period of $f(x) = \cos\frac{\pi}{4}x + \sin\frac{\pi x}{3}$.

(a) 24 (b) 12 (c) 36 (d) 6

14. The function $f: R \to R$ defined by $f(x) = x^2 - 3x + 2$

(a) Onto (b) Into (c) Many-one (d) One-one

15. Given $X = \{1, 2, 3, 4\}$, find all one-one, onto mappings, $f: X \to X$ such that, ...

16. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. The number of onto functions from E to F is **[AIEEE 2002]**

(a) 14 (b) 16 (c) 12 (d) 8

17. Suppose $f(x) = (x+1)^2$ for $x \geq -1$, If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals **[AIEEE 2002]**

(a) $-\sqrt{x} - 1, x \geq 0$ (b) $\frac{1}{(x+1)^2}, x \geq -1$

(c) $\sqrt{x+1}, n \geq -1$ (d) $\sqrt{x} - 1, x \geq 0$

18. Let $f: R \to R$ be defined by $f(x) = 2x + \sin x; x \in R$. Then f is **[AIEEE 2002]**

(a) one to one and onto (b) one to one but not onto

(c) onto but not one-one (d) neither one-to one nor onto

19. The function $f: [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$ is **[AIEEE 2012]**

(a) one-one and onto (b) onto but not one-one

(c) one-one but not onto (d) neither one-one nor onto

20. The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$, $x \in (-1, 1)$ is **[JEE M 2020]**

(a) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

(c) $\frac{1}{4}\left(\log_8^e\right)\log_e\left(\frac{1+x}{1-x}\right)$ (d) $\frac{1}{4}\left(\log_8^e\right)\log_e\left(\frac{1-x}{1+x}\right)$

21. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ given by $f(x) = [\log(\sec x + \tan x)]^3$, then **[JEE M 2014]**

(a) $f(x)$ is odd function (b) $f(x)$ is an one-one function

(c) $f(x)$ is onto function (d) $f(x)$ is even function

22. Let $f(x) = ax^7 + bx^3 + cx - 5$; $a, b, c \in$ constant then find $f(+7)$, if $f(-7) = 7$.

(a) – 16 (b) – 15

(c) – 17 (d) – 20

23. Find the period of the function $f(x) = 2 + 3\cos(3x - 2)$

(a) 2π (b) π (c) — (d) $\frac{\pi}{3}$

24. Find the domain of the function $f(x) = \dfrac{\sqrt{\cos - \frac{1}{\sqrt{2}}}}{\sqrt{\frac{3}{2}x - x^2 - \frac{1}{2}}}$

25. Let $f(x) = \dfrac{x}{1+x}$ defined from $(0, \infty) \to [0, \infty)$ then $f(x)$ is **[AIEEE 2003]**

(a) one-one but not onto (b) one-one and onto

(c) many-one but not onto (d) many-one and onto

Solutions

1. $(1-x^5)$ $\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^5$

Since function is always decreasing is

So, $f'(x) < 0$

$\Rightarrow f(x) = 1 + x^5$ or $f(x) = 1 - x^5$

$f'(x) = 5x^4 > 0$ or $f'(x) = -5x^4 < 0$

$\Rightarrow$ increasing function $\Rightarrow$ decreasing function

Hence, $f(x) = 1 - x^5$ is our required answer.

2. (-3) $\boxed{\text{Using T-1}}$ $f(x) = 1 \pm x^n$

$\because$ $f(3) = -26$

$f(3) = 1 + 3^n$	$f(3) = 1 - 3^n$
$-26 = 1 + 3^n$	$-26 = 1 - 3^n$
$-27 = 3^n$ $\Rightarrow$	$3^n = 27$
Not possible $\Rightarrow$	$\boxed{n = 3}$

Hence, the required function is

$f(x) = 1 - x^3 \Rightarrow f'(x) = -3x^2$

$\Rightarrow f'(1) = -3$

3. $\left(\frac{3\pi}{2}\right)$ $\boxed{\text{Using T-3}}$

$$I = \int_0^{3\pi} \{-2\cos x\}\, dx \qquad \text{... (1)}$$

Apply king property

$$\Rightarrow I = \int_0^{3\pi} \{2\cos x\}\, dx \qquad \text{... (2)}$$

Add eqns. (1) and (2)

$$\Rightarrow 2I = \int_0^{3\pi} (\{2\cos x\} + \{-2\cos x\})\, dx$$

$$\Rightarrow 2I = \int_0^{3\pi} (1)\, dx \Rightarrow I = \frac{3\pi}{2}$$

4. (a) We know that $x - 3 \geq 0 \Rightarrow x \geq 3$

And $7 - x \geq x - 3 \Rightarrow 2x \leq 10 \Rightarrow x \leq 5$

Hence, $x = 3, 4, 5$

Now At $x = 3 \Rightarrow y = {}^4P_0 = 1$

At $x = 4 \Rightarrow y = {}^3P_1 = 3$

At $x = 5 \Rightarrow y = {}^2P_2 = 2$

Hence, range = $\{1, 2, 3\}$.

5. **(d)** As we know that $-\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$

Hence, $\quad -2 \le \sin x - \sqrt{3}\cos x \le 2$

$$-1 \le \sin x - \sqrt{3}\cos x + 1 \le 3$$

Onto $\Rightarrow$ Range = Codomain $\Rightarrow$ $S \in [-1, 3]$.

6. **(b)** $\boxed{\text{Using T-5}}$ Range $\rightarrow y \in R - \left\{\frac{1}{-1}\right\}$

$\Rightarrow \quad y \in R - \{-1\}$

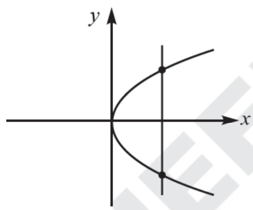
7. **(b)** $\boxed{\text{Using T-5}}$ Range $\rightarrow y \in R - \left\{\frac{3}{2}\right\}$

8. **(b)** $\boxed{\text{Using T-6}}$ $\because \frac{1}{2} < \pi \Rightarrow$ Range is $y \in \left[\cos\frac{1}{2}, 1\right]$

9. **(b)** $\boxed{\text{Using T-6}}$ $\because 5 > \pi \Rightarrow$ Range is $y \in [-1, 1]$

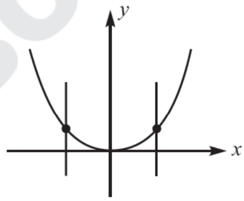
10. **(b, d)** $\boxed{\text{Using T-7}}$

(a)



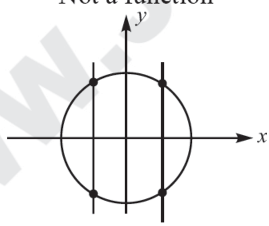
Not a function

(b)



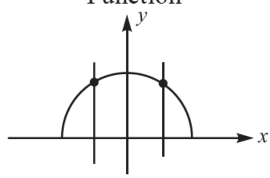
Function

(c)



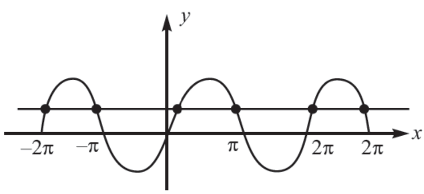
Not a function

(d)



Function

11. **(a, d)** $\boxed{\text{Using T-8}}$



$\Rightarrow$ Many-one

$$\left.\begin{array}{l}\text{Range of } \sin x = y \text{ is } y \in [-1, 1] \\ \text{But given co-domain is } y \in R\end{array}\right\} \Rightarrow \text{Range} \ne \text{Codomain}$$

Hence, into function.

12. **(d)** $\boxed{\text{Using T-15}}$ $a = 10, \quad b = 20$

$\Rightarrow$ Period $= 2\,|20 - 10| = 20.$

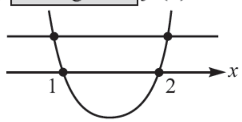
13. **(a)** $f(x) = \underbrace{\sin\frac{\pi x}{4}}_{\downarrow\, T_1} + \underbrace{\sin\frac{\pi x}{3}}_{\downarrow\, T_2}$, Since period of $\sin x = 2\pi$

$\boxed{\text{Using T-13}}$ $T_1 = \dfrac{2\pi}{\frac{\pi}{4}}$ and $T_2 = \dfrac{2\pi}{\frac{\pi}{3}}$

$\boxed{\text{Using T-14}}$ Period $= \text{LCM}(T_1, T_2)$

$= \text{LCM}(8, 6) = 24.$

14. **(b, c)** $\boxed{\text{Using T-8}}$ $f(x) = x^2 - 3x + 2 \;\Rightarrow\;$ Parabola open upwards



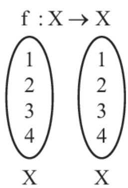
$\Rightarrow$ Many-one function

$\boxed{\text{Using T-9}}$ $\left.\begin{array}{l} x \to \infty \quad \Rightarrow \quad f(x) \to \infty \\ x \to -\infty \quad \Rightarrow \quad f(x) \to \infty \end{array}\right\} \Rightarrow$ Into function

15. **(24)** $\boxed{\text{Using T-10(ii)}}$ $m = 4$ and $n = 4$

$^nP_m = {}^4P_4 = \dfrac{4!}{0!}$

$n = m$



$\Rightarrow$ No. of one-one functions = 24

$\boxed{\text{Using T-10(iv)}}$ Since, $n = m = 4$

$\Rightarrow$ No. of onto functions $= n! = 4! = 24.$

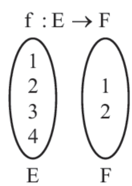
16. **(a)** $\boxed{\text{Using T-10(iv)}}$ $m = 4$ and $n = 2$

Since, $n < m$

$\Rightarrow n^m - {}^nC_1\,(n-1)^m + {}^nC_2\,(n-2)^m - {}^nC_3\,(n-3)^m + \ldots$

$= \; 2^4 - {}^2C_1\,(1)^4 + {}^2C_2 \times 0$

$= 16 - 2 = 14.$



17. **(d)** $\boxed{\text{Using SC-3}}$

Since, $g(x)$ is reflection about $y = x$ line of $f(x)$

$\Rightarrow g(x)$ is inverse of $f(x)$

Hence, $y = (x + 1)^2 \;\Rightarrow\; x + 1 = \sqrt{y} \;\Rightarrow\; x = -1 + \sqrt{y}$

$\Rightarrow f^{-1}(x) = -1 + \sqrt{x}\;;\; x \ge 0.$

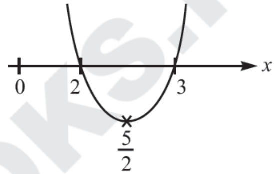
18. **(a)** $\boxed{\text{Using T-8(ii)}}$ Differentiate $f(x)$

$$\Rightarrow \quad f'(x) = 2 + \underbrace{\cos x}_{[-1,1]} \Rightarrow f'(x) > 0 \Rightarrow \text{one-one function}$$

$\boxed{\text{Using T-9}}$ $f(x) = 2x + \underbrace{\sin x}_{[-1,1]}$

$$\left.\begin{array}{lll} x \to \infty & \Rightarrow & f(x) \to \infty \\ x \to -\infty & \Rightarrow & f(x) \to -\infty \end{array}\right\} \Rightarrow \text{Onto function.}$$

19. **(d)** $\boxed{\text{Using T-8(ii)}}$ Differentiate $f(x)$



$$f'(x) = 6x^2 - 30x + 36$$
$$= 6(x^2 - 5x + 6)$$

$$\left.\begin{array}{ll} \text{For,} & x \in [0, 2] \Rightarrow f'(x) > 0 \\ \text{For,} & x \in [2, 3] \Rightarrow f'(x) < 0 \end{array}\right\}$$

Hence, many-one function.

20. **(c)** $\dfrac{y}{1} = f(x) = \dfrac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}; x \in (-1, 1)$

Apply componendo and dividendo

$$\Rightarrow \frac{y+1}{y-1} = \frac{2.8^{2x}}{-2.8^{-2x}} = -8^{4x}$$

Take $\log_8$ to both sides

$$\Rightarrow 4x = \log_8\left(\frac{y+1}{1-y}\right) \Rightarrow f^{-1}(x) = \frac{1}{4}\log_8\left(\frac{y+1}{1-y}\right) \quad \{\text{Change base}\}$$

$$\Rightarrow f^{-1}(x) = \frac{1}{4}\,\frac{\log_e\left(\dfrac{x+1}{1-x}\right)}{\log_e 8}$$

21. **(a, b, c)** For even/odd function

$$\Rightarrow f(-x) = [\log(\sec x - \tan x)]^3$$

$$= \left(\log\left[\frac{\sec^2 x - \tan^2 x}{\sec x + \tan x}\right]\right)^3 = \left[\log\left(\frac{1}{\sec x + \tan x}\right)\right]^3$$

$$= -[\log(\sec x + \tan x)]^3$$

$f(-x) = -f(x) \Rightarrow$ Odd function

For one-one function

$\boxed{\text{Using T-8(ii)}}$ $f'(x) = 3[\log(\sec x + \tan x)]^2 \times \dfrac{1 \cdot \sec x(\sec x + \tan x)}{(\sec x + \tan x)}$

$\Rightarrow f'(x) = 3[\log(\sec x + \tan x)]^3 \underbrace{\sec x}_{>0 \text{ for } \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)}$

Since, $f'(x) > 0$

$\Rightarrow$ One-one function

Since, range = codomain $\in R$. Hence, onto function.

22. **(c)** Put $x = -7$

$\Rightarrow f(-7) = -(a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) - 5 \Rightarrow (a \cdot 7^7 + b \cdot 7^3 + c \cdot 7) = -12$

Now, put $x = 7$

$f(7) = \underbrace{a \cdot 7^7 + b \cdot 7^3 + c \cdot 7} - 5 = -12 - 5 = -17.$

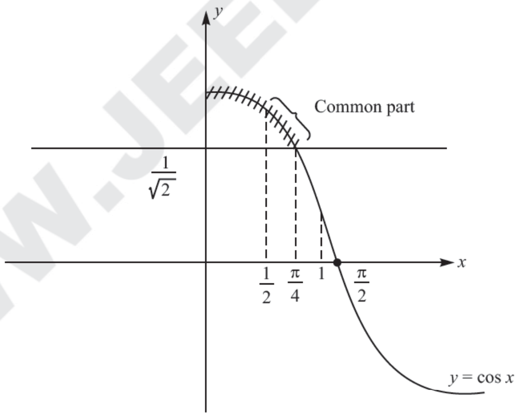
23. **(c)** $\boxed{\text{Using T-13}}$ Since period of $\cos x$ is 2π

$\Rightarrow$ Period of $f(x)$ is $\dfrac{2\pi}{3}$.

24. $x \in \left(\frac{1}{2}, \frac{\pi}{4}\right]$

(1) $\cos x - \dfrac{1}{\sqrt{2}} \geq 0$ (2) $\dfrac{3}{2}x - x^2 - \dfrac{1}{2} > 0$

$\left(x - \dfrac{1}{2}\right)(x-1) < 0$ $x \in \left(+\dfrac{1}{2}, 1\right)$



Hence, the common part is the graph will give the domain of $f(x)$

So the domain is $x \in \left(\dfrac{1}{2}, \dfrac{\pi}{4}\right]$.

25. **(a)** $\boxed{\text{Using T-8(ii)}}$ $f'(x) = \dfrac{(1+x) - x}{(1+x)^2} > 0$

Hence $f(x)$ is one-one function

Since, in the co-domain $\to [0, \infty)$; '0' is included

But in domain $x \neq 0 \Rightarrow f(x) \neq 0$

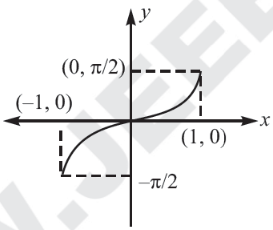
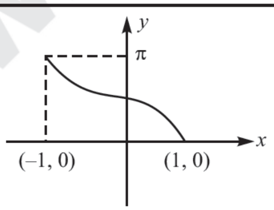
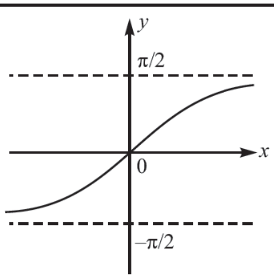
Hence, Range $\neq$ Co-domain $\Rightarrow$ Not onto

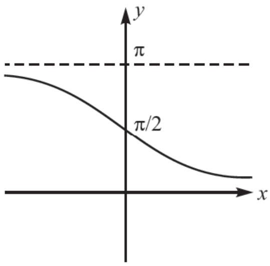
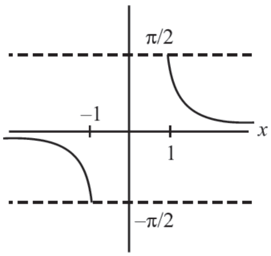
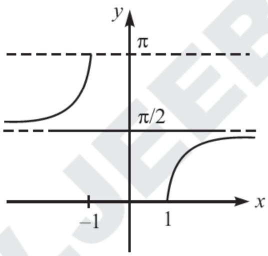
2 Inverse Trigonometric Functions

Review of Key Notes and Formulae

1. **Inverse Trigonometric Function:** As we know that trigonometric functions are not one-one and onto i.e. their natural domain and range, so their inverse do not exist but if we restrict their domain and range, then their inverse may exist.

2. **Graphs of Inverse Trigonometric Functions**

	Function	Graph	Domain	Range
(i)	$y = \sin^{-1} x$		$x \in [-1, 1]$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(ii)	$y = \cos^{-1} x$		$x \in [-1, 1]$	$x \in [0, \pi]$
(iii)	$y = \tan^{-1} x$		$x \in R$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv)	$y = \cot^{-1} x$		$x \in R$	$y \in (0, \pi)$
(v)	$y = \text{cosec}^{-1} x$		$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$
(vi)	$y = \sec^{-1} x$		$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

1. Properties of ITF (Inverse Trigonometric Function)

Property-1

(i) $\sin(\sin^{-1} x) = x$; $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$; $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$; $x \in \mathrm{R}$

(iv) $\cot(\cot^{-1} x) = x$; $x \in \mathrm{R}$

(v) $\text{cosec}(\text{cosec}^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\sec(\sec^{-1} x) = x$; $x \in (-\infty, -1] \cup [1, \infty)$

Property-2

(i) $\sin^{-1}(\sin x) = x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos x) = x$; $x \in [0, \pi]$

(iii) $\tan^{-1}(\tan x) = x$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot x) = x$; $x \in (0, \pi)$

(v) $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x\ ;\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(vi) $\sec^{-1}(\sec x) = x\ ; x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Very Important Note:

If 'x' is not given according to above domain then make it between the above domain by using "$\pm n\pi$"

Ex:

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}.$$

Property-3

(i) $\sin^{-1}(-x) = -\sin^{-1} x \quad \forall\, x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x \quad \forall\, x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x \quad \forall\, x \in R$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x \quad \forall\, x \in R$

(v) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x \quad \forall\, x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x \quad \forall\, x \in (-\infty, -1] \cup [1, \infty)$

Property-4

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\ ; x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\ ; x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}\ ; x \in (-\infty, -1] \cup [1, \infty)$

Property-5

(i) $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right); x \in [-1, 1] - \{0\}$

(ii) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right); x \in [-1, 1] - \{0\}$

(iii) $\tan^{-1} x = \begin{cases} \cot^{-1}\frac{1}{x} & ; x > 0 \\ -\pi + \cot^{-1}\frac{1}{x} & ; x < 0 \end{cases}$

Property-6

(i) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x<0, y<0, xy>1 \\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & xy<1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x>0, y>0, xy>1 \end{cases}$$

(ii) $$\tan^{-1}x - \tan^{-1}y = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & x<0, y<0, xy<-1 \\ \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy>-1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & x>0, y<0 \text{ and } xy>-1 \end{cases}$$

Property-7

$$\sin^{-1}x \pm \sin^{-1}y = \begin{cases} \sin^{-1}\left[x\sqrt{1-y^2} \pm \sqrt{1-x^2}\right]; x, y \geq 0 \text{ and } x^2+y^2 \leq 1 \\ \pi - \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]; x, y \geq 0 \text{ and } x^2+y^2 > 1 \end{cases}$$

Property-7.1

$$\cos^{-1}x \pm \cos^{-1}y = \begin{cases} \cos^{-1}\left[xy \mp \sqrt{1-x^2}\,\sqrt{1-y^2}\right]; x, y > 0 \text{ and } x^2+y^2 \leq 1 \\ \pi - \cos^{-1}\left[xy \mp \sqrt{1-x^2}\,\sqrt{1-y^2}\right]; x, y > 0 \text{ and } x^2+y^2 > 1 \end{cases}$$

Property-8: Simplified Trigonometric Functions

(i) $$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1}x; & x<-1 \\ 2\tan^{-1}x; & -1 \leq x \leq 1 \\ \pi - 2\tan^{-1}x; & x>1 \end{cases}$$

(ii) $$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x; & x<-1 \\ 2\tan^{-1}x; & -1<x<1 \\ -\pi + 2\tan^{-1}x; & x>1 \end{cases}$$

(iii) $$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} -2\tan^{-1}x; & x \leq 0 \\ 2\tan^{-1}x; & x \geq 0 \end{cases}$$

(iv) $$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right);$$

if $x>0, y>0, z>0$ and $(xy+yz+zx)<1$

★**Note:** (i) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then $xy + yz + zx = 1$

(ii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then $xy + yz + zx = xyz$.

Important Points to Remember

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) = \sin^{-1}\frac{x}{a}$

(ii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$

(iii) $\tan^{-1}\left(\frac{3a^2x - x^3}{a(a^2 - 3x^2)}\right) = 3\tan^{-1}\frac{x}{a}$

(iv) $\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots\ldots + \tan^{-1} x_n = \tan^{-1}\left[\frac{S_1 - S_3 + S_5 \ldots\ldots}{1 - S_2 + S_4 - S_6 + \ldots\ldots}\right]$

where, S_k denotes the sum of the product of $x_1, x_2, \ldots, x_n$ taken 'K' at a time.

★**Note:** $\tan^{-1}(\sqrt{2}+1) = \frac{3\pi}{8}$

$\tan^{-1}(2-\sqrt{3}) = \frac{\pi}{12}$

TIPS AND TRICKS: (T-1)

" '0' (zero) or '1' and you are done".

This is a substitution method such that we can substitute the values of x and y be '0' or '1' or vice-versa. This works in maximum cases. In case of any failure check for the other values of the variables (x, y, etc) like 1, 2, 3, ... etc.

★★**Note:** Always back check in this type.

Illustration 1

If $x > 0$, $y > 0$ and $x > y$, then find $\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x+y}{x-y}\right)$.

[AIEEE 2005]

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = 1, \quad y = 1$

$\Rightarrow \quad \tan^{-1}1 + \tan^{-1}(\infty) = \dfrac{\pi}{4} + \dfrac{\pi}{2} = \dfrac{3\pi}{4}$ **Ans. (a)**

Illustration 2

If $\cos^{-1} x - \cos^{-1} \dfrac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2 =$

(a) $-4 \sin^2 \alpha$ (b) $4 \sin^2 \alpha$ (c) 4 (d) $2 \sin 2\alpha$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $x = y = 1 \;\Rightarrow\; \alpha = -\dfrac{\pi}{3}$

Now, put 'α' in the required eqn.

$\Rightarrow 4x^2 - 4xy \cos\left(-\dfrac{\pi}{3}\right) + y^2$ $(\because x = y = 1)$

$\Rightarrow \quad 4 - 4 \times \dfrac{1}{2} + 1 = 3$

Now, check option A, B, C, D for $\alpha = -\dfrac{\pi}{3}$

$\Rightarrow \quad 4 \times \sin^2 \alpha = 4 \times \sin^2\left(-\dfrac{\pi}{3}\right) = 3$ **Ans (b)**

Illustration 3

$$\tan^{-1}\left(\frac{a_1x - y}{a_1y + x}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2a_3}\right) + \ldots$$

$$+ \ldots \tan^{-1}\left(\frac{a_n - a_{n+1}}{1 + a_na_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right).$$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Put $a_1 = a_2 = a_3 = \ldots a_n = 0$

$\Rightarrow \quad -\tan^{-1}\left(\dfrac{y}{x}\right) + 0 + \ldots + 0 + \underbrace{\tan^{-1}(\infty)}_{\pi/2}$

$= \dfrac{\pi}{2} - \tan^{-1}\dfrac{y}{x} = \cot^{-1}\left(\dfrac{y}{x}\right).$

Illustration 4

If $x > 0, y > 0$, then $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

Short-cut solution :

Using T-1 Put $x = y = 1$

$\Rightarrow \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$. **Ans. (b)**

Illustration 5

If $0 \le x \le \frac{1}{2}$, then value of $\tan\left(\sin^{-1}\left(\frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}}\right) - \sin^{-1}x\right)$ is

(a) 0 (b) – 1 (c) 1 (d) $\frac{\pi}{4}$

Short-cut solution :

Using T-1 Since $0 \le x \le \frac{1}{2}$, put $x = 0$

$$\Rightarrow \tan\left(\sin^{-1}\frac{1}{\sqrt{2}}\right) = \tan\frac{\pi}{4} = 1.$$ **Ans. (c)**

Illustration 6

If $x^2 + y^2 + z^2 = k^2$, then value of

$\tan^{-1}\left(\frac{xy}{zk}\right) + \tan^{-1}\left(\frac{xz}{yk}\right) + \tan^{-1}\left(\frac{zy}{xk}\right)$ is equals to

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 0

Short-cut solution :

Using T-1 Put $x = y = z = 1$

$\Rightarrow K^2 = 3 \Rightarrow K = \sqrt{3}$

Hence, $\tan^{-1}\left(\frac{1\cdot 1}{1\cdot\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2}$. **Ans. (a)**

Illustration 7

The value of $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(\frac{x}{y}\right)\right\}+\tan^{-1}\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{x}{y}\right)\right\}$

is equal to

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{2x}{y}$ (d) $\frac{2y}{x}$

Short-cut solution :

Using T-1 Put $x = 1$ and $y = 2$

$$\Rightarrow \tan\left\{\frac{\pi}{4}+\frac{1}{2}\times\frac{\pi}{3}\right\}+\tan\left\{\frac{\pi}{4}-\frac{\pi}{3}\times\frac{1}{2}\right\} = \tan\left\{\frac{5\pi}{12}\right\}+\tan\left\{\frac{\pi}{12}\right\}$$

$$= \tan 75° + \tan 15° = 2+\sqrt{3}+2-\sqrt{3} = 4$$

Now, check for $x = 1$ and $y = 2$ in options (a), (b), (c), (d).

$\Rightarrow \frac{2y}{x} = 4.$ **Ans. (d)**

TIPS AND TRICKS: (T-2)

Short trick to solve series produces in inverse trigonometric functions.
In this we can take value of 'n' be 1, 2 or 3, ... etc. to minimize the steps. And after that check the option for those value of 'n' which has taken.

Illustration 8

The value of

$$\tan^{-1}\left(\frac{x}{1+2x^2}\right)+\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\tan^{-1}\left(\frac{x}{1+12x^2}\right)+\ldots+n \text{ terms}$$

(a) $\tan^{-1}(n+1)x-\tan^{-1}x$ (b) $\tan^{-1}(n+1)x+\tan^{-1}x$
(c) $\tan^{-1}(n-1)-\tan^{-1}x$ (d) $\tan^{-1}(n-1)x-\tan^{-1}x$

Short-cut solution :

Using T-2 Put $n = 1$

$$\Rightarrow \tan^{-1}\left(\frac{x}{1+2x\times x}\right)=\tan^{-1}\left(\frac{2x-x}{1+2x\times x}\right)$$

As we know that $\tan^{-1}\left(\frac{A-B}{1+AB}\right)=\tan^{-1}A-\tan^{-1}B$

$\Rightarrow \tan^{-1}2x-\tan^{-1}x$

Now, check for $n = 1$ in options (a), (b), (c), (d). **Ans. (a)**

Illustration 9

The value of

$$\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\ldots n \text{ terms}$$

(a) $\tan^{-1}(x-n)-\tan^{-1}x$ (b) $\tan^{-1}(x+n)-\tan^{-1}x$

(c) $\tan^{-1}(n-x)-\tan^{-1}x$ (d) $\tan^{-1}(x+n+1)-\tan^{-1}x$

Short-cut solution :

$\boxed{\text{Using T-2}}$ Put $n = 1$

$$\Rightarrow \quad \tan^{-1}\frac{1}{1+x(x+1)}=\tan^{-1}\left(\frac{(x+1)-1}{1+x(x+1)}\right)$$

As we know that $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$\Rightarrow \tan^{-1}(x+1)-\tan^{-1}1$

Check options (a), (b), (c), (d) for $n = 1$. **Ans. (b)**

TIPS AND TRICKS: (T-3)

TIPS & TRICKS!

In this trick first of all we will use substitution method as mentioned in T-1 and then stabilizing the components (options).

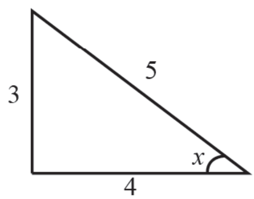
Illustration 10

The value of $\cos^{-1}\left(\frac{4+5\cos x}{5+4\cos x}\right)$.

(a) $2\tan^{-1}\left(\frac{3}{4}\tan\frac{x}{2}\right)$ (b) $2\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$

(c) $2\tan^{-1}\left(\frac{3}{5}\tan x\right)$ (d) $2\tan^{-1}\left(\frac{3}{4}\tan x\right)$

Short-cut solution :



$\boxed{\text{Using T-3}}$ Put $x = 90°$

$$\Rightarrow \quad \cos^{-1}\left(\frac{4+0}{5+0}\right)=\tan^{-1}\left(\frac{3}{4}\right)$$

Now, we will check (a), (b), (c), (d) options.

Stabilizing $\Rightarrow \quad \tan^{-1}\frac{3}{4}=2\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)$

$\Rightarrow$ Take 'tan' to both sides $\Rightarrow \quad \frac{3}{4}=\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}=\frac{3}{4}$. **Ans. (b)**

Illustration 11

If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \dfrac{3\pi}{2}$ then,

(a) $x^{100} + y^{100} + z^{100} - \dfrac{9}{x^{101} + y^{101} + z^{101}} = 0$

(b) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(c) $x^{50} + y^{25} + z^{5} = 0$

(d) $\dfrac{x^{2008} + y^{2008} + z^{2008}}{(xyz)^{2009}} = 0$

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \dfrac{\pi}{2}$

$\Rightarrow x = y = z = 1$

Now, check options (a), (b), (c), (d) for $x = y = z = 1$ **Ans. (a, b)**

Illustration 12

Solve $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \dfrac{\pi}{2}$, then $x =$

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) 0 (d) 1

Short-cut solution :

$\boxed{\text{Using T-1}}$

Check options (a), (b), (c), (d).

Put $x = \dfrac{1}{2} \Rightarrow \cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\dfrac{1}{2} = \dfrac{\pi}{6} + \dfrac{\pi}{3} = \dfrac{\pi}{2}$ **Ans. (b)**

Illustration 13

If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \dfrac{3\pi}{4}$ then the value of $xy + yz + zx$ is,

(a) 3 (b) 2 (c) -3 (d) -2

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $x = y = z = 1$

$\Rightarrow \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \dfrac{3\pi}{4}$

Then, $xy + yz + zx = 3$ **Ans. (a)**

Illustration 14

$$\tan^{-1}\left(\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right); \frac{\pi}{2} \le x < \pi \text{ is equal to}$$

(a) $\frac{x}{2}-\frac{\pi}{2}$ (b) $\frac{\pi}{2}-\frac{x}{2}$ (c) $\pi - x$ (d) $2\pi - x$

Short-cut solution :

Using T-1 Put $x = \frac{\pi}{2}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{0+\sqrt{2}}{0-\sqrt{2}}\right) = -\tan^{-1} 1 = \frac{-\pi}{4}$$

Now, check options (a), (b), (c), (d)

$\Rightarrow \left(\frac{x}{2}-\frac{\pi}{2}=\frac{-\pi}{4}\right).$ **Ans. (a)**

Illustration 15

$$\left[\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right]^{\frac{1}{2}} \text{ is equal to:}$$ **[JEE M 2013]**

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\sqrt{2}$

Short-cut solution :

Using T-1 Put $y = 1$

$\Rightarrow [0^2 + 1] = 1$ **Ans. (a)**

SHORTCUTS: (SC-1)

In order to solve inequilities in inverse trigonometric functions use graphs to minimizing the steps.

Illustration 16

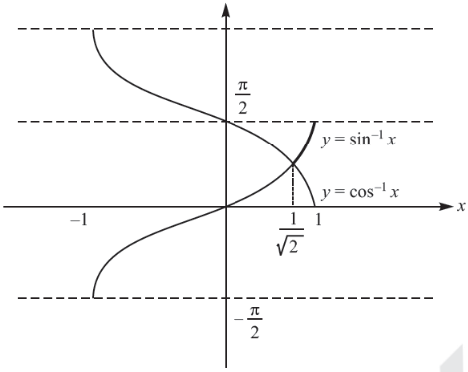
The values of 'x' for which $\sin^{-1} x > \cos^{-1} x \ \forall\ x \in (-1, 1)$.

Short-cut solution :

Using SC-1

We will draw graphs of both functions

$y = \sin^{-1} x$ and $y = \cos^{-1} x$



We will choose that part in the graph which is greater

$$(\sin^{-1} x > \cos^{-1} x)$$

Hence, it is clear from the graph $\Rightarrow \; x \in \left[\frac{1}{\sqrt{2}}, 1\right)$

TECHNIQUE

If the input of inverse trigonometric identity are infinite series and different then change the inputs in simple form using G.P. and then equate.

Illustration 17

If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$,

for $0 < |x| < \sqrt{2}$ then find the value of x.

Short-cut solution :

$\boxed{\text{Using Tech.}}$ $\because \; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... = \frac{x}{1 + \frac{x}{2}} = \frac{2x}{2 + x}$

And $x^2 - \frac{x^4}{2} + \frac{x^6}{4} - .... = \frac{x^2}{1 + \frac{x^2}{2}} = \frac{2x^2}{2 + x^2}$

$\therefore \; \sin^{-1}\left(\frac{2x}{2 + x}\right) + \cos^{-1}\left(\frac{2x^2}{2 + x^2}\right) = \frac{\pi}{2}$ and this is true when

$\frac{2x}{2 + x} = \frac{2x^2}{2 + x^2} \Rightarrow 2x + x^3 = 2x^2 + x^3$

$\Rightarrow \quad 2x^2 - 2x = 0 \Rightarrow x(x - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$

$\Rightarrow \quad x = 1 \qquad$ (x cannot be 0 as $0 < |x| < \sqrt{2}$)

Concept Booster Exercise

1. If $\cos \tan^{-1} \sin \cot^{-1} x = P$; then find 'P'

(a) $\sqrt{\dfrac{x^2+1}{x^2+2}}$ (b) $\sqrt{\dfrac{x^2-1}{x^2+2}}$

(c) $\sqrt{\dfrac{x^2+1}{x^2-2}}$ (d) $\sqrt{\dfrac{x^2+3}{x^2-1}}$

2. $2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right)$; $(a > 0, b > 0)$

(a) $\dfrac{a+b}{1+ab}$ (b) $\dfrac{a+b}{1-ab}$ (c) $\dfrac{a-b}{1+ab}$ (d) $\dfrac{a-b}{1-ab}$

3. If $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, then

(a) $1 + \sin u = \tan \theta$ (b) $1 + \sin u = \tan^2 \theta$

(c) $1 - \sin u = \tan^2 \theta$ (d) $\sin u = \tan^2 \theta$

4. $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$ is equal to:

(a) $1 + \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(b) $1 - \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(c) $\cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$

(d) $\tan(1 + \tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$

5. The value of $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2}$ is equal to **[AIEEE 2008]**

(a) $\dfrac{x}{\sqrt{1+x^2}}$ (b) x (c) $x\sqrt{1+x^2}$ (d) $\dfrac{\sqrt{1+x^2}}{1+x^2}$

6. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ where $|x| < \dfrac{1}{\sqrt{3}}$. Then the value of y is: **[JEE M 2015]**

(a) $\dfrac{3x-x^3}{1+3x^2}$ (b) $\dfrac{3x+x^3}{1+3x^2}$ (c) $\dfrac{3x-x^3}{1-3x^2}$ (d) $\dfrac{3x+x^3}{1-3x^2}$

7. Find value of $\sin^{-1}\cos(\sin^{-1}x)+\cos^{-1}\sin(\cos^{-1}x)$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

8. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$ is equal to

(a) xyz (b) $\frac{1}{xyz}$ (c) 1 (d) 0

9. $\tan\left[\tan^{-1}\left(\frac{2}{1+1\cdot2}\right)+\tan^{-1}\left(\frac{2}{1+2\cdot3}\right)+\ldots+\tan^{-1}\left(\frac{2}{1+n(n+1)}\right)\right]$

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$

10. If p, q, r are positive real numbers and

$$\theta=\tan^{-1}\left(\sqrt{\frac{p(p+q+r)}{qr}}\right)+\tan^{-1}\left(\sqrt{\frac{q(p+q+r)}{pr}}\right)+\tan^{-1}\left(\sqrt{\frac{r(p+q+r)}{pq}}\right),$$

then $\tan\theta$ is equal to

(a) 1 (b) 0 (c) $\frac{p+q+r}{pqr}$ (d) $\frac{pqr}{p+q+r}$

11. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then $x+y+z=$

(a) $xy+yz+zx$ (b) xyz (c) $\frac{1}{xyz}$ (d) $-xyz$

12. Let $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$ then

$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=$

(a) xyz (b) $\frac{1}{2}xyz$ (c) $2xyz$ (d) $\frac{xyz}{3}$

Numerical Value Problems

13. If $\cos^{-1}x+\cos^{-1}y=2\pi$, and $\sin^{-1}x+\sin^{-1}y=P$, then find value of $4+[P]$; where $[x]$ is greatest integer function.

14. Find number of solutions of $\cos^{-1}x=\tan^{-1}x\ \forall\ x\in[-1, 1]$.

15. Find number of solutions of $\sin^{-1}x=1-x\ \forall\ x\in[-1, 1]$.

16. The values of 'x' satisfying the inequality $\cos^{-1}x>\sin^{-1}x\ \forall\ x\in[-1, 1]$ is $[-1, P]$, then find the value of [P]; where [ ] is greatest integer function.

17. The value of 'x' satisfying the inequality $2(\tan^{-1}x)^2 - 5\tan^{-1}x + 3 < 0$ is $(\tan K_1, \tan K_2)$ then find the value of $K_1 + 2K_2$.
↑
(*Two*)

Solutions

1. **(a)** $\boxed{\text{Using T-1}}$ Put $x = 0$

$\Rightarrow$ $\cos\tan^{-1}\sin\cot^{-1}0 = \dfrac{1}{\sqrt{2}} = \text{P}$

Now check options (a), (b), (c), (d) for $x = 0$

2. **(c)** $\boxed{\text{Using T-3}}$ Put $a = b = 1$

$\Rightarrow$ $2\tan^{-1}x = 0 \Rightarrow x = 0$

Now check options (a), (b), (c), (d) for $a = b = 1$

3. **(d)** $\boxed{\text{Using T-1}}$ Put $\theta = 0 \Rightarrow u = 0$

Now check options (a), (b), (c), (d) for $\theta = 0$ and $u = 0$

4. **(c)** $\boxed{\text{Using T-3}}$ Put $x = y = z = 0 \Rightarrow \tan 0 = 0$

Now check options (a), (b), (c), (d) for $x = y = z = 0$

5. **(c)** $\boxed{\text{Using T-1}}$ Put $x = 1$

$$\Rightarrow \sqrt{2}\left[\left\{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right\}^2 - 1\right]^{\frac{1}{2}} = \sqrt{2}$$

Now, check options a, b, c, d for $x = 1$

6. **(c)** $\boxed{\text{Using T-1}}$ Put $x = 1 \Rightarrow \tan^{-1}y = \dfrac{3\pi}{4} \Rightarrow y = -1$

Now, check options (a), (b), (c), (d) for $x = 1$

7. **(a)** $\boxed{\text{Using T-1}}$ Put $x = 0$

$$\Rightarrow \sin^{-1}\cos(0) + \cos^{-1}\sin\frac{\pi}{2} = \frac{\pi}{2}.$$

8. **(c)** $\boxed{\text{Using T-3}}$ Put, $\tan^{-1}x = \tan^{-1}y = \tan^{-1}z = \dfrac{\pi}{3}$

$\Rightarrow$ $x = y = z = \sqrt{3}$

Now, $\dfrac{1}{xy} + \dfrac{1}{yz} + \dfrac{1}{zx} = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3} = 1.$

9. **(b)** $\boxed{\text{Using T-2}}$ Put $n = 1$

$$\Rightarrow \tan\left(\tan^{-1}\left(\frac{2}{3}\right)\right) = \frac{2}{3}.$$

10. **(b)** $\boxed{\text{Using T-2}}$ Put $p = q = r = 1$

$$\Rightarrow \theta = \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3} = \pi.$$

11. **(b)** $\boxed{\text{Using T-3}}$ Put $x = y = z = \sqrt{3}$

$\Rightarrow x + y + z = 3\sqrt{3} = xyz.$

12. **(c)** $\boxed{\text{Using T-3}}$ Put $\sin^{-1}x = \sin^{-1}y = \sin^{-1}z = \frac{\pi}{3}$

$$\Rightarrow x = y = z = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = \frac{3\times\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$$

Now, check options (a), (b), (c), (d) for $x = y = z = \frac{\sqrt{3}}{2}$

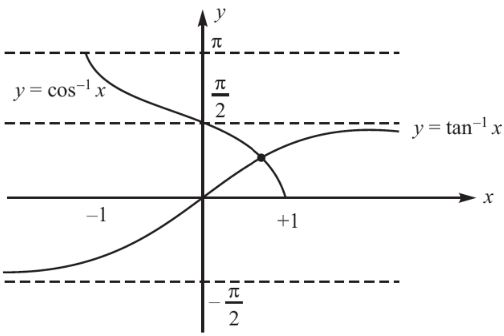
$$\Rightarrow 2\,xyz = 2\times\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{4}$$

13. **(0)** $\boxed{\text{Using T-3}}$ Put $\cos^{-1}x = \cos^{-1}y = \pi \Rightarrow x = y = -1$

Now, $\sin^{-1}(-1) + \sin^{-1}(-1) = P = -\pi$

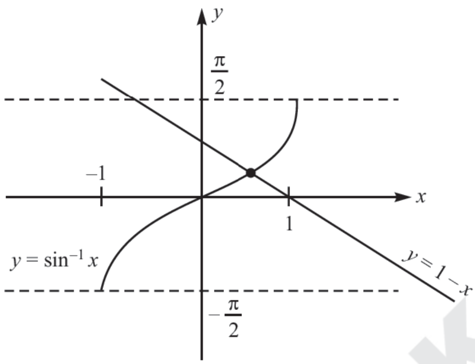
$\Rightarrow 4 + [-\pi] = 4 + (-4) = 0$

14. **(1)** $\boxed{\text{Using SC-1}}$ Drawing graphs of $y = \cos^{-1}x$ and $y = \tan^{-1}x$



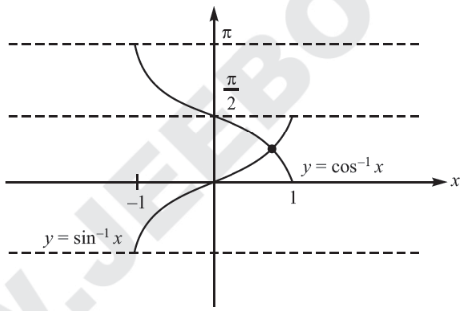
Number of intersection = 1 = No. of solution.

15. **(1)** Using SC-1 Drawing graphs of $y = \sin^{-1} x$ and $y = 1 - x$



Number of intersection = No. of solution = 1.

16. **(0)** Using SC-1 Drawing graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$



Number of intersection = No. of solution = 1.

$\therefore \cos^{-1}x > \sin^{-1}x$ when $x \in \left[-1, \frac{1}{\sqrt{2}}\right]$

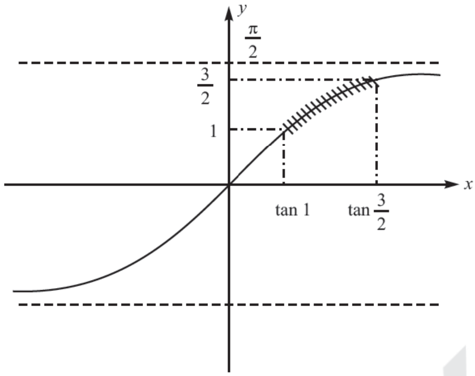
$$\therefore P = \frac{1}{\sqrt{2}} \Rightarrow [P] = \left[\frac{1}{\sqrt{2}}\right] = [0.707] = 0$$

17. **(4)** Using SC-1 $2(\tan^{-1} x)^2 - 5\tan^{-1} x + 3 < 0$

$$\Rightarrow \quad (\tan^{-1} x - 1)\left(\tan^{-1} x - \frac{3}{2}\right) < 0$$

$$\Rightarrow \quad \tan^{-1} x \in \left(1, \frac{3}{2}\right)$$

Now, drawing graph of $y = \tan^{-1} x$



Hence, it is clear from graph that

$$x \in \left(\tan 1, \tan\frac{3}{2}\right) \Rightarrow K_1 = 1, K_2 = \frac{3}{2}.$$

Hence, $$K_1 + 2K_2 = 1 + \frac{3}{2}\cdot 2 = 4.$$

3 Limits

Review of Key Notes and Formulae

1. If $y = f(x)$ function takes indeterminate form at $x = a$, then we use concept of limits.

2. 7 indeterminate forms are: $\left\{\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^\circ, \infty^\circ, 1^\infty\right\}$

3. Process to apply concept of limits is:

$\lim\limits_{x \to a} f(x) \Rightarrow$ Behaviour of $f(x)$ in neighbourhood of $x = a$ $(x \neq a)$

Step 1: Find left hand limit $= \lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a - h)$

Step 2: Find right hand limit $= \lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a + h)$

$$\boxed{\text{If LHL} = \text{RHL} = \text{a finite quantity}}$$

$\Rightarrow$ Limit exists at $x = a$.

(where, 'h' is very small positive quantity)

4. Fundamental Theorem Over Limits

If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} f(x)$ exists and is finite then,

(i) $\lim\limits_{x \to a} f(x) \pm g(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

(ii) $\lim\limits_{x \to a} f(x) \cdot g(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

(iii) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}; \lim\limits_{x \to a} g(x) \neq 0$

(iv) $\underset{x\to a}{\text{Lim}}\left(K f(x)\right) = K \underset{x\to a}{\text{Lim}}\, f(x)$

(v) $\underset{x\to a}{\text{Lim}}\, f\left(g(x)\right) = f\left(\underset{x\to a}{\text{Lim}}\, g(x)\right)$;

provided $f(x)$ is continuous at $x = \underset{x\to a}{\text{Lim}}\, g(x)$

5. Methods of Finding Algebraic Limits

(i) Factorization

(ii) Rationalization

(iii) Double-rationalization

(iv) Use of Binomial theorem (Neglect higher order terms)

6. Limits of Trigonometric Function

(i) $\underset{x\to 0}{\text{Lim}} \dfrac{\sin x}{x} = \underset{x\to 0}{\text{Lim}} \dfrac{x}{\sin x} = \underset{x\to 0}{\text{Lim}} \dfrac{x}{\sin^{-1} x} = \underset{x\to 0}{\text{Lim}} \dfrac{\sin^{-1} x}{x}$

$= \underset{x\to 0}{\text{Lim}} \dfrac{x}{\tan x} = \underset{x\to 0}{\text{Lim}} \dfrac{\tan^{-1} x}{x} = 1$

(ii) $\underset{x\to 0}{\text{Lim}} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

(iii) $\underset{x\to 0}{\text{Lim}} \dfrac{\sin x - x}{x^3} = \dfrac{-1}{6}$

(iv) $\underset{\to}{\text{Lim}} \dfrac{\tan x \quad x}{} \quad \dfrac{1}{}$

(v) $\underset{x\to 0}{\text{Lim}} \dfrac{\sin x}{x} = \dfrac{\pi}{180}$

7. Limits of Logarithmic and Exponential Functions

(i) $\underset{x\to 0}{\text{Lim}} \dfrac{\ln(1+x)}{x} = 1$

(ii) $\underset{x\to 0}{\text{Lim}} \dfrac{\ln(1+x) - x}{x^2} = \dfrac{-1}{2}$

(iii) $\underset{x\to 0}{\text{Lim}} \dfrac{e^x - 1}{x} = 1$

(iv) $\underset{x\to 0}{\text{Lim}} \dfrac{e^x - x - 1}{x^2} = \dfrac{1}{2}$

(v) $\underset{x\to 0}{\text{Lim}} \dfrac{e^x - e^{-x} - 2x}{x^3} = \dfrac{1}{3}$

(vi) $\underset{x\to 0}{\text{Lim}} (1+\lambda x)^{1/x} = e^{\lambda}$

(vii) $\underset{x\to 0}{\text{Lim}} \dfrac{a^x - 1}{c} = \ln a$

8. L' Hospital Rule

Let $f(x)$ and $g(x)$ be two functions such that

(i) $\underset{x\to a}{\text{Lim}}\, f(x) = \underset{x\to a}{\text{Lim}}\, g(x) = 0$ OR Does not exist.

(ii) $f(x)$ and $g(x)$ must be continuous and differentiable at $x = a$.

(iii) $f'(x)$ and $g'(x)$ are continuous at $x = a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} \begin{Bmatrix}\text{Provide } f''(x) \text{ and } g''(x) \\ \text{are continuous at } x = a\end{Bmatrix}$$

★ We can continue this process until indeterminate form vanishes.

TIPS AND TRICKS: (T-1)

In case of algebraic rational expression we can use Binomial theorem to evaluate the limit.

As we know that: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2.1}x^2 + \ldots$

★ We can neglect higher order terms for $x \to 0$

$\Rightarrow \quad (1+x)^n \approx 1 + nx$

Illustration 1

Find the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\quad l = \lim\limits_{x\to 0}\dfrac{1-\frac{x^2}{2}-1-\frac{x^2}{2}}{2x^2} = \dfrac{-1}{2}$

Illustration 2

Find the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-\sqrt{1-2x}}{\sin x}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\quad l = \dfrac{1+\frac{2x}{2}-1+\frac{2x}{2}}{\sin x} = \dfrac{2x}{\sin x} = 2$

Illustration 3

Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation

$\left(\sqrt[3]{1+a}-1\right)x^2 + \left(\sqrt{1+a}-1\right)x + \left(\sqrt[6]{1+a}-1\right) = 0$

where $a > -1$, then $\lim\limits_{a\to 0^+}\alpha(a)$ and $\lim\limits_{a\to 0^+}\beta(a)$ are

(a) $\frac{-5}{2}$ and 1 (b) $\frac{-1}{2}$ and -1

(c) — and 2 (d) $\frac{-9}{2}$ and 3 **[AIEEE 2012]**

Short-cut solution :

Using T-1 $\left(1+\frac{a}{3}-1\right)x^2+\left(1+\frac{a}{2}-1\right)x+\left(1+\frac{a}{6}-1\right)=0$

$\Rightarrow 2ax^2 + 3xa + a = 0 \Rightarrow 2x^2 + 3x + 1 = 0$

$$\Rightarrow x = -1, \frac{-1}{2}$$

Ans. (b)

TIPS AND TRICKS: (T-2)

"Law of Love-We love infinity in denominator and 0 (zero) in numerator".

If $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ and both $f(x)$ and $g(x)$ are polynomial of x. Then, we divide numerator and denominator by the highest power of x and put '0' for $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$ etc.

Illustration 4

Find the value of $\lim\limits_{x\to\infty}\frac{x^4+3x^2+1}{x^2+5}$

Short-cut solution :

Using T-2 Take highest power outside from N^r and D^r.

$$\Rightarrow l = \frac{x^4\left(1+\frac{3}{x^2}+\frac{1}{x^4}\right)}{x^4\left(\frac{1}{x^2}+\frac{5}{x^4}\right)} = \infty$$

Illustration 5

Find the value of $\lim\limits_{x\to\infty}\frac{2x^3-7x^4+2}{5x^4+3x^2+1}$

Short-cut solution :

Using T-2 $l = \frac{x^4\left(\frac{2}{x}-7+\frac{2}{x^4}\right)}{x^4\left(5+\frac{3}{x^2}+\frac{1}{x^4}\right)} = \frac{-7}{5}$

Illustration 6

Find the value of $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $l = \dfrac{x\left[\sqrt{3-\dfrac{1}{x^2}}-\sqrt{2-\dfrac{1}{x^2}}\right]}{x\left[4+\dfrac{3}{x}\right]} = \dfrac{\sqrt{3}-\sqrt{2}}{4}$

TIPS AND TRICKS: (T-3)

Short tricks to evaluate limits of the form:

(i) $\lim\limits_{x\to 0} \dfrac{\cos ax - \cos bx}{x^2} = \dfrac{b^2-a^2}{2}$

(ii) $\lim\limits_{x\to 0} \dfrac{1-\cos ax}{1-\cos bx} = \dfrac{a^2}{b^2}$

(iii) $\lim\limits_{x\to 0} \dfrac{\cos ax - \cos bx}{\cos cx - \cos dx} = \dfrac{a^2-b^2}{c^2-d^2}$

(iv) $\lim\limits_{x\to \infty} \cos\dfrac{x}{2}\cos\dfrac{x}{4}\cos\dfrac{x}{8}\ldots\cos\dfrac{x}{2^n} = \dfrac{\sin x}{x}$

(v) $\lim\limits_{x\to 0} (\cos x + a\sin bx)^{1/x} = e^{ab}$

(vi) $\lim\limits_{m\to \infty} \left(\cos\dfrac{x}{m}\right)^m = 1$

Illustration 7

Find the value of $\lim\limits_{x\to 0} \dfrac{\cos 25x - \cos 9x}{x^2}$

Short-cut solution :

$\boxed{\text{Using T-3(i)}}$ $l = \dfrac{(9)^2-(25)^2}{2} = -272$

Illustration 8

Find the value of $\lim\limits_{x\to 0} \dfrac{1-\cos 3x}{1-\cos 2x}$

Short-cut solution :

$\boxed{\text{Using T-3(ii)}}$ $l = \dfrac{(3)^2}{(2)^2} = \dfrac{9}{4}$

Illustration 9

Find the value of $\lim\limits_{x\to 0} \dfrac{\cos x - \cos 3x}{\cos 2x - \cos 5x}$

Short-cut solution :

$\boxed{\text{Using T-3(iii)}}$ $l = \dfrac{(1)^2 - (3)^2}{(2)^2 - (5)^2} = \dfrac{-8}{-21} = \dfrac{8}{21}$

Illustration 10

Find the value of $\lim\limits_{x\to 0} \left(\cos x + 3\sin 2x\right)^{1/x}$

Short-cut solution :

$\boxed{\text{Using T-3(v)}}$ $l = e^{3\times 2} = e^6$

TIPS AND TRICKS: (T-4)

$$\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \ldots + n^p}{n^q} = \frac{1}{q}\,(p, q, \in N)$$

★ Applicable only when $q - p = 1$

Illustration 11

The value of $l = \lim\limits_{n\to\infty} \left(\dfrac{1}{1-n^4} + \dfrac{8}{1-n^4} + \ldots + \dfrac{n^3}{1-n^4}\right)$ is equal to

Short-cut solution :

$\boxed{\text{Using T-4}}$ $l = -\lim\limits_{n\to\infty} \dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{n^4 - 1}$ $(\because q - p = 1)$

$\Rightarrow\ l = -\dfrac{1}{4}$

Illustration 12

Find the value of $\lim\limits_{n\to\infty} \left(\dfrac{1^4}{n^5} + \dfrac{2^4}{n^5} + \dfrac{3^4}{n^5} + \ldots + \dfrac{n^4}{n^5}\right)$

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\because\ q - p = 1$

$$\Rightarrow\ l = \frac{1}{q} = \frac{1}{5}$$

TIPS AND TRICKS: (T-5)

Short trick to evaluate limits of the form:

$$\lim_{x \to a}\left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$$

Illustration 13

Find the value of $\lim_{x \to 1} \dfrac{x^n - 1}{x^m - 1}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $l = \dfrac{n \cdot 1^{n-1}}{m \cdot 1^{m-1}} = \dfrac{n}{m}$

TIPS AND TRICKS: (T-6)

Short trick to solve limits of the form:

$$\lim_{x \to 0}\left(\frac{k_1^x + k_2^x + k_3^x + \ldots + k_n^x}{n}\right)^{\frac{1}{x}} = \left(k_1 \cdot k_2 \ldots k_n\right)^{1/n}$$

Illustration 14

Find the value of: $\lim_{x \to 0}\left(\dfrac{p^x + q^x + r^x}{3}\right)^{\frac{1}{x}}$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $l = \left(p \cdot q \cdot r\right)^{1/3}$

Illustration 15

Find the value of $\lim_{x \to \infty}\left(\dfrac{a_1^{1/x} + a_2^{1/x} + \ldots + a_n^{1/x}}{n}\right)^{nx}$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $l = (a_1 \cdot a_2 \cdot a_3 \dots a_n)^{n \times \frac{1}{n}}$

$= a_1 \cdot a_2 \cdot a_3 \dots a_n$

TIPS AND TRICKS: (T-7)

Sum of the infinite series of the type:

$$\lim_{n\to\infty}\left[\frac{1}{p_1 q_1}+\frac{1}{p_2 q_2}+\frac{1}{p_3 q_3}+\dots\right]=\frac{1}{p_1(q_1-p_1)}$$

where $p_1, p_2, p_3 \dots$ are in AP,

$q_1, q_2, q_3, \dots$ are in other AP.

and common difference of both AP must be same

$q_1 - p_1 = q_2 - p_2 = \dots.$

Illustration 16

Find the value of $\lim_{n\to\infty}\left[\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\dots+\frac{1}{(3n-1)(3n+2)}\right]$

Short-cut solution :

$\boxed{\text{Using T-7}}$ Since $5-2=8-5=11-8\dots$ are same

$\therefore$ Both series [2, 5, 8, ...] and [5, 8, 11, ...] are in A.P.

$$\Rightarrow l = \frac{1}{p_1(q_1-p_1)} = \frac{1}{2(3)} = \frac{1}{6}$$

Illustration 17

Find the value of $\lim_{n\to\infty}\left[\frac{1}{3.9}+\frac{1}{9.15}+\frac{1}{15.21}+\dots+\frac{1}{(6n-3)(6n+3)}\right]$

Short-cut solution :

$\boxed{\text{Using T-7}}$ $l = \frac{1}{p_1(q_1-p_1)} = \frac{1}{3(9-3)} = \frac{1}{3.6}$

$$\Rightarrow \frac{1}{18}$$

TIPS AND TRICKS: (T-8)

In ∞° and 0° indeterminate form, best way is to take log to both sides.

OR

$$\lim_{x\to a} f(x)^{g(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}$$

Illustration 18

Evaluate $\lim\limits_{x\to 0} (x)^x$

Short-cut solution :

$\boxed{\text{Using T-8}}$ Let $l = \lim\limits_{x\to 0} (x)^x$

$$\log_e(l) = \lim_{x\to 0} x \log_e x = \lim_{x\to 0} \frac{\log_e x}{1/x}$$

Now, Apply L' Hospital Rule

$$\Rightarrow \log_e l = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0 \Rightarrow l = e^\circ = 1$$

Illustration 19

Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim\limits_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals

(a) 1 (b) $e^{1/2}$ (c) e^2 (d) e^3

Short-cut solution :

$\boxed{\text{Using T-8}}$ Given $f: R \to R$, $f(1) = 3$ and $f'(1) = 6$

Then $\lim\limits_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x}$

$$= e^{\lim\limits_{x\to 0} \frac{1}{x}[\log f(1+x) - \log f(1)]}$$

$$= e^{\lim\limits_{x\to 0} \frac{\frac{1}{f(1+x)} f'(1+x)}{1}} \quad \text{[using L' Hospital rule]}$$

$$= e^{\frac{f'(1)}{f(1)}} = e^{6/3} = e^2$$

SHORTCUTS: (SC-1)

Use of Expansion in Limits.

We need not to learn any expansion series.

Use Taylor's Series: (In order to expand $f(x)$)

$$\text{Exp }(f(x)) = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \frac{f'''(0)\cdot x^2}{3!} + \ldots$$

This is the Master shortcut to remember any expansion series.

Illustration 20

Expansion of $\sin x = f(x)$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $f(x) = f(0) + \dfrac{f'(0)\cdot x}{1!} + \dfrac{f''(0)\cdot x^2}{2!} + \dfrac{f'''(0)\cdot x^3}{3!} + \ldots$

$$\Rightarrow \sin x = 0 + \frac{x}{1!} + \left(\frac{-x^3}{3!}\right) + \frac{x^5}{5!} + \left(\frac{-x^7}{7!}\right) + \ldots$$

$$\Rightarrow x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Illustration 21

Expansion of $f(x) = \cos^{-1} x$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $f(0) = \dfrac{\pi}{2},\ f'(0) = -1,\ f''(0) = 0, \ldots$

$$\Rightarrow \text{Exp }(\cos^{-1} x) = \frac{\pi}{2} - \left(x + \frac{x^3}{3!} + \frac{9x^5}{5!} + \ldots\right)$$

Illustration 22

Expansion of $f(n) = (1 + x)^n$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $f(0) = 1,\ f'(0) = n,\ f''(0) = n(n-1), \ldots$

$$\Rightarrow (1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

Illustration 23

Expansion of $f(x) = \log_e (1 + x)$

Short-cut solution :

Using SC-1 $f(0) = 0,\ f'(0) = 1,\ f''(0) = -1, \ldots$

$$\Rightarrow \log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^5}{5} + \ldots$$

Illustration 24

Let $f(x)$ be a function such that $\lim\limits_{x\to 0} \frac{f(x)}{x} = 1$ and

$\lim\limits_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{\{f(x)\}^3} = 1$, then $b - 3a$ is equal to

Short-cut solution :

Using SC-1 $\lim\limits_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{\{f(x)\}^3} = 1$

$$\Rightarrow \lim_{x\to 0} \frac{x + ax\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\right\} - b\left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right\}}{\{f(x)\}^3} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\frac{1+a-b}{x^2} + \left(-\frac{a}{2!} + \frac{b}{3!}\right) + x^2\left(\frac{a}{4!} - \frac{b}{5!}\right) + \ldots\ldots}{\left\{\frac{f(x)}{x}\right\}^3} = 1$$

$\Rightarrow 1 + a - b = 0$ and $-\frac{a}{2!} + \frac{b}{3!} = 1 \Rightarrow a = -\frac{5}{2}$ and $b = -\frac{3}{2}$. Thus $b - 3a = 6$

SHORTCUTS: (SC-2)

Limits of the form 1^∞

Let, $\lim\limits_{x\to a} (f(x))^{g(x)}$ is of the form 1^∞

where, $\lim\limits_{x\to a} f(x) = 1$ and $\lim\limits_{x\to a} g(x) = \infty$

Then, $\lim\limits_{x\to a} \left(f(x)\right)^{g(x)} = e^{\lim\limits_{x\to a} (f(x)-1)\, g(x)}$

Illustration 25

$$\lim_{x\to 0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x} \text{ is equal to :}$$

(a) e (b) 2 (c) 1 (d) e^2

Short-cut solution :

$\boxed{\text{Using SC-2}}$ $\lim_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{1/x}$

$$\Rightarrow e^{\lim_{x\to 0}\frac{1}{x}\left[\tan\left(\frac{\pi}{4}+x\right)-1\right]} \Rightarrow e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{1+\tan x}{1-\tan x}-1\right)}$$

$$\Rightarrow e^{\lim_{x\to 0}\left(\frac{2\tan x}{1-\tan x}\right)\frac{1}{x}} = e^{\lim_{x\to 0}\left(\frac{\tan x}{x}\right)\left(\frac{2}{1-\tan x}\right)} = e^2$$

Ans. (d)

Illustration 26

$$\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{1/x^2} \text{ is equal to:}$$

(a) $\frac{1}{e}$ (b) $\frac{1}{e^2}$

(c) e^2 (d) e

Short-cut solution :

$\boxed{\text{Using SC-2}}$ Let R = $\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}} = e^{\lim_{x\to 0}\frac{1}{x^2}\left\{\frac{3x^2+2}{7x^2+2}-1\right\}}$

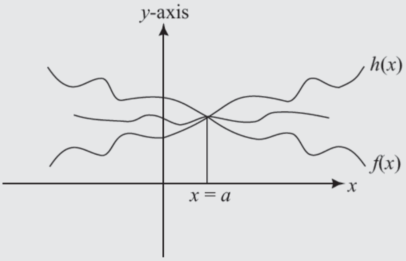
$$= e^{\lim_{x\to 0}\frac{1}{x^2}\left\{\frac{-4x^2}{7x^2+2}\right\}} = e^{\frac{-4}{2}} = e^{-2} = \frac{1}{e^2}$$

Ans. (b)

TECHNIQUE

Sandwich Theorem (Squeeze Play Theorem)

Sandwich theorem helps in calculating the limits, when limits can not be calculated using any formula.



Sandwich theorem: If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that,

$f(x) \le g(x) \le h(x)\ \forall\ x \in$ neighborhood of $x = a$.

and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} (h(x)) = l$ (say)

Then $\lim\limits_{x\to a} g(x) = l$

Illustration 27

$$\lim_{x\to\infty} \frac{\log x}{[x]}$$

Short-cut solution :

$\boxed{\text{Using Tech.}}$ We can have $f(x) = \dfrac{\log x}{x}$ and $h(x) = \dfrac{\log x}{x-1}$ as $x-1 < [x] \le x$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{(using L' Hospital's rule)}$$

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\log x}{x-1} = \lim_{x\to\infty} \frac{1/x}{1} = 0 \quad \text{(using L' Hospital's rule)}$$

$$\therefore \lim_{x\to\infty} \frac{\log x}{[x]} = 0$$

Illustration 28

Find the value of $\lim\limits_{n\to\infty} \underbrace{\left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \ldots + \frac{n}{n^2+n} \right)}_{f(n)} = l$

Short-cut solution :

Using Tech.

$$\Rightarrow \quad \frac{n^2}{n^2+n} \quad < \quad f(n) \quad < \quad \frac{n^2}{n^2+1}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 = l = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Hence $l = 1$

Illustration 29

Find $\lim_{n\to\infty} \frac{1}{n^3}\left\{[1^2 x]+[2^2 x]+[3^3 x]+\ldots+[n^2 x]\right\}$

where $[x]$ is greatest integer function.

Short-cut solution :

Using Tech.

Assume greatest integer function as a simple function, since $[x] \le x$

$$\Rightarrow \quad \lim_{x\to\infty} \frac{1^2 x + 2^2 x + 3^2 x + \ldots + n^2 x}{n^3}$$

$$\Rightarrow \quad \lim_{x\to\infty} \frac{(1^2+2^2+3^2+\ldots+n^2)x}{n^3}$$

Now, $\frac{1}{3}x \quad (\because \; q-p=1)$

$\Rightarrow \frac{x}{3}$

Concept Booster Exercise

1. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+9}-3}$ is equal to

(a) 3 (b) 4 (c) 1 (d) 2

2. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ is equal to **[JEE M 2019]**

(a) $\dfrac{1}{4\sqrt{2}}$ (b) $\dfrac{1}{2\sqrt{2}}$

(c) $\dfrac{1}{2\sqrt{2}\,(\sqrt{2}+1)}$ (d) Does not exist

3. $L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}\,\dfrac{-x^2}{4}}{x^4}$; L is finite, $a>0$ then 'a' is equal to

[AIEEE 2009]

(a) 2 (b) 1 (c) $\dfrac{1}{64}$ (d) $\dfrac{1}{32}$

4. $\lim\limits_{x\to \infty}\dfrac{\sqrt{x^6-5x^3+2x+7}}{x^5+4x^4-11x^3}$ is equal to

(a) $\dfrac{1}{2}$ (b) 1 (c) 0 (d) Does not exist

5. $\lim\limits_{x\to 0}\dfrac{\cos\left((2m+1)x\right)-\cos\left((2n-1)x\right)}{x^2}$ is equal to

(a) $2\,[(n-m)\,(m+n+1)]$
(b) $(n+m)\,(n-m-1)$
(c) $2\,[(n+m)\,(n-m-1)]$
(d) $(n-m)\,(m+n+1)$

6. $\lim\limits_{n\to \infty}\left[\dfrac{1}{1-n^3}+\dfrac{2^2}{1-n^3}+\dfrac{3^2}{1-n^3}+\ldots+\dfrac{n}{1-n^3}\right]$ is equal to

(a) $-\dfrac{1}{2}$ (b) $-\dfrac{1}{4}$ (c) $\dfrac{1}{3}$ (d) $-\dfrac{1}{3}$

7. $\lim\limits_{x\to 2}\left(\dfrac{x^{100}-2^{100}}{x-2}\right)$ is equal to

(a) 99×2^{99} (b) 100×2^{99} (c) 101×2^{99} (d) 100×2^{100}

8. $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+\ldots+n^x}{n}\right)^{1/x}$ is equal to

(a) $(n)^{1/n}$ (b) $(n!)^{1/n}$ (c) $(n-1)^{1/n}$ (d) $((n-1)!)^{1/n}$

9. $\lim\limits_{n\to\infty}\left(\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots+\dfrac{1}{n\,(n+1)}\right)$ is equal to

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{5}$

10. $\lim\limits_{n\to\infty}\dfrac{1}{n^2+1}+\dfrac{2}{n^2+2}+\ldots+\dfrac{n}{n^2+n}$

(a) 0 (b) 1 (c) 2 (d) 3

11. $\lim\limits_{n\to 0}\left[\dfrac{e^x-1}{x}\right]$, where $[x]$ is greatest integer function is equal to

(a) 0 (b) 1 (c) -1 (d) Does not exist

NUMERICAL VALUE PROBLEMS

12. $\lim\limits_{x\to 2}\dfrac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}}$ is equal to ________ **[JEE M 2020]**

13. Let the variable x_n be determined by the following law of formula

$$x_0=\sqrt{a},\ x_1=\sqrt{a+\sqrt{a}},\ x_2=\sqrt{a+\sqrt{a+\sqrt{a}}},\ x_3=\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$$

where, $a>0$, $\lim\limits_{x\to\infty}x_n=\dfrac{p+\sqrt{1+qa}}{2}$, then find the value of $p+q$.

14. Evaluate: $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}\right)^{x-1}$

15. If $\lim\limits_{x\to 0}\left(\dfrac{A\cos x+Bx\sin x-5}{x^4}\right)$ exists and finite, then find the value of $\dfrac{A}{B}$.

16. $\lim\limits_{n\to\infty}\cos\dfrac{1}{2}\cos\dfrac{1}{4}\cos\dfrac{1}{8}\ldots\cos\dfrac{1}{2^n}$ is equal to y. Then find the value of $[y]$, where $[x]$ G.I.F.

Solutions

1. **(a)** Using T-1 $\lim\limits_{x\to 0}\dfrac{(1+x^2)^{1/2}-1}{3\left(1+\dfrac{x^2}{9}\right)^{1/2}-3}=\lim\limits_{x\to 0}\dfrac{1+\dfrac{x^2}{2}-1}{3+\dfrac{3x^2}{2.9}-3}=3$

2. **(a)** Using T-1

$$\lim_{x\to 0}\frac{\left(1+1+\dfrac{y^4}{2}\right)^{1/2}-\sqrt{2}}{y^4}=\lim_{x\to 0}\frac{\sqrt{2}\left[1+\dfrac{y^4}{4.2}-1\right]}{y^4}=\frac{1}{4\sqrt{2}}$$

3. **(a)** Using T-1

$$\lim_{x\to 0}\frac{a\left[1-1+\dfrac{x^2}{2a^2}+\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{x^4}{2a^4}\right]-\dfrac{x^2}{4}}{x^4}=\lim_{x\to 0}\frac{\dfrac{x^2}{2a}+\dfrac{x^4}{8a^3}-\dfrac{x^2}{4}}{x^4}$$

Since L is finite $=\lim\limits_{x\to 0}\dfrac{x^2\left(\dfrac{1}{2a}-\dfrac{1}{4}\right)+\dfrac{x^4}{8a^3}}{x^4}\Rightarrow a=2.$

$\Rightarrow \text{L}=\dfrac{1}{64}$

4. **(c)** Using T-2 Take highest power outside

$$\Rightarrow \lim_{x\to\infty}\frac{x^3\sqrt{1-\dfrac{5}{x^3}+\dfrac{2}{x^5}+\dfrac{7}{x^6}}}{x^5\left(1+\dfrac{4}{x}-\dfrac{11}{x^2}\right)}=0$$

5. **(c)** Using T-3(i) $l=\dfrac{b^2-a^2}{2}=2\,(n+m)\,(n-m-1)$

6. **(d)** Using T-4 $\because (q-p=1)$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^3-1}+\frac{2^2}{n^3-1}+\frac{3^2}{n^3-1}+\ldots+\frac{n}{n^3-1}=\frac{-1}{q}=\frac{-1}{3}$$

7. **(b)** $\boxed{\text{Using T-5}}$ Here, $a = 2$ and $n = 100$

$$\Rightarrow\ l = n \cdot a^{n-1} = 100 \cdot 2^{99}$$

8. **(b)** $\boxed{\text{Using T-6}}$ $l = (k_1 \cdot k_2 \cdot k_3 \ldots k_n)^{1/n} = (1 \cdot 2 \cdot 3 \ldots n)^{1/n} = (n!)^{1/n}$

9. **(a)** $\boxed{\text{Using T-7}}$ $\because (q_1 - p_1 = 1 = q_2 - p_2)$

$$\Rightarrow\ \frac{1}{p_1(q_1 - p_1)} = \frac{1}{2 \cdot (3-2)} = \frac{1}{2}$$

10. **(b)** $\boxed{\text{Using SC-2}}$ Sandwitch theorem

$$\frac{n^2}{n^2+n} < \underbrace{\frac{1}{n^2+1} + \frac{2}{n^2+2} + \ldots + \frac{n}{n^2+n} < \frac{n^2}{n^2+1}}_{f(n)}$$

$$\lim_{n\to\infty} \frac{n^2}{n+n} \quad < \quad \lim_{n\to\infty} f(n) \quad < \quad \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

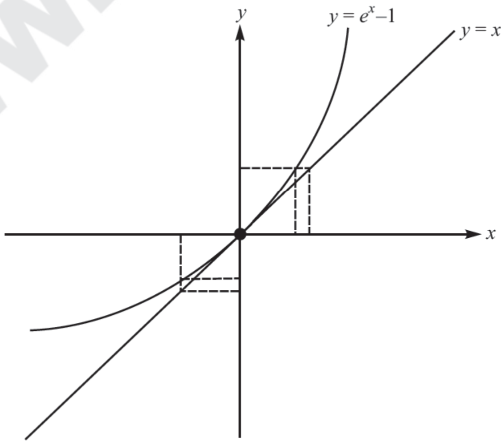
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$1 \qquad\qquad\qquad\qquad 1$$

Hence, L = 1

11. **(d)** In this we solve using graph.

We draw graphs of $y = e^x - 1 =$ and $y = x$



$$\left.\begin{array}{l}\text{For RHL, } x>0,\ e^x-1 > x \Rightarrow \lim\limits_{x\to 0}\left[\dfrac{e^x-1}{x}\right]=1 \\ \qquad \Rightarrow \dfrac{e^x-1}{x}>1 \\ \text{For LHL, } x<0,\ e^x-1 > x \Rightarrow \lim\limits_{x\to 0}\left[\dfrac{e^x-1}{x}\right]=0 \\ \qquad \Rightarrow \dfrac{e^x-1}{x}<1\end{array}\right\} \Rightarrow \begin{array}{l}\text{LHL} \neq \text{RHL} \\ \text{Limit does not exist.}\end{array}$$

12. **(36)** Put $3^{x/2}=t \Rightarrow \operatorname*{Lim}\limits_{t\to 3}\dfrac{t^2+\dfrac{27}{t^2}-12}{\dfrac{1}{t}-\dfrac{3}{t^2}}$

$$= \operatorname*{Lim}_{t\to 3}\frac{t^4+27-12t^2}{t-3} = \operatorname*{Lim}_{t\to 3} 4t^3-24t\,[\text{L' Hospital rule}]$$

$$= 4(27)-24(3)=36.$$

13. **(5)** Let $x_n=\sqrt{a+\sqrt{a+\sqrt{a+\ldots}}} \quad \Rightarrow x_n=\sqrt{a+x_n}$

$$\Rightarrow x_n^2-x_n-a=0 \Rightarrow x_n=\frac{1\pm\sqrt{1+4a}}{2}$$

Negative sign will be rejected.

$$\Rightarrow \operatorname*{Lim}_{x\to\infty} x_n=\frac{1+\sqrt{4a+1}}{2} \quad \Rightarrow p=1, q=4$$

Hence, $p+q=5$

14. **(1)** Using T-8 Since it is ∞° form

$\Rightarrow$ Take log to both sides

$$\Rightarrow \ln(l)=\operatorname*{Lim}_{x\to 1}(x-1)\ln\left(\frac{1}{x-1}\right)$$

$$\Rightarrow \ln(l) = \lim_{x\to 1} \frac{\ln\left(\frac{1}{x-1}\right)}{\left(\frac{1}{(x-1)}\right)} = \lim_{x\to 1} \frac{(x-1)\left(-\frac{1}{(x-1)^2}\right)}{\frac{1}{(x-1)^2}} = 0$$

$$\Rightarrow \ln(l) = 0 \;\Rightarrow\; l = e^{\circ} = 1$$

15. (2) Using SC-1 Use expansion of $\sin x$ and $\cos x$

$$\Rightarrow \lim_{x\to 0} \frac{A\left(1-\frac{x^2}{2}+\frac{x^4}{4!}\right)+Bx\left(\frac{x}{1!}-\frac{x^3}{3!}\right)-5}{x^4} \left\{\begin{matrix}\text{Neglecting higher}\\ \text{order terms.}\end{matrix}\right\}$$

$$\Rightarrow A-5=0 \quad \text{and} \quad \frac{-A}{2}+B=0 \quad \Rightarrow \frac{A}{B}=2$$

16. (0) Using T-3(iv) Here, $x = 1$

$$\Rightarrow \left[\frac{\sin 1}{1}\right] = 0$$

4 Continuity

Review of Key Notes and Formulae

1. **Continuity of a Function at a Point :**

   A function $f(x)$ is said to be continuous at $x = a$, if

   $$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

   Otherwise, it is said to be discontinuous.

2. **Continuity of a Function is an Interval :**

   (i) A function $f(x)$ is said to be continuous in an open interval (a, b), if $f(x)$ is continuous at every point of the interval.

   (ii) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if $f(x)$ is continuous in (a, b). In addition, $f(x)$ is continuous at $x = a$ from right and $f(x)$ is continuous at $x = b$ from left.

   ★ **Note :** To determine function's continuity in an interval, the best way is to draw graph.

   ⇒ If graph of a function has no break or gap, then it is continuous, otherwise it will be discontinuous function.

3. **Reason's of Discontinuity:**

   (i) $\lim_{x\to a} f(x)$ does not exist.

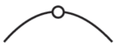
   (ii) $\lim_{x\to a} f(x)$ exists $\neq f(a)$

   (iii) $f(a)$ is not defined.

4.

Types of discontinuity

- Removable type
  - Missing point

    * $f(0)$ Not defined

    $Eg: f(x) = \frac{\sin x}{x}$ $x = 0$
  - Isolated point

    * $\lim_{x \to a} f(x) \neq f(a)$

    $Eg: f(x) = \frac{x^2 - 16}{x - 4}$

    $x \neq 4,\ f(x) = 9$
- Non-removable type
  - Finite type

    Both LHL and RHL are finite but LHL $\neq$ RHL

    Eg: $f(x) = \{x\}$ at all integral x.
  - Infinite

    At least one of LHL OR RHL is 'do not defined'

    $Eg: f(x) = \frac{1}{x - 4}$ at $x = 4$
  - Oscillatory

    Limit oscillates b/w two finite values but we cannot determine it

    $Eg: f(x) = \sin \frac{1}{x}$ at $x = 0$

5. Jump of Discontinuity :

Jump = | LHL – RHL|, provided both LHL and RHL are finite.

6. Theorems Continuity : At $x = a$.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x).\,g(n)$	$\frac{f(x)}{g(x)}; g(a) \neq 0$
C	C	C	C	C
C	D	D	C or D	C or D
D	C	D	C or D	C or D
D	D	C or D	C or D	C or D

Where C $\rightarrow$ Continuous, D $\rightarrow$ Discontinuous function.

7. Continuity of Composite Function:

If $f(x)$ is continuous at $x = a$ and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.

TIPS! & TRICKS

TIPS AND TRICKS: (T-1)

If $f(x)$ is continuous in $x \in [a, b]$ and $f(a), f(b)$ are opposite in sign, then there exists at least one root of equation $f(x) = 0$ in $x \in (a, b)$

Illustration 1

Show that $x = a \sin x + b$, where $0 < a < 1$, $b > 0$ has at least one positive root which doesn't exceed $b + a$.

Short-cut solution :

$\boxed{\text{Using T-1}}$ Let $f(x) = x - a \sin x - b$ is continuous function.

Now, $f(0) = -b < 0$

$f(a + b) = a - a \sin (a + b) = a(1 - \sin (a + b)) \geq 0$

Hence, one positive root in $[0, a + b]$

Illustration 2

Show that $f(x) = x^3 + 2x - 1$ has root in the interval $x \in [0, 1]$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $f(x)$ is continous in $x \in [0, 1]$

Since, $\left.\begin{matrix} f(0) = -1 < 0 \\ f(1) = 2 > 0 \end{matrix}\right\} \Rightarrow$ There exists at least one c in $(0,1)$ such that $f(c)=0$.

TIPS & TRICKS

TIPS AND TRICKS: (T-2)

$f : R \to R$, if $f(x)$ is even degree polynomial whose leading coefficient and absolute constant term are of opposite in sign, then $f(x) = 0$ has at least 2 real roots.

Illustration 3

If $f : R \to R, f(x) = 2x^6 - 3x^5 + 4x^3 - x - 7$. Then prove that $f(x) = 0$ must have at least two real roots.

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because$ Leading coefficient >0 (2)

Absolute term <0 (-7)

Hence, $f(x) = 0$ must have at least 2 real roots.

TIPS & TRICKS

TIPS AND TRICKS: (T-3)

If $y = (x)$ is continuous function and takes rational values for all x then $f(x)$ will be a constant function.

Illustration 4

Let $f(x)$ be a continuous functions for $x \in [1, 3]$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then value of $f(1.5)$ is

(a) 7.5 (b) 10

(c) 8 (d) 9

Short-cut solution :

Using T-3 $\because f(x)$ is continuous and takes rational values.

$\Rightarrow \boxed{f(x)=10} \Rightarrow f(1.5)=10$ **Ans. (b)**

TIPS AND TRICKS: (T-4)

If $y = f(x)$ is monotonically increasing and continuous on an interval $x \in$ (a, b) then $f^{-1}(y)$ exists and continuous and monotonically increasing

Illustration 5

Check whether inverse of $f(x) = \sin x + 1 \ \forall \ x \in (-\pi^2, \pi^2)$ is continuous or not.

Short-cut solution :

Using T-4 $\because f(x)$ is continuous in $x \in \left(-\pi^2, \pi^2\right)$

$\Rightarrow f^{-1}(x)$ is also continuous

TECHNIQUE

Intermediate value theorem (IVT)

Let $f(x)$ be continuous in closed interval $[a, b]$ then $f(x)$ will attain the least value (say m) and the greatest value (say M) for $x \in [a, b]$ then there exists at least one $c \in [a, b]$ such that $f(c)=\dfrac{\lambda_1 m+\lambda_2 M}{\lambda_1+\lambda_2}; \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1+\lambda_2 \neq 0$

Illustration 6

Does function $f(x)=\dfrac{x^3}{4}-\sin(\pi x)+3$, take on value $2\dfrac{1}{3}$ within the interval $[-2, 2]$

Short-cut solution :

Using Tech. Since, $f(x)$ is continuous in $[-2, 2]$ and $f(-2) = 1$, $f(2) = 5$

Hence, by IVT, it takes the value $2\dfrac{1}{3}$

Concept Booster Exercise

1. The equation $2x^3 - 6x + 1 = 0$ on $x \in (1, 2)$ has
   (a) no solution
   (b) at least one real solution
   (c) infinite solution
   (d) None of these

2. The equation $2 \cos x + 6x - 3$ has
   (a) no solution
   (b) at least one real solution in $x \in [0, \pi^3]$
   (c) infinite solution
   (d) None of these

3. The equation $\frac{a_1}{x-\lambda_1}+\frac{a_2}{x-\lambda_2}+\frac{a_3}{x-\lambda_3}=0$ where $a_1, a_2\, a_3 > 0$ and $\lambda_1 < \lambda_2 < \lambda_1$ has
   (a) no solution
   (b) one real solution
   (c) two real roots
   (d) infinite roots

4. If $f(x)=\begin{cases} x;\, x\in Q \\ -x;\, x\notin Q \end{cases}$, then $f(x)$ is continous at -

   [AIEEE 2002]
   (a) Only at zero
   (b) Only at 0, 1
   (c) All real numbers
   (d) All rational numbers

5. If $f(x)=\frac{x}{2}-1$, then on the interval $[0, \pi]$, then
   (a) $f^{-1}(x)$ is continuous
   (b) $f^{-1}(0) = 0$
   (c) $f^{-1}(x)$ is discontinuous
   (d) None of these

Numerical Value Problems

6. If $f(x)=\begin{cases} x^2+cx+1; & x\in Q \\ ax^2+2x+b; & x\notin Q \end{cases}$ is continuous at $x = 1, 2, 3$, then find $a + b + c$

   [AIEEE 2010]

7. Let $f : R \to R$ be a continuous function defined by

   $f(x)=\frac{1}{e^x+2e^{-x}}$, there $0 < f(x) < \frac{1}{C\sqrt{C}}$, $\forall\, x \in R$

   Therefore 'C' is equal to —

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ Let $f(x) = 2x^3 - 6x + 1$ $\because f(x)$ is continuous
Hence, $f(1) = -3 < 0$ and $f(2) = 5 > 0$
$\Rightarrow \exists$ at least one root in $x \in (0, 2)$

2. **(b)** $\boxed{\text{Using T-1}}$ Let $f(x) = 2\cos x + 6x - 3$ $\because f(x)$ is continuous function

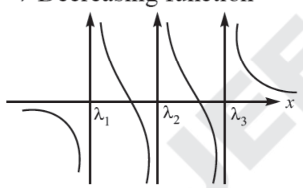
And, $f(0) = -1 < 0$ and $f\left(\frac{\pi}{3}\right) = 2(\pi - 1) > 0$

$\Rightarrow \exists$ at least one root in $x \in \left(0, \frac{\pi}{3}\right)$

3. **(c)** Let $f(x) = \dfrac{a_1}{x - \lambda_1} + \dfrac{a_2}{x - \lambda_2} + \dfrac{a_3}{x - \lambda_3}$

Now, $f'(x) = -\dfrac{a_1}{(x - \lambda_1)^2} - \dfrac{a_2}{(x - \lambda_2)^2} - \dfrac{a_3}{(x - \lambda_3)^2} < 0$

$\Rightarrow$ Decreasing function



Hence, two real roots.

4. **(a)** $\boxed{\text{Using T-3}}$ One point is common *i.e.* 0
Hence, $f(x)$ is continous at $x = 0$

5. **(a)** $\boxed{\text{Using T-4}}$ $\because f(x)$ is continuous is $x \in [0, \pi]$
$\Rightarrow f^{-1}(x)$ will also be continuous

6. **(4)** Since, $f(x)$ is continuous

$$\Rightarrow (a-1)x^2 + (2-c)x + (b-1) = 0 \begin{matrix} \nearrow 1 \\ \rightarrow 2 \\ \searrow 3 \end{matrix} \text{ (on subtracting)}$$

1, 2, 3, must be roots of above equation
$\Rightarrow$ Equation becomes identity (Because more than one root)
$\Rightarrow a = 1, b = 1, c = 2$; Hence, $a + b + c = 4$

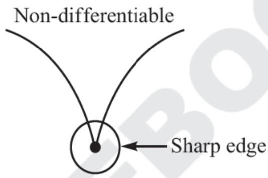
7. **(2)** As we know that AM $\geq$ GM $\Rightarrow \dfrac{e^x + \frac{2}{e^x}}{2} \geq (2)^{1/2} \Rightarrow \dfrac{e^x}{e^{2x} + 2} \leq \dfrac{1}{2\sqrt{2}}$

Hence, $0 < \dfrac{1}{e^x + 2e^{-x}} < \dfrac{1}{2\sqrt{2}} \Rightarrow C = 2$

5 Differentiability

Review of Key Notes and Formulae

1. If function $y = f(x)$ is differentiable if the curve has no break point and no sharp edge.



2. **Differentiability at a Point:**

A function $f(x)$ is said to be differentiable at a point 'a' in its domain, if

$$\underbrace{\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}}_{\substack{\text{Left hand derivative} \\ \text{(LHD)}}} = \underbrace{\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}}_{\substack{\text{Right hand derivative} \\ \text{(RHD)}}} = \text{finite} = f'(a)$$

★**Note:** If $f(x)$ is differentiable $\Rightarrow$ Unique tangent of finite slope exists.

3. **Relation between Continuity and Differentiability → at** $x = a$

Differentiable	$\Rightarrow$	Continuous
Continuous	$\not\Rightarrow$	Differentiable
Discontinuous	$\Rightarrow$	Non-differentiable
Non-differentiable	$\not\Rightarrow$	Discontinuous

4. **Differentiability of a Function in an Interval**

(i) A function $f(x)$ is said to be differentiable in an interval (a, b), if $f(x)$ is differentiable at every point in (a, b).

(ii) Above point and in addition $f(x)$ is differentiable at $x = a$ from right and $x = b$ from left.

Theorems over differentiability $\rightarrow$ at $x = a$

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \cdot g(x)$	$\frac{f(x)}{g(x)}$; $g(x) \neq 0$
D	D	D	D	D
D	ND	ND	N·D· or D	N·D· or D
ND	D	ND	N·D· or D	N·D· or D
ND	ND	ND or D	ND or D	ND or D

Determination of differentiable $f(x)$ satisfying a given functional rule

Step 1: Start with the equation $\Rightarrow \quad f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Step 2: Manipulate this term and use functional rule

Step 3: Express $f'(x)$ in terms of 'x'.

Step 4: Now, $f'(x)$ is obtained, integrate and get $f(x)$.

TIPS AND TRICKS: (T-1)

Short trick to check differentiability. Find derivative of given function and put the value on which differentiability is to be checked. Then,

If $f'(x)$ = finite $\Rightarrow f(x)$ is differentiable at $x = a$.

Illustration 1

Check differentiability of $f(x) = |x - 2|$ at $x = 2$;

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because \quad f(x) = \begin{cases} x - 2 & ; \quad x \geq 2 \\ -x + 2 & ; \quad x < 2 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 1 & ; \quad x \geq 2 \quad \Rightarrow \quad \text{RHD} = 1 \\ -1 & ; \quad x < 2 \quad \Rightarrow \quad \text{LHD} = -1 \end{cases}$$

Hence, $f(x)$ is not differentiable at $x = 2$.

Illustration 2

Check differentiability of $f(x) = \begin{cases} x[x] & ; \quad 0 \leq x < 2 \\ (x-1)[x] & ; \quad 2 \leq x < 3 \end{cases}$

where, $[x]$ is greatest integer function.

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because\ f(x)=\begin{cases} x-1 & ; \quad 0\le x<2 \\ 2(x-1) & ; \quad 2\le x<3 \end{cases}$

$\Rightarrow\ f'(x)=\begin{cases} 1 & ; \quad 0\le x<2 \\ 2 & ; \quad 2\le x<3 \end{cases}$

Hence, $f(x)$ is not differentiable function.

Illustration 3

Check differentiability of $f(x)=\begin{cases} 1+\sin x & ; \quad 0\le x<\dfrac{\pi}{2} \\ 2+\left(x-\dfrac{\pi}{2}\right)^2 & ; \quad x\ge\dfrac{\pi}{2} \end{cases}$ at $x=\dfrac{\pi}{2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $f'(x)=\begin{cases} 1+\cos x & ; \quad 0\le x<\dfrac{\pi}{2} \\ 2\left(x-\dfrac{\pi}{2}\right) & ; \quad x\ge\dfrac{\pi}{2} \end{cases}$

Now, $f'\left(\dfrac{\pi}{2}\right)=\begin{cases} 0 & \Rightarrow \text{LHD} \\ 2\left(\dfrac{\pi}{2}-\dfrac{\pi}{2}\right)=0 & \Rightarrow \text{RHD} \end{cases}$

Hence, $f(x)$ is differentiable at $x=\dfrac{\pi}{2}$

TIPS AND TRICKS: (T-2)

If LHD and RHD both are finite but are unequal then function $f(x)$ is continuous at $x=a$.

Illustration 4

Check the continuity and differentiability of **[AIEEE 2011]**

$$f(x)=\begin{cases} -x-\dfrac{\pi}{2} & ; \quad x\le-\dfrac{\pi}{2} \\ -\cos x & ; \quad -\dfrac{\pi}{2}<x\le 0 \\ x-1 & ; \quad 0<x\le 1 \\ \ln x & ; \quad x>1 \end{cases}$$

(a) $f(x)$ is continuous at $x = \frac{-\pi}{2}$

(b) $f(x)$ is non-differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 1$

(d) $f(x)$ is differentiable at $x = \frac{-3}{2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $f'(x) = \begin{cases} -1 & ; \quad x \le -\frac{\pi}{2} \\ \sin x & ; \quad -\frac{\pi}{2} < x \le 0 \\ 1 & ; \quad 0 < x \le 1 \\ \frac{1}{x} & ; \quad x > 1 \end{cases}$

Now, $f'\left(\frac{-\pi}{2}\right) = -1 \Rightarrow$ continuous function

$\boxed{\text{Using T-2}}$ $f'(0) \left.\begin{matrix} \nearrow \text{LHD} = 0 \\ \searrow \text{RHD} = 1 \end{matrix}\right\} \Rightarrow$ Non-differentiable but continuous

$\boxed{\text{Using T-1}}$ $f'(1) = 1$ and $f'\left(\frac{-3}{2}\right) \Rightarrow$ Differentiable **Ans. (a, b, c, d)**

SHORTCUTS: (SC-1)

Checking Differentiability using graphs.

If $y = f(x)$ curve has:

★ No break and no sharp edge $\Rightarrow$ Differentiable function

★ Break or sharp edge $\Rightarrow$ Non-differentiable function

Illustration 5

If $f : R \to R$, $f(x) = \max. \{x, x^3\}$ then set of points where $f(x)$ is not differentiable.

(a) $\{-1, 1\}$

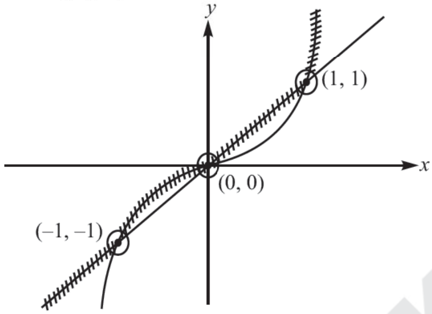
(b) $\{-1, 0\}$

(c) $\{0, 1\}$

(d) $\{-1, 0, 1\}$ **[AIEEE 2001]**

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Drawing graph:



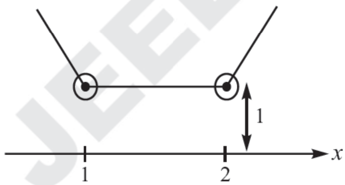
As shown in the figure there are sharp edges at $x = -1, 0, 1$ **Ans. (d)**

Illustration 6

Discuss the differentiability of $f(x) = |x - 1| + |x - 2|$ at $x = 1, 2$.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Drawing graph.



$\Rightarrow$ Figure shows, non-differentiable at $x = 1, 2$.

Concept Booster Exercise

1. $f(x)=\begin{cases} 3^x & ; \quad -1\le x\le 1 \\ 4-x & ; \quad 1<x<4 \end{cases}$ at $x = 1$, is

(a) Differentiable (b) Non-differentiable

(c) Discontinuous (d) None of these

2. The function defined by $f(x)=\begin{cases} |x-3| & ; \quad x\ge 1 \\ \frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4} & ; \quad x<1 \end{cases}$ **[AIEEE 2006]**

(a) Continuous at $x = 1$ (b) Differentiable at $x = 1$

(c) Continuous at $x = 3$ (d) All of these

3. If $f(x) = \min\{1, x^2, x^3\}$, then, **[AIEEE 2012]**

(a) $f(x)$ is continuous $\forall\, x \in R$

(b) $f'(x)>0 \ \forall\ x>1$

(c) Non-differentiable but continuous $\forall\, x \in R$

(d) $f(x)$ is differentiable for two values of 'x'.

4. Consider the function $f(x)=|x-2|+|x-5| \ \forall\ x\in R,$ then

(a) $f'(4) = (0)$ (b) $f(x)$ is continuous in $[2, 5]$

(c) $f(2) = f(5)$ (d) $f(x)$ is differentiable in $(2, 5)$

5. If $f(x) = \begin{cases} ax+b & ; \quad x\le -1 \\ ax^3+x+2b & ; \quad x>-1 \end{cases}$ is differentiable at $x = -1$, then find a and b.

(a) $a = 1, b = 1$ (b) $a = 1, b = \frac{-1}{2}$

(c) $a = \frac{-1}{2}, b = 1$ (d) $a = -1, b = 1$

Solutions

1. **(b)** Using T-1 $f'(x)=\begin{cases}3^x\log_e 3 & ; & -1\le x\le 1\\ -1 & ; & 1<x<4\end{cases}$

$\Rightarrow f'(1)=\begin{cases}3\log_e 3 & \Rightarrow & \text{LHD}\\ -1 & \Rightarrow & \text{RHD}\end{cases}$

Hence, function is non-differentiable.

2. **(d)** Using T-1 $\because\ f(x)=\begin{cases}-x+3 & ; & x\ge 1\\ \dfrac{x^2}{4}-\dfrac{3}{2}x+\dfrac{13}{4} & ; & x<1\end{cases}$

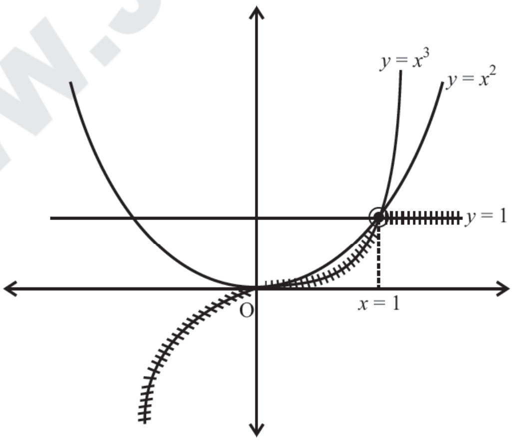
Now, $f'(x)=\begin{cases}-1 & ; & x\ge 1\\ \dfrac{x}{2}-\dfrac{3}{2} & ; & x<1\end{cases}$

$\Rightarrow f'(1)=\left.\begin{cases}-1 \Rightarrow \text{RHD}\\ -1 \Rightarrow \text{LHD}\end{cases}\right\}\begin{array}{l}\Rightarrow \text{Function is differentiable at } x=1\\ \Rightarrow \text{Continuous at } x=1\end{array}$

Now for $x = 3$

$f'(3) = -1$ (LHD and RHD both) $\Rightarrow$ Continuous at $x = 3$

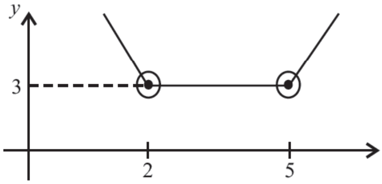
3. **(a, c)** Using SC-1 Drawing graph



As shown in the figure required function is continuous in $x \in R$ but not differentiable at $x = 1$.

4. **(a, b, c, d)**

Using SC-1 Drawing graph.



Hence, as shown in the figure:

$f'(4) = 0, f(x)$ is continuous in $x \in [2, 5]$

$f(2) = f(5), f(x)$ is differentiable in $x \in (2, 5)$

5. **(c)** Using T-1 $\because f(x)$ is differentiable and continuous

$$\Rightarrow f'(x) = \begin{cases} a & ; \quad x \le -1 \\ 3ax^2 + 1 & ; \quad x > -1 \end{cases}$$

$$\Rightarrow \text{LHD} = \text{RHD} \Rightarrow a = 3a + 1 \Rightarrow a = \frac{-1}{2}$$

$\because$ Continuous

$$\Rightarrow a(-1) + b = a(-1)^3 - 1 + 2b \quad \Rightarrow b = 1 \left(\because a = \frac{-1}{2}\right)$$

6 Methods of Differentiation

Review of Key Notes and Formulae

1. Differentiation

The rate of change of quantity y with respect to another quantity x is called the derivative or differential coefficient of 'y' with respect to 'x'. The process of finding derivative of a function is called differentiation.

2. Derivatives of Standard Functions

$f(x)$	$\frac{d}{dx}(f(x))$	$f(x)$	$\frac{d}{dx}(f(x))$
x^n	$nx^{n-1}; n \in R$	$\sec x$	$\sec x \tan x, x \neq (2n+1)\frac{\pi}{2}$
e^{ax}	$\log_e^a; a \in R$	$\operatorname{cosec} x$ $\cot x$	$-\operatorname{cosec} x \cot x; x \neq n\pi$ $-\operatorname{cosec}^2 x, x \neq n\pi$
a^x	$a^x \log_e a; a > 0, a \neq 1$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}; -1 < x < 1$
$\log_e x$	$\frac{1}{x}; x > 0$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}; -1 < x < 1$
$\log_a x$	$\frac{1}{x \log_e a}; x > 0$	$\tan^{-1} x$	$\frac{1}{1+x^2}; x \in R$
$\sin x$	$\cos x$	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}; \lvert x \rvert > 1$
$\cos x$	$-\sin x$	$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x \rvert \sqrt{x^2-1}}; \lvert x \rvert > 1$
$\tan x$	$\sec^2 x;$ $x \neq (2n+1)\frac{\pi}{2}, n \in I$	$\cot^{-1} x$	$\frac{-1}{1+x^2}; x \in R$

3. Rules for Differentiation

(R-1) $\frac{d}{dx}(K\,f(x)) = K \cdot \frac{d}{dx}(f(x))$; where K is constant

(R-2) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(R-3) **Product Rule :** $\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$

(R-4) **Quotient Rule :** $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

(R-5) **Chain Rule :**

If y is a function of u, u is a function of v and v is a function of x. Then this impties

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}}$$

(R-6) **Parametric differentiation**

If $x = P(t)$, $y = Q(t)$, where 't' is parameter then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(Q(t))}{\frac{d}{dt}(P(t))} = \frac{Q'(t)}{P'(t)}$$

(R-7) Differentiation of one function w.r.t. other function

$$\frac{d(f(x))}{d(g(x))} = \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))} = \frac{f'(x)}{g'(x)}$$

(R-8) **Logarithmic differentiation :**

It is applicable if

(i) $y = \overbrace{f_1 \cdot f_2 \cdot f_3 \ldots f_n}^{\text{All are functions of 'x'}}$ (product, divide or power form)

(ii) $y = (f(x))^{g(x)}$

★ Take log to both sides and then differentiate.

(R-9) **Successive differentiation :**

Differential coefficient of $\frac{dy}{dx} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

Differential coefficient of $\frac{d^2y}{dx^2} \Rightarrow \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

Similarly we can proceed for higher orders of differential coefficients.

(R-10) **Differentiation using substitutions :**

If required, we can reduce the given function in a simple form using trigonometrical substitutions.

Function	Substitution	Function	Substitution
(i) $\sqrt{a^2 - x^2}$	$x = a \sin\theta$ / $a\cos\theta$	(v) $\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$	$x = a\sqrt{\cos 2\theta / a \tan\theta}$
(ii) $\sqrt{x^2 + a^2}$	$x = a\tan\theta$ / $a\cot\theta$	(vi) $\sqrt{\dfrac{x}{a + x}}$	$x = a\tan^2\theta$
(iii) $\sqrt{x^2 - a^2}$	$x = a\sec\theta$ / $a\,\mathrm{cosec}\,\theta$	(vii) $\sqrt{\dfrac{x}{a - x}}$	$x = a\sin^2\theta$
(iv) $\sqrt{\dfrac{a - x}{a + x}}$	$x = a\cos 2\theta$		

TIPS AND TRICKS: (T-1)

Short trick to solve differentiation of the form:

$$\frac{d}{dx}\left(\frac{a\,f(x)+b}{c\,f(x)+d}\right) = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}.f'(x)}{\left(c\,f(x)+d\right)^2}$$

Illustration 1

If $y = \dfrac{3\log x + 8}{-7\log_e x + 9}$, find $\dfrac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\dfrac{dy}{dx} = \dfrac{(27 + 56)\cdot\dfrac{1}{x}}{(-7\log_e x + 9)^2}$

Illustration 2

If $y = \dfrac{2+3\cos x}{\cos x - 5}$, then find $\dfrac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $y = \dfrac{3\cos x + 2}{\cos x - 5}$

$$\Rightarrow \frac{dy}{dx} = \frac{(-15-2)(-\sin x)}{(\cos x - 5)^2} = \frac{+17\sin x}{(\cos x - 5)^2}$$

TIPS AND TRICKS: (T-2)

Short trick on Derivative of Implicit Function

If $f(x, y) = 0$ is a implicit function then,

$$\frac{dy}{dx} = \frac{-f_x}{f_y}$$

f_x: diff. it w.r.t. to x and keeping 'y' constant

f_y: diff. it w.r.t. to y and keeping 'x' constant

Illustration 3

Find $\dfrac{dy}{dx}$ if, $x^3 + y^3 + 2xy + \sin y - 100 = 0$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\dfrac{dy}{dx} = \dfrac{-\left(3x^2+0+2y+0\right)}{0+3y^2+2x+\cos y} = \dfrac{-\left(3x^2+2y\right)}{3y^2+2x+\cos y}$

Illustration 4

Find $\dfrac{dy}{dx}$ if, $\cos(xy) - xy^3 + 5y = 0$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\dfrac{dy}{dx} = \dfrac{-(-\sin(xy)\cdot y + y^3 + 0}{-\sin(xy)\cdot x - 3xy^2 + 5}$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy + y^3}{-x\sin xy - 3xy^2 + 5}$$

TIPS AND TRICKS: (T-3)

Short trick to find differentiation of the form:

$$\frac{d}{dx}\left((f(x))^{g(x)}\right)=\left((f(x))^{g(x)}\right)\left\{\frac{g(x)}{f(x)}\cdot f'(x)+\log_e\left(f(x)\right)\cdot g'(x)\right\}$$

Illustration 5

If $y=x^{\sin x}$, find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-3}}$ $x^{\sin x}\left\{\frac{\sin x}{x}\times 1+\log_e x\ (\cos x)\right\}=\frac{dy}{dx}$

Illustration 6

If $y=(\sec x)^{e^x}$, find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-3}}$ $(\sec x)^{e^x}\left\{\frac{e^x}{\sec x}(\sec x\tan x)+\log_e(\sec x)\cdot e^x\right\}=\frac{dy}{dx}$

Illustration 7

If $y=(\sin^{-1}x)^{\cos x}$, find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-3}}$

$$\frac{dy}{dx}=(\sin^{-1}x)^{\cos x}\left\{\frac{\cos x}{\sin^{-1}x}\times\frac{1}{\sqrt{1-x^2}}+\log_e(\sin^{-1}x)\cdot(-\sin x)\right\}$$

TIPS AND TRICKS: (T-4)

Differentiation of the form:

If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+\ldots\infty}}}$ then, $\frac{dy}{dx}=\frac{f'(x)}{2y-1}$

Illustration 8

If $y=\sqrt{\sin^2 x+\sqrt{\sin^2 x+\sqrt{\sin^2 x+\ldots\infty}}}$, then find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\frac{dy}{dx}=\frac{2\sin x\cos x}{2y-1}=\frac{\sin 2x}{2y-1}$

Illustration 9

If $y=\sqrt{\log_e^2 x+\sqrt{\log_e^2 x+\sqrt{\log_e^2 x+\ldots\infty}}}$; then find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\frac{dy}{dx}=\frac{2(\log_e x)\frac{1}{x}}{2y-1}=\frac{\frac{2}{x}(\log_e x)}{2y-1}$

TIPS AND TRICKS: (T-5)

Infinite series differentiation

$y=(f(x))^{(f(x))^{(f(x))\ \ldots\ldots\infty}}$, then $\frac{dy}{dx}=\frac{y^2 f'(x)}{f(x)\{1-y\log_e(f(x))\}}$

Illustration 10

If $y=x^{x^{x\ldots\ldots\infty}}$, then find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\frac{dy}{dx}=\frac{y^2\times 1}{x\{1-y\log_e x\}}$

Illustration 11

If $y=(\sin x)^{(\sin x)\ldots\ldots\infty}$, then find $\frac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\frac{dy}{dx}=\frac{y^2\times\cos x}{\sin x\{1-y\log_e \sin x\}}$

TIPS AND TRICKS: (T-6)

If $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ldots\ldots \infty}}$, then $\dfrac{dy}{dx} = \dfrac{y\, f'(x)}{2y - f(x)}$

Illustration 12

If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots\infty}}}$, then find $\dfrac{dy}{dx}$

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\dfrac{dy}{dx} = \dfrac{y \times 1}{2y - x} = \dfrac{y}{2y - x}$ $\qquad (\because f(x) = x)$

TIPS AND TRICKS: (T-7)

Short trick on n^{th} order derivative:

To find r^{th} derivative of $f(x) = (ax + b)^n$

Then, if (i) $n \geq r$, $f^r(x) = a^r \cdot P_r\, (ax + b)^{n-r}$ (ii) If $n < r, f^r(x) = 0$

Illustration 13

If $f(x) = 2x^{19} - x^7$, then find $f^{10}(x)$

Short-cut solution :

$\boxed{\text{Using T-7}}$ $f^{10}(x) = 2^{10}\, ({}^{19}P_{10})\, x^{19-10} - 0$

$\Rightarrow f^{10}(x) = 2^{10} \cdot {}^{19}P_{10}\, x^9$

Illustration 14

If $f(x) = x^n$, then value of **[AIEEE 2003]**

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n + f^n(1)}{n!} \text{ is}$$

Short-cut solution :

$$1 - \frac{n}{1} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \ldots + (-1)^n$$

$\Rightarrow \quad {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\, {}^nC_n$

$\Rightarrow \quad 0$

TIPS & TRICKS

TIPS AND TRICKS: (T-8)

Use substitution method to reduce the calculations.

Assume $f(x)$ a function in simplest form or put constant = 0, 1, 2, 3, ... etc. for simplification.

Illustration 15

Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A.P. then $f'(a), f'(b)$ and $f'(c)$ are in

(a) AP (b) GP (c) AGP (d) HP

Short-cut solution :

$\boxed{\text{Using T-8}}$ Let $f(x) = x^2 \Rightarrow f'(x) = 2x$

Now, $f'(a) = 2a$, $f'(b) = 2b$ and $f'(c) = 2c$

Hence, are in AP.

Illustration 16

If $y = \sin^2\theta + \cos^2(\theta+\delta) + 2\sin\theta\sin\delta\cos(\theta+\delta)$, then $\dfrac{d^3y}{dx^3}$ is

(a) $\dfrac{\sin^3(\theta+\delta)}{\cos\theta}$ (b) $\cos(\theta+3\delta)$ (c) 0 (d) none of these

Short-cut solution :

$\boxed{\text{Using T-8}}$ Let $\delta = 0 \Rightarrow y = \sin^2\theta + \cos^2\theta = 1 \Rightarrow \dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{d^3y}{dx^3} = 0$

Here check the options (a), (b), (c), (d) for $\delta = 0$. **Ans. (c)**

SHORTCUTS: (SC-1)

Let $f(x)$ be a polynomial and $x = \alpha$ be 'r' times repeated root of $f(x) = 0$ then,

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(r-1)}(\alpha) = 0$$

★ **Note:** Reverse is also true.

Illustration 17

If $f(x)$ has two times repeated roots then find the function.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ If $f(x)$ has two times repeated roots

$\Rightarrow f(x) = (x - \alpha)^2 g(x)$; where '$\alpha$' is any root of $f(x)$.

SHORTCUTS: (SC-2)

Derivative of inverse of a function

Let $g(x)$ be the inverse of $f(x)$

$$\Rightarrow \quad \boxed{f(g(x)) = x \quad \text{or} \quad g(f(x)) = x}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$f'(g(x)) \cdot g'(x) = 1 \qquad g'(f(x)) \cdot f'(x) = 1$$

Illustration 18

If $f(x) = e^{x^3 + x^2 + x}$ and $g(x) = f^{-1}(x)$, then find $g'(e^3)$

Short-cut solution :

$\boxed{\text{Using SC-2}}$ Let $g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1$

Put $x = 1$; $g'(f(1)) \cdot f'(1) = 1$

$$\Rightarrow g'(e^3) = \frac{1}{6e^3} \qquad (\because f(1) = e^3)$$

Illustration 19

If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$ then the value of $g'(1)$ is **[AIEEE 2009]**

Short-cut solution :

$\boxed{\text{Using SC-2}}$ Let $g(f(x)) = x$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$

Now, Put $x = 0 \Rightarrow g'(1) \cdot f'(0) = 1$

$$\Rightarrow \quad g'(1) = 2 \left(\because f'(0) = \frac{1}{2}\right)$$

TECHNIQUE

To find the derivative of the product of finite number of functions.

$f(x) = g_1(x)\, g_2(x)\, g_3(x)...g_n(x)$

Take log on both sides

$\log f(x) = \log[g_1(x) + g_2(x)...g_n(x)]$

$\log f(x) = \log g_1(x) + \log g_2(x) + ... + \log_n(x)$

$$\frac{1}{f(x)} f'(x) = \frac{g'_1(x)}{g_1(x)} + \frac{g'_2(x)}{g_2(x)} + ... + \frac{g'_n(x)}{g_n(x)}$$

$$f'(x) = f(x)\left[\frac{g'_1(x)}{g_1(x)} + \frac{g'_2(x)}{g_2(x)} + ... + \frac{g'_n(x)}{g_n(x)}\right]$$

Illustration 20

If $f(x) = (x + 1)(x + 2)(x + 3)...(x + n)$, then find $f'(0)$.

Short-cut solution :

Using Tech.

$$f'(x) = f(x)\left[\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + ... + \frac{1}{x+n}\right]$$

$$\therefore f'(0) = f(0)\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]$$

$$= (1\cdot 2\cdot 3\cdot ....n)\left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]$$

$$= n!\left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]$$

Illustration 21

If $f(x) = (1 + x)\,(3 + x^2)^{1/2}\,(9 + x^3)^{1/3}$, then find $f'(x)$.

Short-cut solution :

Using Tech.

$$f'(x) = f(x)\left[\frac{1}{1+x} + \frac{1\times 2x}{2(3+x^2)^{1/2}\,(3+x^2)^{\frac{1}{2}}} + \frac{3x^2}{3(9+x^3)}\right]$$

Illustration 22

If $f(x) = \dfrac{\sqrt{1-x^2}\,(2x+3)^{1/2}}{(x^2+2)^{2/3}}$, then find $f'(0)$

Short-cut solution :

$\boxed{\text{Using Tech.}}$

$\because \quad f(0) = \dfrac{3^{1/2}}{2^{2/3}}$

$\Rightarrow f'(x) = f(x)\left[\dfrac{1(-2x)}{2(1-x^2)} + \dfrac{1}{2} \times \dfrac{2}{(2x+3)} - \dfrac{2}{3} \times \dfrac{1}{(x^2+2)} \times 2x\right]$

Put, $\quad x = 0$

$\Rightarrow f'(0) = \dfrac{3^{1/2}}{2^{2/3}}\left[0 + \dfrac{1}{3}\right] = \dfrac{1}{2^{2/3}.3^{1/2}}$

Concept Booster Exercise

1. If $f(x) = \dfrac{\sin(x^2+1)+2}{3\sin(x^2+1)-5}$, then $f'(0)$ is equal to

(a) 1 (b) 0 (c) – 1 (d) 2

2. If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\dfrac{dy}{dx}$ is equal to

(a) 0 (b) 1 (c) 2 (d) –1

3. If $f(x) = x^{x^{x^4}}$, then $f'(1)$ is equal to

(a) 0 (b) – 1 (c) 1 (d) 2

4. If $y = \sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}$ then $\left.\dfrac{dy}{dx}\right]_{x=0}$ is equal to

(a) 0 (b) – 1 (c) 1 (d) 2

5. If $x = (e^y)^{(e^y)^{e^y \ldots\ldots\infty}}$, then find $\dfrac{dx}{dy}$

(a) $\dfrac{x^2}{1-xy}$ (b) $\dfrac{y^2}{1-xy}$ (c) $\dfrac{x^2}{1+xy}$ (d) $\dfrac{x^2}{1-y}$

6. If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \ldots\ldots\infty}}}$, then find $\dfrac{dx}{dy}$

(a) $\dfrac{2xy}{x^2-xy}$ (b) $\dfrac{2xy}{2y-x^2}$ (c) $\dfrac{2y}{2y-x^2}$ (d) $\dfrac{2x}{2y-x^2}$

7. If $f(x) = 3x^{11} + 5x^6$; the find $f^{10}(x)$ (10^{th} order derivative)

(a) $3^9\ {}^{11}P_{10}\,(3x)$ (b) $3^{11}\ {}^{11}P_{10}\,(3x)$

(c) $3^{10}\ {}^{11}P_{10}\,(3x)$ (d) $3^{10}\ {}^{11}P_{10}\,(3x)^2$

8. If $y = \cos^{-1}(2x^2 - 1)$, then find $\dfrac{dy}{dx}$

(a) $\dfrac{2}{\sqrt{1-x^2}}$ (b) $\dfrac{2}{\sqrt{1+x^2}}$ (c) $\dfrac{-2}{\sqrt{1+x^2}}$ (d) $\dfrac{-2}{\sqrt{1-x^2}}$

9. If $y = (x^2+1)\,e^{2x}\cos x$, then find $\left.\dfrac{dy}{dx}\right]_{x=0}$

(a) 2 (b) – 2 (c) 3 (d) – 3

10. If 'g' is the inverse of a function f and $f'(x) = \dfrac{1}{1+x^5}$ then $g'(x)$ is equal to

(a) $1 + x^5$ (b) $5x^4$ (c) $\dfrac{1}{1+(g(x))^5}$ (d) $1 + \{g(x)\}^5$

11. If $f(x)$ be a polynomial which has $x = \alpha$ be four times repeated roots then $f'''(\alpha)$ is equal to

(a) 1 (b) 0

(c) 2 (d) Insufficient information

12. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is equal to

(a) 1 (b) –1 (c) 2 (d) 0

13. If $f(x) = \sum_{n=1}^{100} (x-n)^{n(101-n)}$; find $\dfrac{f'(101)}{f(101)}$

(a) 5051 (b) 5050 (c) 4050 (d) 4051

14. If $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}$, then $\dfrac{dy}{dx}$ is equal to

(a) y (b) $y + \dfrac{x^n}{n!}$ (c) $y - \dfrac{x^n}{n!}$ (d) $y - 1 - \dfrac{x^n}{n!}$

15. $\dfrac{d}{dx}[\sin^n x \cdot \cos(nx)]$ is equal to

(a) $n \sin^{n-1} x \cos(n + 1)x$ (b) $n \sin^{n-1} x \cos nx$

(c) $n \sin^{n-1} x \cos(n - 1)x$ (d) $n \sin^{n-1} x \sin(n + 1)x$

Numerical Value Problems

16. If $y = \sqrt{\cos^2 x + \sqrt{\cos^2 x + \sqrt{\cos^2 x + \ldots\ldots \infty}}}$ then find $\left.\dfrac{dy}{dx}\right]_{x=0}$

17. If $y = (\cos x)^{e^x}$, then find $\left.\dfrac{dy}{dx}\right]_{x=0}$

Solutions

1. (b) Using T-1 $f'(x) = \dfrac{(-5-6)\cdot\cos(x^2+1)\cdot 2x}{\{3\sin(x^2+1)-5\}^2}$

$\Rightarrow f'(0) = 0$

2. (b) Using T-2 $\because f(x, y) = xe^{xy} - y - \sin^2 x$

$$\Rightarrow f'(x) = \frac{-f_x}{f_y} = \frac{-(e^{xy} + xe^{xy}.y - 2\sin x\cos x}{x^2 e^{xy} - 1}$$

$\Rightarrow f'(0) = 1$

3. (c) Using T-3 $f'(x) = x^{x^{x^4}}\left[\dfrac{x^{x^4}}{x} + \ln x\dfrac{d}{dn}(x^{x^4})\right]$

$$\Rightarrow f'(x) = x^{x^{x^4}}\left[\frac{x^{x^4}}{x} + (\ln x)\, x^{x^4}(4x^3\ln x + x^3)\right] \Rightarrow f'(1) = 1$$

4. (b) Using T-4 $\dfrac{dy}{dx} = \dfrac{1}{2y-1}(\because f(x) = x)$

Now, put $x = 0$ in the original eqn $\Rightarrow y = 0$

Hence, $\left.\dfrac{dy}{dx}\right]_{x=0} = -1$

5. (a) Using T-5 This is similar to T-5 but in this dependent variable is 'x' and independent variable is 'y'.

$$\Rightarrow \frac{dx}{dy} = \frac{x^2 f'(y)}{f(y)\{1 - x\log_e(f(y))\}}, \quad \text{Here } f(y) = e^y$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^2 e^y}{e^y\{1 - x\log_e e^y\}} = \frac{x^2}{1-xy}$$

6. (b) Using T-6 $\because \quad f(x) = x^2$

$$\Rightarrow \frac{dy}{dx} = \frac{y\cdot 2x}{2y - x^2} = \frac{2xy}{2y - x^2}$$

7. (c) Using T-7 Here, $n = 11, \ r = 10$

$\Rightarrow f^{10}(x) = 3^{10}\ {}^{11}P_{10}(3x)^1 + 0$

8. (d) $\boxed{\text{Using T-8}}$ Put $x = \cos\theta \Rightarrow y = \cos^{-1}\cos 2\theta$

$\Rightarrow y = 2\theta \Rightarrow y = 2\cos^{-1} x$

Hence, $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1-x^2}}$

9. (a) $\boxed{\text{Using Tech}}$ Let $y = f(x)$

$\Rightarrow f'(x) = (x^2+1)\,e^{2x}\cdot\cos x\left[\dfrac{2x}{x^2+1} + \dfrac{2e^{2x}}{e^{2x}} - \dfrac{\sin x}{\cos x}\right]$

$\Rightarrow f'(0) = 2$

10. (d) $\boxed{\text{Using SC-2}}$ $fog\,(x) = x$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\dfrac{1}{1+\{g(x)\}^5}} \qquad \left(\because f'(x) = \frac{1}{1+x^5}\right)$$

$\Rightarrow g'(x) = 1 + \{g(x)\}^5$

11. (b) $\boxed{\text{Using SC-1}}$

If four times repeated roots then,

$$f(\alpha) = f'(\alpha) = f''(\alpha) = f'''(\alpha) = 0$$

12. (a) $\boxed{\text{Using T-2}}$ Here, $f(x, y) = \log(x+y) - 2xy$

$$\text{Now, } \frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-\dfrac{1}{x+y} + 2y}{\dfrac{1}{x+y} - 2x} = 1 \qquad (\because y(0) = 1)$$

13. (b) $\boxed{\text{Using Tech.}}$

$$\frac{f'(x)}{f(x)} = \sum_{n=1}^{100} \frac{(x-n)^{n(101-n)-1}\cdot n(101-n)}{(x-n)^{n(101-n)}} = \sum_{n=1}^{100} \frac{n(101-n)}{(x-n)}$$

$$\Rightarrow \frac{f'(101)}{f(101)} = \sum_{n=1}^{100} n = 5050$$

14. (c) $\boxed{\text{Using T-8}}$ Put $n = 1 \Rightarrow y = 1 + x$

Now, $\dfrac{dy}{dx} = 1$

Here, check the options (a), (b), (c), (d) for $n = 1$

$\Rightarrow \quad y - x = 1 + x - x = 1$

15. (a) $\boxed{\text{Using T-8}}$ Put $n = 1 \Rightarrow \dfrac{d}{dx}(\sin x \cos x) = \cos 2x$

Now, check options (a), (b), (c), (d) for $n = 1$

$\Rightarrow \quad \cos 2x$

16. (0) $\boxed{\text{Using T-4}}$ $\because \; f(x) = \cos^2 x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\cos x\,(-\sin x)}{2y-1} \Rightarrow \left.\frac{dy}{dx}\right]_{x=0} = 0$$

17. (0) $\boxed{\text{Using T-3}}$ Here, $f(x) = \cos x$ and $g(x) = e^x$

$$\text{Now,} \quad \frac{dy}{dx} = (\cos x)^{e^x}\left[\frac{e^x}{\cos x}\cdot(-\sin x) + \log_e(\cos x)\cdot e^x\right]$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{x=0} = 0$$

7 Application of Derivatives

Review of Key Notes and Formulae

Let the tangent and normal drawn at point $P(x_1, y_1)$ to the curve $y = f(x)$.

Equation of tangent: $y - y_1 = \left.\dfrac{dy}{dx}\right]_P (x - x_1)$

Equation of normal: $y - y_1 = \dfrac{-1}{\left.\dfrac{dy}{dx}\right]_P}(x - x_1)$

★ **Note:**

(i) If tangent is parallel to x-axis then

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

(ii) If tangent is parallel to y-axis then

$$\Rightarrow \quad \frac{dy}{dx} = \infty$$

(iii) If tangent makes equal angles with the axes

$$\Rightarrow \quad \frac{dy}{dx} = \pm 1$$

(iv) If normal is parallel to x-axis then

$$\Rightarrow \quad \frac{dy}{dx} = \infty$$

(v) If normal is parallel to y-axis then

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

(vi) If normal is equally inclined from both the axes or cuts equal intercept then

$$\Rightarrow \quad \left(\frac{dy}{dx}\right) = \pm 1$$

Length of Sub-tangent, Sub-normal, Tangent and Normal

(i) Length of sub-tangent $= \left| \dfrac{y}{\left(\dfrac{dy}{dx}\right)} \right|$

(ii) Length of sub-normal $= \left| y\left(\dfrac{dy}{dx}\right) \right|$

(iii) Length of tangent $= \left| \dfrac{y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}{\left(\dfrac{dy}{dx}\right)} \right|$

(iv) Length of normal $= \left| y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} \right|$

where, $\dfrac{dy}{dx}$ is slope of tangent.

Monotonicity

1. **Monotonic Functions:**

(i) *Monotonically increasing function*: The value of $f(x)$ should increase (decrease) or remain equal by increasing (decreasing) the value of x.

Let x_1, x_2 belongs to domain of the function

Then; $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

Or $x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

(ii) *Monotonically decreasing function:* The value of $f(x)$ should decrease (increase) or remain equal by increasing (decreasing) the value of x.

Then; $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$

Or $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$

★**Note:**

(i) If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in D$, then $f(x)$ is strictly increasing in domain D.

(ii) If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in D$, then $f(x)$ is strictly decreasing in domain D.

Properties of Monotonic Functions

(i) If $f(x)$ is strictly increasing (decreasing) function on an interval $[a, b]$, then f^{-1} exists and also a strictly increasing (decreasing) function.

(ii) If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ for each $c \in (a, b)$ then

$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$
↑ increasing	↑ increasing	↑ increasing	↑ increasing
↑ increasing	↓ decreasing	↓ decreasing	↓ decreasing
↓ decreasing	↓ decreasing	↑ increasing	↑ increasing

2. Maxima and Minima:

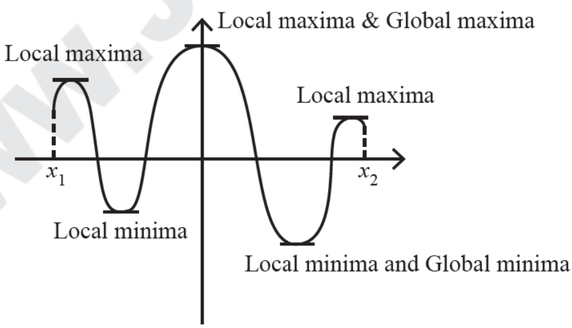
(i) *Local maxima:* A function $f(x)$ is said to have local maxima at point $x = a$, if there exists a very small positive quantity h, such that $f(x) \leq f(a)\ \forall\ x \in (a - h, a + h)$.

(ii) *Local minima:* A function $f(x)$ is said to have local minima at point $x = b$, if there exists a very small positive quantity h, such that $f(x) \geq f(b)\ \forall\ x \in (b - h, b + h)$.

★The point $x = a$ is called the point of local maxima of the function.

(iii) *Global maxima:* A function 'f' has a global maxima at 'c', if $f(c) \geq f(x)$ for all $x \in$ D.

(iv) *Global minima:* A function 'f' has a global minima at 'd', if $f(d) \leq f(x)$ for all $x \in$ D.



3. Methods for Testing Maxima/Minima:

(i) *First derivative test:* Find the critical points (say $x = c$) by putting $f'(x) = 0$

(a) If $f'(x)$ changes sign from positive to negative at $x = c$, then the point is said to be point of local maxima.

(b) If $f'(x)$ changes sign from negative to positive at $x = c$, then the point is said to be point of local minima.

(ii) *Second order derivative test:*

Step 1: Find $f'(x)$ and equals it to zero ($f'(x) = 0$)

Step 2: Find the real values of x by putting $f'(x) = 0$ (say $c_1, c_2, c_3, \ldots$)

Step 3: Find $f''(x)$ and substitute the values of $c_1, c_2, c_3, \ldots$ in it and get the sign of $f''(x)$.

Step 4: (a) If $f''(c) < 0$; then it has local maxima at $x = c$.

(b) If $f''(c) > 0$; then it has local minima at $x = c$.

4. **Greatest and Least Values of a Function:** If a function $f(x)$ is defined in an interval $[a, b]$ then

Greatest value of $f(x) = \text{Max.}\{f(a), f(b), f(c)\}$

Least value of $f(x) = \text{Min.}\{f(a), f(b), f(c)\}$

where $x = c$ is a point such that $f'(c) = 0$

TIPS & TRICKS

TIPS AND TRICKS: (T-1)

If the curve passes through the origin, then the equation of tangent at the origin can be directly written by equating to zero, the lowest degree terms appearing in the equation of the curve.

Illustration 1

Find the equation of tangent at origin.

(i) $x^2 + y^2 + 2gx + 2fy = 0$

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

Short-cut solution :

(i) Using T-1 Lowest degree term $= 2gx + 2fy = 0$

$\Rightarrow$ $gx = -fy$

(ii) Using T-1 Lowest degree term $= x^2 - y^2 = 0$

$\Rightarrow$ $y = \pm x$

TIPS & TRICKS

TIPS AND TRICKS: (T-2)

If OP and OQ are intercept made by a tangent to the curve $x^m + y^m = a^m$ on the coordinate axes then, $(OP)^n + (OQ)^n = a^n$

Illustration 2

The sum of intercepts made by a tangent to the curve $x^{1/3} + y^{1/3} = 2$ at point (8, 8) on the coordinate axes is ___.

Short-cut solution :

$\boxed{\text{Using T-2}}$ Curve: $x^{1/3} + y^{1/3} = 8^{1/3}$

$\Rightarrow$ (OP) + (OQ) = 8

Illustration 3

If the tangents at any point on the curve $x^{2/3} + y^{2/3} = p^{2/3}$ cuts off the intercepts 'a' and 'b' on the coordinate axes, then find the value of $a^{-5/3} + b^{-5/3}$.

Short-cut solution :

$\boxed{\text{Using T-2}}$ Curve; $x^{2/3} + y^{2/3} = p^{2/3}$

Hence, $a^{-5/3} + b^{-5/3} = p^{-5/3}$

TIPS AND TRICKS: (T-3)

Equation of tangent to the curve $x^n + y^n = a^n$ at (x_1, y_1) is

$xx_1^{n-1} + yy_1^{n-1} = a^n$

Illustration 4

The equation of tangent to the curve $x^{2/3} + y^{2/3} = 2^{2/3}$ at point (x_1, y_1)

Short-cut solution :

$\boxed{\text{Using T-3}}$ Put $n = \dfrac{2}{3}$ $\Rightarrow$ $x(x_1)^{\frac{2}{3}-1} + y(y_1)^{\frac{2}{3}-1} = 2^{\frac{2}{3}}$

Equation of tangent is $\Rightarrow \dfrac{x}{x_1^{1/3}} + \dfrac{y}{y_1^{1/3}} = 2^{2/3}$

Illustration 5

The tangent to the curve, $y = xe^{x^2}$ passing through the point (1, e) also passes through the point: **[JEE M 2019]**

(a) (2, 3e) (b) $\left(\dfrac{4}{3}, 2e\right)$ (c) $\left(\dfrac{5}{3}, 2e\right)$ (d) (3, 6e)

Short-cut solution :

The equation of curve $y = xe^{x^2}$

$$\Rightarrow \frac{dy}{dx} = e^{x^2}.1 + x.e^{x^2}.2x$$

Since (1, e) lies on the curve $y = xe^{x^2}$, then equation of tangent at (1, e) is

$$y - e = (e^{x^2}(1+2x^2))_{x=1}(x-1)$$

$$y - e = 3e(x - 1)$$
$$3ex - y = 2e$$

So, equation of tangent to the curve passes through the point $\left(\frac{4}{3}, 2e\right)$ **Ans. (b)**

TIPS AND TRICKS: (T-4)

If two curves $p_1x^2 + q_1y^2 = 1$ and $p_2x^2 + q_2y^2 = 1$ cuts orthogonally to each other then

$$\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} - \frac{1}{q_2}$$

Illustration 6

If curves $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ and $\frac{x^2}{m^2} - \frac{y^2}{n^2} = 1$ intersects orthogonally, then find the condition:

Short-cut solution :

Using T-4 $\frac{1}{\frac{1}{p^2}} - \frac{1}{\frac{1}{q^2}} = \frac{1}{\frac{1}{m^2}} - \frac{1}{\frac{-1}{n^2}}$

$\Rightarrow p^2 - q^2 = m^2 + n^2$

TIPS AND TRICKS: (T-5)

Area of triangle formed by tangent to any point of curve $2xy = p^2$ with coordinates axes is constant and it is equal to p^2.

Illustration 7

Area of triangle formed by tangent at any point of curve $xy = 9$ with coordinate axes is _____.

Short-cut solution :

Using T-5 $\because$ Curve is $2xy = 18$

$\Rightarrow$ Area = 18 sq. units

TIPS AND TRICKS: (T-6)

Let $f(x) = \dfrac{p.\sin x + q.\cos x}{r.\sin x + s.\cos x}$

(i) If $\begin{vmatrix} p & q \\ r & s \end{vmatrix} > 0 \Rightarrow$ Increasing function

(ii) If $\begin{vmatrix} p & q \\ r & s \end{vmatrix} < 0 \Rightarrow$ Decreasing function

Illustration 8

If $f(x) = \dfrac{\sin x + p\cos x}{2\sin x + 3\cos x}$ is increasing function then find the interval of 'a'.

Short-cut solution :

Using T-6(i) $\begin{vmatrix} a & 2 \\ 5 & 6 \end{vmatrix} > 0$

$\Rightarrow 6a - 10 > 0 \Rightarrow a > \dfrac{5}{3}$

Illustration 9

If $f(x) = \dfrac{\sin x + p\cos x}{2\sin x + 3\cos x}$ is decreasing function then find the interval of 'p'.

Short-cut solution :

Using T-6(ii) $\begin{vmatrix} 1 & p \\ 2 & 3 \end{vmatrix} < 0$

$\Rightarrow 3 - 2p < 0 \Rightarrow p > \dfrac{3}{2}$

Illustration 10

The function, $f(x) = (3x-7)x^{2/3}$, $x \in \mathbf{R}$, is increasing for all x lying in : **[JEE M 2020]**

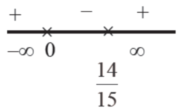
(a) $(-\infty, 0) \cup \left(\dfrac{14}{15}, \infty\right)$ (b) $(-\infty, 0) \cup \left(\dfrac{3}{7}, \infty\right)$

(c) $\left(-\infty, \dfrac{14}{15}\right)$ (d) $\left(-\infty, -\dfrac{14}{15}\right) \cup (0, \infty)$

Short-cut solution :

$$f(x) = (3x - 7) \cdot x^{2/3}$$

$$f'(x) = 3x^{2/3} + (3x - 7) \cdot \frac{2}{3} x^{-1/3} = \frac{15x - 14}{3x^{1/3}}$$



For increasing function

$f'(x) > 0$ then $x \in (-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$ **Ans. (a)**

Illustration 11

If $f(x) = xe^{x(1-x)}$, then $f(x)$ is **[AIEEE 2001S]**

(a) increasing on $[-1/2, 1]$ (b) decreasing on R

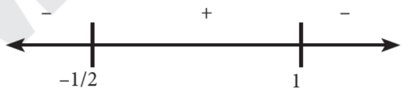
(c) increasing on R (d) decreasing on $[-1/2, 1]$

Short-cut solution :

$f(x) = xe^{x(1-x)}$

$\Rightarrow$ $f'(x) = e^{x(1-x)} + (1 - 2x)\, x\, e^{x(1-x)} = -e^{x(1-x)}(2x + 1)(x - 1)$

Critical point are $x = -\frac{1}{2}$ and 1



Hence, $f(x)$ is increasing on $[-1/2, 1]$. **Ans. (a)**

TIPS AND TRICKS: (T-7)

Let $f(x)$ be a polynomial function, then between any two roots of $f(x)$, there exists one root of its derivative i.e. $f'(x)$.

Illustration 12

Prove that between the roots of $x^2 - 3x + 2 = 0$ there exist one root of its derivative.

Short-cut solution :

$\boxed{\text{Using T-7}}$ $\because x^2 - 3x + 2 = 0$

$\Rightarrow$ $x = 1, 2$

Hence, $2x - 3 = 0$

$\Rightarrow \quad x = \dfrac{3}{2}$ which is between 1 and 2.

TIPS AND TRICKS: (T-8)

Descarts rule of sign for the roots of a polynomial.

Rule 1: The maximum number of positive real roots of a polynomial equation. $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ is number of changes of the signs of coefficients from positive to negative and negative to positive.

Example: $f(x) = \underset{\oplus}{x^3} \underset{\oplus}{+3x^2} \underset{\oplus}{+7x} \underset{\ominus}{-11} = 0$

$\Rightarrow$ Only one change of sign, then the number of positive roots of above equation is atmost '1'.

Rule 2: The maximum number of negative real roots of the polynomial equation $f(x) = 0$ is the number of changes from positive to negative and negative to positive in the sign coefficients of the equation $f(-x) = 0$

Example: $f(x) = x^5 + 2x^3 + 7x^2 - x + 12 = 0$

$\Rightarrow f(-x) = \underset{\ominus}{-x^5} \underset{\ominus}{-2x^3} \underset{\oplus}{+7x^2} \underset{\oplus}{+x} \underset{\oplus}{+12} = 0$

$\Rightarrow$ Only one change of sign, then negative roots is atmost '1'.

TIPS AND TRICKS: (T-9)

For the trigonometric functions of the type:

(i) If $f(x) = a \sin x \pm \text{b} \cos x$ →
- Maximum value $= \sqrt{a^2 + b^2}$
- Minimum value $= -\sqrt{a^2 + b^2}$

(ii) If $f(x) = a^2\sec^2 x + b^2\text{cosec}^2 x \longrightarrow$ Minimum value $= (a + b)^2$

(iii) If $f(x) = a \sec x + b \,\text{cosec}\, x \longrightarrow$ Minimum value $= (a^{2/3} + b^{2/3})^{3/2}$

Illustration 13

For the curve $r^2 = \dfrac{a^2}{\sin^2\theta} + \dfrac{b^2}{\cos^2\theta}$, then minimum value of 'r' is equal to

Short-cut solution :

$\boxed{\text{Using T-9(ii)}}$ $r = \sqrt{a^2\text{cosec}^2\theta + b^2\sec^2\theta}$

$\Rightarrow r_{\min} = a + b$

Illustration 14

The greatest value of the function $y = \dfrac{\sin 2x}{\sin\left(x+\dfrac{\pi}{4}\right)}; x \in \left(0, \dfrac{\pi}{2}\right)$ is equal to:

Short-cut solution :

$\boxed{\text{Using T-9(iii)}}$ $y = \dfrac{2\sin x\cos x}{\dfrac{\sin x}{\sqrt{2}}+\dfrac{\cos x}{\sqrt{2}}} = \dfrac{2\sqrt{2}}{\sec x+\operatorname{cosec} x}$

$\Rightarrow y_{\max} = \dfrac{2\sqrt{2}}{(\sec x+\operatorname{cosec} x)_{\text{Min.}}} \Rightarrow y_{\max} = \dfrac{2\sqrt{2}}{(1^{2/3}+1^{2/3})^{3/2}} = \dfrac{2\sqrt{2}}{2\sqrt{2}} = 1$

Illustration 15

The maximum value of $3\cos\theta + 5\sin\left(\theta - \dfrac{\pi}{6}\right)$ for any real value of θ is:

[JEE M 2019]

(a) $\sqrt{19}$ (b) $\dfrac{\sqrt{79}}{2}$

(c) $\sqrt{34}$ (d) $\sqrt{31}$

Short-cut solution :

Let, the functions is,

$f(\theta) = 3\cos\theta + 5\sin\theta\cdot\cos\dfrac{\pi}{6} - 5\sin\dfrac{\pi}{6}\cos\theta$

$= 3\cos\theta + 5\times\dfrac{\sqrt{3}}{2}\sin\theta - 5\times\dfrac{1}{2}\cos\theta$

$= \left(3-\dfrac{5}{2}\right)\cos\theta + 5\times\dfrac{\sqrt{3}}{2}\sin\theta = \dfrac{1}{2}\cos\theta + \dfrac{5\sqrt{3}}{2}\sin\theta$

$\max f(\theta) = \sqrt{\dfrac{1}{4}+\dfrac{25}{4}\times 3} = \sqrt{\dfrac{76}{4}} = \sqrt{19}$ **Ans. (a)**

TIPS & TRICKS!

TIPS AND TRICKS: (T-10)

The sum of two natural numbers is 'a' then their product is maximum, if numbers are $x = \dfrac{a}{2}$ and $y = a - \dfrac{a}{2}$

Illustration 16

The sum of two natural numbers is 20 and their product is maximum. Find the numbers.

Short-cut solution :

$\boxed{\text{Using T-10}}$ Here, $a = 20$

Then, numbers are $x = \frac{a}{2} = 10$ and $y = a - \frac{a}{2} = 10$

Hence, numbers are $x = 10$ and $y = 10$

Illustration 17

The sum of two natural numbers is 30. Then their product is maximum if numbers are ______

Short-cut solution :

$\boxed{\text{Using T-10}}$ Here, $a = 30$

Then numbers are $x = \frac{30}{2} = 15$ and $y = 30 - 15 = 15$

TIPS & TRICKS!

TIPS AND TRICKS: (T-11)

Maximum area of rectangle whose perimeter is 'P' is equal to $\left(\frac{P}{4}\right)^2$.

Illustration 18

Maximum area of rectangle whose perimeter is 24 is equal to ____

Short-cut solution :

$\boxed{\text{Using T-11}}$ Here, $P = 24$

$\Rightarrow$ Maximum area $= \left(\frac{P}{4}\right)^2 = 36$

Illustration 19

Maximum area of rectangle whose perimeter is 36 is equal to ____

Short-cut solution :

$\boxed{\text{Using T-11}}$ Here, $P = 36$

$\Rightarrow$ Maximum area $= \left(\frac{P}{4}\right)^2 = 81$

TIPS AND TRICKS: (T-12)

Least perimeter of rectangle having area 'A' square units is equal to $4\sqrt{A}$ units.

Illustration 20

Least perimeter of rectangle having area 49 m^2 is equal to ____

Short-cut solution :

Using T-12 Here, A = 49

$\Rightarrow$ Least perimeter = $4\sqrt{A} = 4\sqrt{49} = 28$ m

Illustration 21

Least perimeter of rectangle having area $\frac{81}{4}$ m^2 is equal to ____

Short-cut solution :

Using T-12 Here, $A = \frac{81}{4}$

$\Rightarrow$ Least perimeter = $4\sqrt{A} = 4\sqrt{\frac{81}{4}} = 18$ m

TIPS AND TRICKS: (T-13)

If the triangle is inscribed in a circle of radius 'r', then the area of inscribed triangle will be maximum if triangle is an equilateral triangle.

$\Rightarrow$ Maximum area = $\frac{\sqrt{3}}{4}(r\sqrt{3})^2$

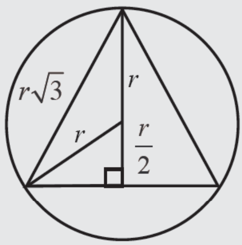


Illustration 22

Maximum area of triangle inscribed in a circle of radius 5 m is equal to

Short-cut solution :

Using T-13 Here, $r = 5$

$\Rightarrow$ Maximum area = $\frac{\sqrt{3}}{4}(5\sqrt{3})^2 = \frac{75\sqrt{3}}{4}$ m^2

TIPS AND TRICKS: (T-14)

Maximum and minimum value of $y = ax^2 + bx + c$ is as follows:

- If $a > 0$ (coeff. of x^2)
  - Max. value: $\boxed{\infty}$
  - Min. value: $\boxed{\dfrac{4ac - b^2}{4a}}$
- If $a < 0$ (coeff. of x^2)
  - Max. value: $\boxed{\dfrac{4ac - b^2}{4a}}$
  - Min. value: $\boxed{-\infty}$

Illustration 23

Find maximum and minimum value of $y = x^2 + x + 1$.

Short-cut solution :

$\boxed{\text{Using T-14}}$ Here, $a > 0$

$\Rightarrow$ Minimum value $= \dfrac{4-1}{4} = \dfrac{3}{4}$

Maximum value is ∞.

Illustration 24

Find the maximum and minimum value of $y = -2x^2 + 3x + 1$.

Short-cut solution :

$\boxed{\text{Using T-14}}$ Here, $a < 0$

$\Rightarrow$ Minimum value is $-\infty$

Maximum value $= \dfrac{4\times(-2)\times 1 - 9}{4(-2)} = \dfrac{17}{8}$

Illustration 25

The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is: **[JEE M 2016]**

(a) $\dfrac{\sqrt{15}}{2}$ (b) $\sqrt{\dfrac{19}{2}}$

(c) $\sqrt{\dfrac{15}{2}}$ (d) $\dfrac{\sqrt{19}}{2}$

Short-cut solution :

$D = \sqrt{\alpha^2 + (\alpha^2 - 4)^2}$

$D^2 = \alpha^2 + \alpha^4 + 16 - 8\alpha^2 = \alpha^4 - 7\alpha^2 + 16$

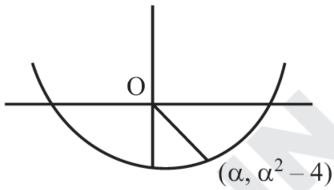
$\dfrac{dD^2}{d\alpha} = 4\alpha^3 - 14\alpha = 0$

$2\alpha(2\alpha^2 - 7) = 0$

$\alpha^2 = \dfrac{7}{2}$

$D^2 = \dfrac{49}{4} - \dfrac{49}{2} + 16 = -\dfrac{49}{4} + 16 = \dfrac{15}{4}$

$D = \dfrac{\sqrt{15}}{2}$



Ans. (a)

SHORTCUTS: (SC-1)

To find whether $f(x) > g(x)$ or $g(x) > f(x)$ i.e. to establish inequalities.
The concept of AM ≥ GM can be used ⇒ Split the function such that on multiplication (GM) it becomes unity.

Illustration 26

Prove that: $2\sin x + \tan x \geq 3x,\ x \in \left[0, \dfrac{\pi}{2}\right]$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Let $f(x) = 2\sin x + \tan x - 3x$

$\because f'(x) = 2\cos x + \sec^2 x - 3$

$= \underbrace{\cos x + \cos x + \sec^2 x} - 3$

Now, apply AM ≥ GM ⇒ $\dfrac{\cos x + \cos x + \sec^2 x}{3} \geq (\cos^2 x . \sec^2 x)^{1/3}$

$\Rightarrow 2\cos x + \sec^2 x \geq 3 \Rightarrow f'(x) \geq 0$

Hence, $2\sin x + \tan x \geq 3x$. Hence Proved.

SHORTCUTS: (SC-2)

In order to find condition for roots of cubic polynomial

$f(x) = ax^3 + bx^2 + cx + d = 0$ if $f(x)$ is non-monotonic function then follow the steps.

Step 1: Differentiate $f(x) \Rightarrow f'(x) = 3ax^2 + 2bx + c = 0 \begin{matrix} \nearrow x_1 \\ \searrow x_2 \end{matrix}$

Now, quadratic equation must have 2 real roots (D > 0). Let the roots be x_1 and x_2.

Step 2: If $f(x_1).f(x_2) > 0 \Rightarrow$ 1 real and 2 imaginary root.

If $f(x_1).f(x_2) = 0 \Rightarrow$ 3 real (2 coincident roots)

If $f(x_1).f(x_2) < 0 \Rightarrow$ 3 real (Distinct roots)

Illustration 27

If the cubic $y = x^3 + px + q$ has 3 distinct real roots, then

(a) $4p^3 - 27q^2 < 0$ (b) $4p^2 - 27q^3 > 0$

(c) $4p^3 + 27q^2 < 0$ (d) None of these

Short-cut solution :

$\boxed{\text{Using SC-2}}$ $\because y' = 3x^2 + p = 0 \Rightarrow x = \pm\sqrt{\frac{-p}{3}}$

Now, $y = x(x^2 + p) + q$

$$\Rightarrow y\left(\sqrt{\frac{-p}{3}}\right).y\left(-\sqrt{\frac{-p}{3}}\right) = \left[\sqrt{\frac{-p}{3}}\left(\frac{2p}{3}\right)+q\right]\left[-\sqrt{\frac{-p}{3}}\left(\frac{2p}{3}\right)+q\right] < 0$$

$$= \frac{-4p^2}{9}\left(\frac{-p}{3}\right) + q^2 < 0$$

$\Rightarrow 4p^3 + 27q^2 < 0$ **Ans. (c)**

SHORTCUTS: (SC-3)

Maximum and minimum value can be find by using AM ≥ GM.

Split the terms such its multiplication (GM) becomes unity.

Illustration 28

If $p^2x^4 + q^2y^4 = c^6$, then find the maximum value of xy.

Short-cut solution :

$\boxed{\text{Using SC-3}}$ $\because$ AM $\geq$ GM

$$\Rightarrow \frac{p^2x^4 + q^2y^4}{2} \geq (p^2x^4q^2y^4)^{1/2}$$

$$\Rightarrow \frac{c^6}{2} \geq pqx^2y^2$$

$$\Rightarrow xy \leq \frac{c^3}{\sqrt{2pq}}$$

Illustration 29

If $xy = c^4$, then find the minimum value of $px + qy$ $(p > 0, q > 0)$.

Short-cut solution :

$\boxed{\text{Using SC-3}}$ $\because$ AM $\geq$ GM $\Rightarrow \dfrac{px+qy}{2} \geq (pqxy)^{1/2}$

$\Rightarrow px + qy \geq 2c^2\sqrt{xy}$ $\qquad (\because xy = c^4)$

Illustration 30

If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of $(p + q)$ is _____ **[AIEEE 2007]**

Short-cut solution :

$\boxed{\text{Using SC-3}}$ Let $p = \sin\theta$ and $q = \cos\theta$

$\Rightarrow p + q = \sin\theta + \cos\theta$

$\Rightarrow (p + q)_{\max} = \sqrt{2}$

TECHNIQUE

Angle of intersection of two curves:

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$ respectively and θ be the angle between intersection of tangents of curves C_1 and C_2. Then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}, \text{ where } m_1 = \left(\frac{dy}{dx}\right)_{C_1} \text{ and } m_2 = \left(\frac{dy}{dx}\right)_{C_2}$$

Illustration 31

Find the angle between the curves $y = \sin x$ and $y = \cos x$.

Short-cut solution :

$\boxed{\text{Using Tech.}}$ Point of intersection, $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

$$y = \sin x \Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}$$

$$y = \cos x \Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = -\frac{1}{\sqrt{2}}$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{2}{\sqrt{2}} \times \frac{2}{1} = 2\sqrt{2}$$

$$\therefore \qquad \theta = \tan^{-1}(2\sqrt{2})$$

Concept Booster Exercise

1. Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 0$ at origin.
(a) $xy = 1$ (b) $xy = 0$ (c) $xy = -1$ (d) $xy = 2$

2. If OA and OB are intercepts made by a tangent to the curve $\sqrt{x}+\sqrt{y}=4$ at point (4, 4) on coordinate axes is:
(a) $4\sqrt{2}$ (b) $8\sqrt{2}$ (c) $\sqrt{2}$ (d) $\sqrt{256}$

3. The equation of tangent to the curve $\frac{x^2}{9}+\frac{y^2}{4}=1$ at point $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ is equal to
(a) $2x + 3y = 6\sqrt{2}$ (b) $2x - 3y = 6\sqrt{2}$
(c) $2x + 3y = 3\sqrt{2}$ (d) $2x - 3y = 3\sqrt{2}$

4. If $2x^2 + 3y^2 = 1$ and $px^2 + 2y^2 = 1$ cuts orthogonally to each other then value of 'p' is equal to
(a) $\frac{2}{3}$ (b) $\frac{-3}{2}$ (c) $\frac{-2}{3}$ (d) $\frac{3}{2}$

5. Area of triangle formed by tangent at any point to the curve $5xy = \frac{p^2}{2}$ is
(a) p^2 (b) $\frac{p^2}{5}$ (c) $\frac{p^2}{2}$ (d) $\frac{p^2}{3}$

6. The value of 'b' for which $f(x) = \frac{\sin x + b\cos x}{2\sin x + 3\cos x}$ is increasing, is equal to
(a) $b > \frac{3}{2}$ (b) $b > \frac{2}{3}$ (c) $b < \frac{3}{2}$ (d) $b < \frac{2}{3}$

7. The number of real roots of $(x + 3)^2 + (x + 5)^2 = 16$ is
(a) 0 (b) 2 (c) 4 (d) None of these

8. For the curve $\frac{c}{r^2} = a^2\text{cosec}^2\theta + b^2\sec^2\theta$, then maximum value of 'r' is:
(a) $\frac{\sqrt{c}}{a-b}$ (b) $\frac{c}{a+b}$ (c) $\frac{c}{a-b}$ (d) $\frac{\sqrt{c}}{a+b}$

9. The sum of two natural numbers is 10 and the product of numbers are maximum, then the numbers are
(a) 2, 8 (b) 5, 5 (c) 4, 6 (d) 3, 7

10. The maximum of a rectangle whose perimeter is $\sqrt{3}$, is equal to

(a) $\frac{3}{4}$ (b) $\frac{3}{16}$ (c) $\frac{3}{8}$ (d) $\frac{3}{2}$

11. Least perimeter of rectangle whose area is 9 square units is equal to

(a) 12 (b) 24 (c) 6 (d) 36

12. Maximum area of triangle inscribed in a circle of radius $\sqrt{3}$ m is equal to

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{9}{4}$ (c) $\frac{\sqrt{3}}{4}$ (d) $\frac{9\sqrt{3}}{4}$

13. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min. $f(x) >$ max. $g(x)$, then relation between b and c is **[AIEEE 2003]**

(a) No real value of b & c (b) $0 < c < b\sqrt{2}$

(c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$

14. If A > 0, B > 0 and $A + B = \frac{\pi}{3}$, then maximum value of tan A . tan B is:

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

15. For $0 < a < x$, then minimum value of function $\log_a x + \log_x a$ is

(a) $\frac{1}{2}$ (b) 2 (c) 3 (d) 4

Numerical Value Problems

16. Let x and y be two real variables such that $x > 0$ and $xy = 1$. Find minimum value of $x + y$ is ________

17. If $ax^2 + \frac{b}{x} \geq c$ for all positive 'x' where $a > 0$ and $b > 0$, then $27ab^2 \geq \lambda c^3$ then, y is equal to ________

18. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at $x =$ ________ **[AIEEE 2006]**

19. Let $f: R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in AP, the minimum value of $f(x)$ is ________ **[JEE M 2020]**

20. The value of 'a' so that sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value, is ________

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ Putting lowest degree term = 0

   $\Rightarrow \quad xy = 0$

2. **(d)** $\boxed{\text{Using T-2}}$ Curve: $x^{1/2} + y^{1/2} = 16^{1/2}$

   $\Rightarrow$ Sum of intercepts $(OA)^1 + (OB)^1 = (16)^1 = \sqrt{256}$

3. **(a)** $\boxed{\text{Using T-3}}$ Putting $n = 2$ and $x_1 = \dfrac{3}{\sqrt{2}}, y_1 = \sqrt{2}$

   $\Rightarrow \dfrac{x.3}{9.\sqrt{2}} + \dfrac{y.\sqrt{2}}{4} = 1 \Rightarrow 2x + 3y = 6\sqrt{2}$

4. **(d)** $\boxed{\text{Using T-4}}$ Here, $p_1 = 2, p_2 = p, q_1 = 3, q_2 = 2$

   $\Rightarrow \dfrac{1}{2} - \dfrac{1}{p} = \dfrac{1}{3} - \dfrac{1}{2} \Rightarrow p = \dfrac{3}{2}$

5. **(b)** $\boxed{\text{Using T-5}}$ $2xy = \dfrac{p^2}{5}$

   Hence, area $= \dfrac{p^2}{5}$

6. **(c)** $\boxed{\text{Using T-6(i)}}$ $\begin{vmatrix} 1 & b \\ 2 & 3 \end{vmatrix} > 0$

   $\Rightarrow 3 - 2b > 0 \quad \Rightarrow b < \dfrac{3}{2}$

7. **(b)** $\boxed{\text{Using T-8(i)}}$ By Descarts rule:

   $\Rightarrow x^2 + 9 + 6x + x^2 + 25 + 10x - 16 = 0$

   $\Rightarrow x^2 + 8x + 9 = 0$

   Rule 1: $(f(x))$ $\quad (+) \quad (+) \quad (+)$ (No sign change)

   Rule 2: $(f(-x))$ $\quad x^2 - 8x + 9 = 0$

   $(+) \quad (-) \quad (+)$ (Two sign change)

   $\Rightarrow$ 2 real roots.

8. **(d)** $\boxed{\text{Using T-8(ii)}}$ $\because \; r^2_{\text{Max.}} = \dfrac{c}{(a^2 \operatorname{cosec}^2\theta + b^2 \sec^2\theta)_{\text{Min.}}}$

   $\Rightarrow r^2_{\text{Max.}} = \dfrac{c}{(a+b)^2} \Rightarrow r_{\text{max.}} = \dfrac{\sqrt{c}}{a+b}$

9. **(b)** $\boxed{\text{Using T-10}}$ Numbers are $\frac{10}{2}$ and $10 - \frac{10}{2} \Rightarrow 5, 5$

10. **(b)** $\boxed{\text{Using T-11}}$ $\text{Area}_{(\text{Max})} = \left(\frac{P}{4}\right)^2 = \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{3}{16}$

11. **(a)** $\boxed{\text{Using T-12}}$ Perimeter (Least) = $4\sqrt{\text{A}} \quad 4\sqrt{9} = 12$ units

12. **(d)** $\boxed{\text{Using T-13}}$ $\text{Area}_{(\text{Max})} = \frac{\sqrt{3}}{4}(r\sqrt{3})^2 = \frac{9\sqrt{3}}{4}$ m^2

13. **(d)** $\boxed{\text{Using T-14}}$ $\because$ Min. $(f(x))$ > Max. $(g(x))$

$$\Rightarrow \frac{4(2c^2) - 4b^2}{4} > \frac{4(-b^2) - 4c^2}{-4}$$

$$\Rightarrow 2b^2 < c^2 \quad \Rightarrow \quad |c| > |b|\sqrt{2}$$

14. **(b)** For maximum $\text{A} = \text{B} = \frac{\pi}{6}$

$$\Rightarrow \tan\text{A}.\tan\text{B} = \tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{6}\right) = \frac{1}{3}$$

15. **(b)** $\boxed{\text{Using SC-3}}$ AM $\geq$ GM $\qquad \left(\because \log_a x = \frac{1}{\log_x a}\right)$

$$\Rightarrow \frac{\log_a x + \frac{1}{\log_x a}}{2} \geq (1)^{1/2} \Rightarrow (\log_a x + \log_x a)_{\text{Min}} = 2$$

16. **(2)** $\boxed{\text{Using SC-3}}$ AM $\geq$ GM

$$\Rightarrow \frac{x+y}{2} \geq (xy)^{1/2} \Rightarrow x + y \geq 2 \qquad (\because xy = 1)$$

17. **(4)** $\boxed{\text{Using SC-1}}$ AM $\geq$ GM

$$\Rightarrow \frac{ax^2 + \frac{b}{2x} + \frac{b}{2x}}{3} \geq \frac{(ab^2)^{1/3}}{4^{1/3}} \qquad \left(\because ax^2 + \frac{b}{x} \geq c\right)$$

$$\Rightarrow 27ab^2 \geq 4c^3 \qquad \text{(On cubing both sides)}$$

18. **(2)** $\boxed{\text{Using SC-3}}$ AM $\geq$ GM

$$\Rightarrow \frac{\frac{x}{2}+\frac{2}{x}}{2} \geq (1)^{1/2} \Rightarrow \left(\frac{x}{2}+\frac{2}{x}\right)_{\text{Min}} = 2$$

19. **(3)** $\boxed{\text{Using SC-3}}$ Since $(2^{1+x} + 2^{1-x}), f(x), (3^x + 3^{-x})$ are in A.P.

$$\Rightarrow f(x) = \frac{2^{1-x}+2^{1-x}+3^x+3^{-x}}{2}$$

Now, applying AM $\geq$ GM

$$\Rightarrow \frac{\frac{2^{1+x}}{2}+\frac{2^{1+x}}{2}+\frac{2^{1-x}}{2}+\frac{2^{1-x}}{2}+3^x+3^{-x}}{6} \geq (1)^{1/6}$$

$$\Rightarrow 2^{1+x} + 2^{1-x} + 3^x + 3^{-x} \geq 6$$

$$\Rightarrow \frac{2^{1+x}+2^{1-x}+3^x+3^{-x}}{2} \geq 3$$

20. **(1)** $\boxed{\text{Using T-14}}$ Let α, β be the roots of the quadratic equation

$$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\,\alpha\,\beta = (a-2)^2 - 2(1-a) = a^2 - 2a + 2$$

$$(\alpha^2 + \beta^2)_{\text{Min}} = \frac{4\times 2\times 1-4}{4} = 1$$

8 Indefinite Integration

Review of Key Notes and Formulae

1. **Definition:** Reverse process of differentiation

$$\underbrace{\int f(x)\,dx}_{\text{Integrand}} = g(x) + C \rightarrow \text{Indefinite Integration Constant.}$$

$g(x) \longrightarrow$ Integral / Primitive / Anti-derivative

2. **Standard Integration to Remember**

Integrands	**Integrals**	**Integrands**	**Integrals**
(i) $\int x^n\,dx$	$\dfrac{x^{n+1}}{n+1} + C$	(xi) $\int \cot x\,dx$	$-\log_e\lvert\cos x\rvert + C$
(ii) $\int \frac{1}{x}\,dx$	$\log_e\lvert x\rvert + C$	(xii) $\int \tan x\,dx$	$\log_e\lvert\sec x\rvert + C$
(iii) $\int e^x\,dx$	$e^x + C$	(xiii) $\int \sec x\,dx$	$\log_e\lvert\sec x + \tan x\rvert + C$
(iv) $\int a^x\,dx$	$\dfrac{a^x}{\log_e a} + C$	(xiv) $\int \operatorname{cosec} x\,dx$	$\log_e\lvert\operatorname{cosec} x - \cot x\rvert + C$
(v) $\int \sin x\,dx$	$-\cos x + C$	(xv) $\int \dfrac{dx}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\dfrac{x}{a} + C$
(vi) $\int \cos x\,dx$	$\sin x + C$	(xvi) $\int \dfrac{-dx}{\sqrt{a^2 - x^2}}$	$\cos^{-1}\dfrac{x}{a} + C$
(vii) $\int \sec^2 x\,dx$	$\tan x + C$	(xvii) $\int \dfrac{dx}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$
(viii) $\int \operatorname{cosec}^2 x\,dx$	$-\cot x + C$	(xviii) $\int \dfrac{-dx}{a^2 + x^2}$	$\dfrac{1}{a}\sec^{-1}\dfrac{x}{a} + C$

Integrands	Integrals	Integrands	Integrals
(ix) $\int \sec x \tan x \, dx$	$\sec x + C$	(xix) $\int \frac{dx}{\lvert x \rvert \sqrt{x^2 \quad a^2}}$	$\frac{1}{a} \sec^{-1} \frac{x}{a} + C$
(x) $\int \operatorname{cosec} x \cot x \, dx$	$-\operatorname{cosec} x + C$	(xx) $\int \frac{-dx}{\lvert x \rvert \sqrt{x^2 - a^2}}$	$\frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

3. Methods of Integration

(A) Integration by substitution method:

For integral $\int f'\{g(x)\}\, g'(x)\, dx,$ we create a new variable $t = g(x)$, so that $g'(x) = \frac{dt}{dx}$

★ Special Integrals in Substitution Method

(i) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log_e \left| \frac{a+x}{a-x} \right| + C$

(ii) $\int \frac{dx}{x^2 - a^2} dx = \frac{1}{2a} \log_e \left| \frac{x-a}{x+a} \right| + C$

(iii) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e \left| x + \sqrt{x^2 - a^2} \right| + C$

(iv) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log_e \left| x + \sqrt{x^2 + a^2} \right| + C$

Different forms:

FORM-I : $\int \frac{px+q}{ax^2+bx+c} dx$ OR $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$

$\Rightarrow$ Put $px + q = K_1 \frac{d}{dx}(ax^2 + bx + c) + K_2$

Now, find K_1 & K_2 and integrate it.

(B) Integration by parts method:

We use this method when there is a product of two functions.

$$\int \underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{g(x)}\, dx = f(x) \int g(x)\, dx - \int \left\{ \frac{d}{dx} f(x) \int g(x)\, dx \right\} dx$$

★ Order to follow: Inverse $\rightarrow$ Logarithm $\rightarrow$ Algebraic $\rightarrow$ Trigonometric $\rightarrow$ Exponential.

★ Some Special Integrals in By-parts Method

(i) $\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2 \log\left|x+\sqrt{x^2+a^2}\right|\right] + C$

(ii) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$

(iii) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2-a^2} - a^2 \log\left|x+\sqrt{x^2-a^2}\right|\right] + C$

Different forms:

FORM-I : $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

$\Rightarrow$ Put $px+q = K_1\left[\frac{d}{dx}(ax^2+bx+c)\right] + K_2$

Now, find K_1 & K_2 and then integrate it.

(C) Integration by partial fraction method:

If degree of numerator < degree of denominator then partial fraction method will be applicable. Decompose $\frac{f(x)}{g(x)}$ into partial fraction.

Ex. (i) $\frac{px+q}{(x-a)(x-b)^2(x-c)^3} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$

$+ \frac{D}{(x-c)} + \frac{E}{(x-c)^2} + \frac{F}{(x-c)^3}$

(ii) $\frac{px2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Now, find all values of constants (assumed) and then integrate it.

TIPS & TRICKS

TIPS AND TRICKS: (T-1)

Short trick to solve linear form integration.

$$\int\left(\frac{ax+b}{cx+d}\right)dx = \frac{ax}{c} - \frac{(ad-bc)}{c^2}\log_e|cx+d| + C$$

Illustration 1

Solve: $\int \frac{2x+7}{4x+5}\,dx$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\int \frac{2x+7}{4x+5}dx = \frac{2x}{4} - \frac{(10-28)}{16}\log_e|4x+5|+C$

$= \frac{x}{2} + \frac{9}{8}\log_e|4x+5|+C$

Illustration 2

Solve: $\int \frac{7x-15}{4x+11}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\int \frac{7x-15}{4x+11}dx = \frac{7x}{4} - \frac{(77+60)}{16}\log_e|4x+11|+C$

$= \frac{7x}{4} - \frac{137}{16}\log_e|4x+11|+C$

TIPS AND TRICKS: (T-2)

Short trick to solve integration of the form:

$$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{d}{dx}\frac{(ax^2+bx+c)}{\sqrt{4ac-b^2}}\right)+C$$

where, $4ac - b^2 > 0$

Illustration 3

Solve: $\int \frac{dx}{5x^2+6x+5}$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because\ 4ac - b^2 = 100 - 36 = 64 > 0$

$$\Rightarrow \int \frac{dx}{5x^2+6x+5} = \frac{2}{\sqrt{64}}\tan^{-1}\left(\frac{10x+6}{\sqrt{64}}\right)+C = \frac{1}{4}\tan^{-1}\left(\frac{5x+3}{4}\right)+C$$

Illustration 4

Solve: $\int \frac{dx}{x^2+4x+7}$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because\ 4ac-b^2=28-16=12>0$

$$\Rightarrow \int \frac{dx}{x^2+4x+7}=\frac{2}{\sqrt{12}}\tan^{-1}\left(\frac{2x+4}{\sqrt{12}}\right)+C=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+C$$

TIPS AND TRICKS: (T-3)

Short trick to solve integration of the form:

$\int x^n f(x)\,dx$; where $f(x)$ is **trigonometric or exponential function**.

Tabular Method : (Process as follows)

Ex: $\mathrm{I}=\int \underset{\mathrm{II}}{(\sin 2x)}\ \underset{\mathrm{I}}{x^2}\,dx$

Derivative of I		**Integral of II**	
x^2		$\sin 2x$	
	$\xrightarrow{+}$	$-\frac{1}{2}\cos 2x$	Multiply the terms as arrow indicates sign changes alternatively
$2x$			
	$\xrightarrow{-}$	$-\frac{1}{4}\sin 2x$	
2			
	$\xrightarrow{+}$	$+\frac{1}{8}\cos 2x$	
0			

$$\Rightarrow\ \mathrm{I}=-\frac{x^2}{2}\cos 2x+\frac{2x}{4}\sin 2x+\frac{2}{8}\cos 2x$$

Illustration 5

Solve: $\int \underset{\mathrm{I}}{x^4}\ \underset{\mathrm{II}}{e^x}\,dx$

Short-cut solution :

$\boxed{\text{Using T-3}}$

Differentiation		Integration
x^4	$+$	e^x
$4x^3$	$-$	e^x
$12x^2$	$+$	e^x
$24x$	$-$	e^x
24	$+$	e^x
0		e^x

$\Rightarrow\ x^4e^x - 4x^3e^x + 12x^2e^x - 24x\, e^x + 24e^x$

TIPS AND TRICKS: (T-4)

Short trick to solve integration of the form:

$$\int \frac{dx}{a+b\cos^2 x} = \frac{1}{a}\left\{\sqrt{\frac{a}{a+b}}\tan^{-1}\left(\sqrt{\frac{a}{a+b}}\tan x\right)\right\} + C$$

★ **Note:** In case of '$\sin^2 x$' use $\sin^2 x = 1 - \cos^2 x$

Illustration 6

$\int \frac{dx}{3+2\cos^2 x}$, solve the integration

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\because\ a = 3, b = 2$

$$\Rightarrow \int \frac{1}{a+b\cos^2 x}dx = \frac{1}{3}\left\{\sqrt{\frac{3}{5}}\tan^{-1}\left(\sqrt{\frac{3}{5}}\tan x\right)\right\} + C$$

Illustration 7

$\int \frac{dx}{6+4\sin^2 x}$, solve the integration

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\because \int \frac{dx}{10-4\cos^2 x}$

Here, $a = 10, b = -4$

$$\Rightarrow \int \frac{dx}{6+4\sin^2 x} = \int \frac{dx}{10-4\cos^2 x} = \frac{1}{10}\left\{\sqrt{\frac{10}{6}}\tan^{-1}\left(\sqrt{\frac{10}{6}}\tan x\right)\right\} + C$$

TIPS AND TRICKS: (T-5)

Short trick to solve integration of the form:

$$\int \frac{ae^{kx}+b}{ce^{kx}+d}dx = \frac{bx}{d} + \frac{1}{k}\frac{(ad-bc)}{cd}\log_e|ce^{kx}+d| + C$$

Illustration 8

Solve: $\int \frac{3e^x+5}{2e^x+7}dx$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\int \frac{3e^x+5}{2e^x+7}dx = \frac{5x}{7} + \frac{(21-10)}{14}\log_e|2e^x+7| + C$

Illustration 9

Solve: $\int \frac{7e^{5x}+10}{-8e^{5x}+3}dx$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\int \frac{7e^{5x}+10}{-8e^{5x}+3}dx = \frac{10x}{3} + \frac{1}{5}\left(\frac{101}{-24}\right)\log_e|3-8e^{5x}| + C$

TIPS AND TRICKS: (T-6)

Short trick of the form:

(i) $\int e^{mx}\sin(nx)\,dx = \frac{e^{mx}}{m^2+n^2}\big(m\sin(nx) - n\cos(nx)\big) + C$

(ii) $\int e^{mx}\cos(nx)\,dx = \frac{e^{mx}}{m^2+n^2}\big(m\cos(nx) + n\sin(nx)\big) + C$

Illustration 10

Solve: $I = \int e^{3x} \cdot \sin 2x \, dx$

Short-cut solution :

$\boxed{\text{Using T-6}}$ Here, $m = 3, n = 2$

$$\Rightarrow \int e^{mx} \sin(nx)\, dx = \frac{e^{3x}}{9+4}(3 \sin 2x - 2 \cos 2x) + C$$

Illustration 11

Solve: $I = \int e^{-5x} \cdot \cos 7x \, dx$

Short-cut solution :

$\boxed{\text{Using T-6}}$ Here, $m = -5, n = 7$

$$\Rightarrow \int e^{mx} \cos(nx)\, dx = \frac{e^{-5}}{49+25}\left(-5 \cos (7x) + 7 \sin (7x)\right) + C$$

TIPS AND TRICKS: (T-7)

Short trick to solve integration in the partial fraction form:

★ This trick is valid only when denominator can be factorized into the linear form.

Ex. $\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$

Step 1. Put $x - 1 = 0 \Rightarrow x = 1$ and substitute $x = 1$ in rest of the factors, multiplied by $\log_e$ (factor which has avoided).

Step 2. Similarly repeat the process for other factors which are in denominator i.e. $x = -2, x = 3$ and add all the parts.

$$\Rightarrow \frac{2(1)-1}{(1+2)(1-3)} \ln|x-1| + \frac{(2(-2)-1)}{(-2-1)(-2-3)} \ln|x+2| + \frac{2(3)-1}{(3-1)(3+2)} \ln|x-3| + C$$

Illustration 12

Solve: $\int \frac{x^2+4}{x^3-3x^2+2x}\,dx$

Short-cut solution :

$\boxed{\text{Using T-7}}$ $\because \int \frac{x^2+4}{x(x-1)(x-2)}$

(Put $x = 0, x = 1, x = 2$ systematically as discussed earlier)

$$\Rightarrow \int \frac{x^2+4}{x^3-3x^2+2x}\,dx = \int \frac{x^2+4}{x(x-1)(x-2)}\,dx = \frac{0+4}{(0-1)(0-2)}$$

$$\ln|x| + \frac{1+4}{1(1-2)}\ln|x-1| + \frac{4+4}{2(2-1)}\ln|x-2| + C$$

$$\Rightarrow\ 2\ln|x| - 5\ln|x-1| + 4\ln|x-2| + C$$

TIPS AND TRICKS: (T-8)

Short trick to solve integration of the form:

(i) $\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\log_e\left|\frac{x^n}{x^n+1}\right| + C;\ n \in N$

(ii) $\int \frac{dx}{x(x^n-1)} = \log_e\left|\frac{x^n-1}{x^n}\right| + C;\ n \in N$

Illustration 13

Solve: $\int \frac{dx}{x(x^6+1)}$

Short-cut solution :

$\boxed{\text{Using T-8 (i)}}$ $\because\ n = 6$

$$\Rightarrow \int \frac{dx}{x\left(x^6+1\right)} = \frac{1}{6}\ln\left|\frac{x^6}{x^6+1}\right| + C$$

Illustration 14

Solve: $\int \frac{dx}{x(x^4-1)}$

Short-cut solution :

$\boxed{\text{Using T-8 (ii)}}$ $\because\ n = 4$

$$\Rightarrow \int \frac{dx}{x\left(x^4-1\right)} = \ln\left|\frac{x^4-1}{x^4}\right| + C$$

TIPS AND TRICKS: (T-9)

Short trick to solve integration of the form:

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\,dx = \left(\frac{ac+bd}{c^2+d^2}\right)x + \left(\frac{ad-bc}{c^2+d^2}\right)\ln\text{(Denominator)} + C$$

Illustration 15

Solve: $\int \frac{2\cos x + \sin x}{4\cos x + 3\sin x}\,dx$

Short-cut solution :

$\boxed{\text{Using T-9}}$ Here, $a = 2, b = 1, c = 4, d = 3$

$$\Rightarrow \int \frac{2\cos x + \sin x}{4\cos x + 3\sin x}\,dx = \left(\frac{8+3}{25}\right)x + \left(\frac{6-4}{25}\right)\ln|4\cos x + 3\sin x| + C$$

SHORTCUTS: (SC-1)

Reverse process of integration i.e. differentiation. If we have to find the unknowns then differentiate the given integration both sides and reach the answer.

Illustration 16

If $\int \frac{5\tan x}{\tan x - 2}\,dx = x + a\ln|\sin x - 2\cos x| + K$ **[AIEEE 2012]**

Then 'a' is equal to

(a) 1 (b) 2 (c) -1 (d) -2

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Taking $\frac{d}{dx}$ of both sides

$$\Rightarrow \frac{5\tan x}{\tan x-2}=1+\frac{a(\cos x+2\sin x)}{\sin x-2\cos x}$$

$$\Rightarrow \frac{5\tan x}{\tan x-2}-1=\frac{a(\cos x+2\sin x)}{\sin x-2\cos x}$$

(Divide N^r and D^r by $\cos x$ in RHS)

$$\Rightarrow \frac{4\tan x+2}{\tan x-2}=\frac{a+2a\tan x}{\tan x-2}$$

On comparing both sides $\Rightarrow a=2$ **Ans. (b)**

Illustration 17

If $I_n=\int \tan^n x\,dx\ (n>1)$ and $I_4+I_6=a\tan^5 x+bx^5+C$; then find a, b. **[JEE M 2017]**

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $\because \int \tan^4 x\,dx+\int \tan^6 x\,dx=a\tan^5 x+bx^5+C$

On differentiating both sides, we get

$\Rightarrow \tan^4 x+\tan^6 x=5a\tan^4 x\cdot\sec^2 x+5bx^4$

$\Rightarrow \tan^4 x+\tan^6 x=5a\tan^4 x+5a\cdot\tan^6 x+5bx^4$

On comparing both sides $\Rightarrow a=\frac{1}{5}, b=0$

SHORTCUTS: (SC-2)

Integrals of the form $\int \sin^n x\cos^m x\,dx$

Case-I: If n is odd, put $\cos x=t$

Case-II: If m is odd, put $\sin x=t$

Case-III: If m and n both odd, put $\sin x$ or $\cos x=t$

Case-IV: If $m+n$ is negative even integer, put $\tan x=t$.

Illustration 18

Solve: $\int \sin^{99} x \cos^3 x \, dx$

Short-cut solution :

$\boxed{\text{Using SC-2}}$ $m = 3$ and $n = 99 \Rightarrow$ Case -III

Hence, put $\sin x = t \Rightarrow \dfrac{dt}{dx} = \cos x$

$$\Rightarrow \int t^{99}(1-t^2)\, dt = \frac{t^{100}}{100} - \frac{t^{102}}{102}$$

$$\Rightarrow \frac{(\sin x)^{100}}{100} - \frac{(\sin x)^{102}}{102} + C$$

Illustration 19

Solve: $\displaystyle\int \frac{dx}{\sin^{1/2} x \cos^{7/2} x}$

Short-cut solution :

$\boxed{\text{Using SC-2}}$ $\because m + n = -4 \Rightarrow$ Case-IV

Hence, convert in terms of $\tan x$.

$$\Rightarrow \int \frac{\sec^4 x}{\sqrt{\dfrac{\sin x}{\cos x}}} dx = \int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}} dx$$

Now, put $\tan x = t \Rightarrow \dfrac{dt}{dx} = \sec^2 x$

$$\Rightarrow \int \frac{1+t^2}{\sqrt{t}} dt = 2\sqrt{t} + \frac{2}{5}(t)^{5/2} = 2\sqrt{\tan x} + \frac{2}{5}(\tan x)^{5/2} + C$$

SHORTCUTS: (SC-3)

Integrals of the form:

(i) $\int e^x (f(x) + f'(x))\, dx = e^x f(x) + C$

(ii) $\int (x f'(x) + f(x))\, dx = x f(x) + C$

Illustration 20

Solve: $\int \frac{x+\sin x}{1+\cos x}\,dx$

Short-cut solution :

$\boxed{\text{Using SC-3 (ii)}}$

$$\int\left(\frac{x}{1+2\cos^2\frac{x}{2}-1}+\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}\right)dx$$

$$\Rightarrow \int\left(\underbrace{x\cdot\frac{1}{2}\sec^2\frac{x}{2}}_{f'(x)}+\underbrace{\tan\frac{x}{2}}_{f(x)}\right)dx \;\Rightarrow\; x\tan\frac{x}{2}+C$$

Illustration 21

Solve: $\int \frac{e^x(x^2+5x+7)}{(x+3)^2}\,dx$

Short-cut solution :

$\boxed{\text{Using SC-3 (i)}}$ $\int e^x\left\{\underbrace{\left(\frac{x+2}{x+3}\right)}_{f(x)}+\underbrace{\frac{1}{(x+3)^2}}_{f'(x)}\right\}dx \;\Rightarrow\; e^x\left(\frac{x+2}{x+3}\right)+C$

TECHNIQUE: (Tech.1)

Algebraic Twins : To find the integral of the form

$\int\frac{x^2+a^2}{x^4+\lambda x^2+a^4}dx$ and $\int\frac{x^2-a^2}{x^4+\lambda x^2+a^4}dx$

To evaluate the integral of these above forms, first we divide both numerator and denominator by x^2 and then express the denominator in the form $\left(x-\frac{a^2}{x}\right)^2\pm k^2$ and $\left(x+\frac{a^2}{x}\right)^2\pm k^2$ respectively. Then put $x-\frac{a^2}{x}=t$ and $x+\frac{a^2}{x}=t$ respectively.

Illustration 22

Evaluate $\int \frac{x^2+1}{x^4+x^2+1}\,dx$

Short-cut solution :

Using Tech.

$$I=\int \frac{x^2+1}{x^4+x^2+1}\,dx=\int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx=\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3}$$

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx=dt$

$$\therefore I=\int \frac{dt}{t^2+3}=\int \frac{dt}{t^2+(\sqrt{3})^2}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}+C$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C$$

TECHNIQUE: (Tech. 2)

To find the integral of the form $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P and Q both are pure quadratic expression in x. Such that $P = ax^2 + b$, $Q = cx^2 + d$.

Then, put $x=\frac{1}{t}$ and $c + dt^2 = u^2$.

Illustration 23:

Evaluate $\int \frac{1}{\left(1-x^2\right)\sqrt{1+x^2}}\,dx$

Short-cut solution :

Using Tech.

Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$I = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\left(1-\frac{1}{t^2}\right)\sqrt{1+\frac{1}{t^2}}} = -\int \frac{t\,dt}{\left(t^2-1\right)\sqrt{t^2+1}}$$

Let $t^2+1=u^2$, we get $2t\,dt = 2u\,du$

$$I = -\int \frac{du}{u^2-(\sqrt{2})^2} = \frac{-1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|+c$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}}\right|+c = -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}\right|+c$$

$$= -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x}\right|+c$$

Concept Booster Exercise

1. $\int \frac{x-1}{x+1} dx$ is equal to

(a) $x - \log_e |x+1| + C$
(b) $x - 2\log_e |x+1| + C$
(c) $-2\log_e |x+1| + C$
(d) $-\log_e |x+1| + C$

2. $\int \frac{dx}{(x+1)(x+2)(x+3)}$ is equal to

(a) $\frac{1}{2}\ln|(x+1)(x+3)| - \ln|x+2| + C$

(b) $\frac{1}{2}\ln|x+1| + C$

(c) $\frac{1}{2}\ln|x+2| - \ln|x+1| + C$

(d) $\frac{1}{2}\ln|x+3| - \ln|x+2| + C$

3. $\int \frac{dx}{2+\sin^2 x}$ is equal to

(a) $\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$
(b) $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$
(c) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{2}}\right) + C$
(d) None of these

4. $\int 2x^3 e^{-x} dx$ is equal to

(a) $e^x(-x^3 - 3x^2 - 6x - 6)$
(b) $e^{-x}(x^3 + 3x^2 + 6x + 6)$
(c) $e^{-x}(-x^3 - 3x^2 - 6x - 6)$
(d) None of these

5. $\int \frac{10 \cdot e^{2x} + 5}{11 \cdot e^{2x} - 2} dx$ is equal to

(a) $\frac{-5x}{2} + \frac{75}{44}\log_e|11 \cdot e^{2x} - 2| + C$

(b) $\frac{75}{44}\log_e|11 \cdot e^{2x} - 2| + C$

(c) $\dfrac{-5x}{2}+\dfrac{2}{9}\log_e|10\cdot e^{2x}+5|+C$

(d) $\dfrac{2}{9}\log_e|10\cdot e^{2x}+5|+C$

6. $\int e^x\cdot\sin 3x\,dx$ is equal to

(a) $\dfrac{e^x}{9}(\sin 3x-3\cos 3x)+C$ (b) $\dfrac{e^x}{10}(\sin 3x-3\cos 3x)+C$

(c) $\dfrac{e^x}{8}(\sin 3x-3\cos 3x)+C$ (d) $\dfrac{e^x}{10}(\sin 3x+3\cos 3x)+C$

7. $\int\dfrac{dx}{x^2+3x+4}$ is equal to

(a) $2\tan^{-1}\left(\dfrac{2x+3}{\sqrt{7}}\right)+C$ (b) $\dfrac{1}{\sqrt{7}}\tan^{-1}\left(\dfrac{2x+3}{\sqrt{7}}\right)+C$

(c) $\dfrac{2}{\sqrt{7}}\tan^{-1}\left(\dfrac{2x-3}{\sqrt{7}}\right)+C$ (d) $\dfrac{2}{\sqrt{7}}\tan^{-1}\left(\dfrac{2x+3}{\sqrt{7}}\right)+C$

8. $\int\dfrac{1}{x(1+x^7)}dx$ is equal to

(a) $\log_e\left|\dfrac{x^7}{1+x^7}\right|+C$ (b) $\dfrac{1}{7}\log_e\left|\dfrac{x^7}{1+x^7}\right|+C$

(c) $-\dfrac{1}{7}\log_e\left|\dfrac{x^7}{1+x^7}\right|+C$ (d) $7\log_e\left|\dfrac{x^7}{1+x^7}\right|+C$

9. $\int\dfrac{\sin x}{\sin x-\cos x}dx$ is equal to

(a) $\dfrac{1}{2}\log|\sin x+\cos x|+C$ (b) $\log|\sin x-\cos x|+C$

(c) $\dfrac{1}{2}\log|\sin x-\cos x|+\dfrac{x}{2}+C$ (d) $\dfrac{1}{2}\log|\sin x+\cos x|+\dfrac{x}{2}+C$

10. If $\int\dfrac{\sin x}{\sin(x-\alpha)}dx=Ax+B\log\cdot\sin(x-\alpha)+C,$ then value of (A, B) is

[AIEEE 2004]

(a) $(\sin\alpha, \cos\alpha)$ (b) $(\cos\alpha, \sin\alpha)$

(c) $(-\sin\alpha, \cos\alpha)$ (d) $(-\cos\alpha, \sin\alpha)$

11. If $f\left(\frac{x-4}{x+2}\right) = 2x+1; x \in R - \{1, -2\}$, then $\int f(x)\, dx$ is equal to

[JEE M 2018]

(a) $12 \ln|1-x| - 3x + C$ (b) $-12 \ln|1-x| - 3x + C$
(c) $-12 \ln|1-x| + 3x + C$ (d) $12 \ln|1-x| + 3x + C$

12. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx = Ax + B \log(9e^{2x} - 4) + C$, then, '$A$' and '$B$' are

(a) $\frac{-3}{2}, \frac{35}{36}$ (b) $\frac{-2}{3}, \frac{35}{36}$
(c) $\frac{-3}{2}, \frac{36}{35}$ (d) $\frac{-2}{3}, \frac{36}{35}$

13. $\int \sin^5 x \cos^2 x\, dx$ is equal to

(a) $-\frac{\cos^7 x}{7} - \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3}$ (b) $-\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3}$
(c) $\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} + \frac{\cos^3 x}{3}$ (d) None of these

14. $\int \frac{(x-1)\, e^x}{(x+1)^3}\, dx$ is equal to

(a) $\frac{e^x}{(x-1)^2} + C$ (b) $\frac{e^x}{(x+1)^2} + C$
(c) $\frac{e^x}{x-1} + C$ (d) $\frac{e^x}{x+1} + C$

Numerical Value Problems

15. Let $f(x) = \int e^x (x-1)(x-2)\, dx$, then '$f$' decreases in the interval (a, b), then $a + b$ is ________ . **[JEE M 2020]**

16. If $\int e^{3x} \cos 4x\, dx = e^{3x}(A \sin 4x + B \cos 4x) + C$, then $4A + 3B$ is equal to ________ .

17. If $\int \frac{e^x + (1+x^2)\, e^x \tan^{-1} x}{1+x^2}\, dx = A \cdot e^x \tan^{-1} x + C$, then '$A$' is equal to ________ .

18. If $\int \frac{2x+5}{\sqrt{7-6x-x^2}}\, dx = A\sqrt{7-6x-x^2} + B \sin^{-1}\left(\frac{x+3}{4}\right) + C$ their ordered pair is (A, B), then $|A + B|$ is ________ .

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ $\because\ a=1, b=-1, c=1, d=1$

$$\Rightarrow\ x-\frac{2}{1}\log_e|x+1|+C$$

2. **(a)** $\boxed{\text{Using T-7}}$

Putting $x = -1$

$$\Rightarrow\ \frac{1}{2}\ln|x+1|$$

Putting $x = -2$

$$\Rightarrow\ -\ln|x+2|$$

Putting $x = -3$

$$\Rightarrow\ \frac{1}{2}\ln|x+3|$$

Hence, $\displaystyle\int\frac{1}{(x+1)(x+2)(x+3)}dx$

$$=\frac{1}{2}\ln(x+1)(x+3)-\ln(x+2)+C$$

3. **(c)** $\boxed{\text{Using T-4}}$ $\displaystyle\int\frac{dx}{3-\cos^2 x}$ $(\because\ \sin^2 x = 1-\cos^2 x)$

$$\Rightarrow\ a=3, b=-1\ \Rightarrow\int\frac{dx}{3-\cos^2 x}=\frac{1}{3}\left\{\sqrt{\frac{3}{2}}\tan^{-1}\left(\sqrt{\frac{3}{2}}\tan x\right)\right\}+C$$

4. **(c)** $\boxed{\text{Using T-3}}$

$$\begin{array}{lcl} x^3 & \xrightarrow{+} & e^{-x} \\ 3x^2 & \xrightarrow{-} & -e^{-x} \\ 6x & \xrightarrow{+} & e^{-x} \\ 6 & \xrightarrow{-} & -e^{-x} \\ 0 & & e^{-x} \end{array}$$

$$\Rightarrow\ \int 2x^3 e^{-x}\,dx = -x^3 e^{-x}-3x^2\cdot e^{-x}-6x\,e^{-x}-6e^{-x}$$

$$\Rightarrow\ e^{-x}(-x^3-3x^2-6x-6)$$

5. **(a)** $\boxed{\text{Using T-5}}$ $\because\ a=10, b=5, c=11, d=-2, k=2$

$$\Rightarrow\int\frac{10\cdot e^{2x}+5}{11\cdot e^{2x}-2}dx=\frac{5x}{-2}+\frac{1}{2}\left(\frac{-75}{-22}\right)\log_e|11\cdot e^{2x}-2|+C$$

6. **(b)** $\boxed{\text{Using T-6 (i)}}$ $m = 1, n = 3$

$$\Rightarrow \int e^x \sin(3x)\,dx = \frac{e^x}{1+9}(\sin(3x) - 3\cos(3x)) + C$$

7. **(d)** $\boxed{\text{Using T-2}}$ $\because\ 4ac - b^2 = 16 - 9 = 7 > 0$

$$\Rightarrow \int \frac{dx}{x^2+3x+4} = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+3}{\sqrt{7}}\right) + C$$

8. **(b)** $\boxed{\text{Using T-8 (i)}}$ $\because\ n = 7$

$$\Rightarrow \int \frac{1}{x\left(1+x^7\right)}dx = \frac{1}{7}\log_e\left|\frac{x^7}{1+x^7}\right| + C$$

9. **(c)** $\boxed{\text{Using T-9}}$ Here, $\int \frac{0\cdot\cos x + 1\cdot\sin x}{-\cos x + 1\cdot\sin x}dx$

$\because\ a = 0, b = 1, c = -1, d = 1$

$$\Rightarrow \int \frac{\sin x}{\sin x - \cos x}dx = \left(\frac{1}{1+1}\right)x + \left(\frac{1}{2}\right)\ln|\sin x - \cos x| + C$$

10. **(b)** $\boxed{\text{Using SC-1}}$ $\int \frac{\sin x}{\sin(x-\alpha)}dx = Ax + B\log\sin(x-\alpha) + C$

Differentiating both sides

$$\Rightarrow \frac{\sin x}{\sin(x-\alpha)} = A + \frac{B\cdot\cos(x-\alpha)}{\sin(x-\alpha)}$$

$$\frac{\sin x}{\sin(x-\alpha)} = \frac{\sin x\,[A\cos\alpha + B\sin\alpha] + \cos x\,[B\cos\alpha - A\sin\alpha]}{\sin(x-\alpha)}$$

On comparing both sides $\Rightarrow A = \cos\alpha, B = \sin\alpha$

11. **(b)** $\boxed{\text{Using T-1}}$ Put $\frac{x-4}{x+2} = t \Rightarrow x = \frac{2t+4}{1-t}$

$$\Rightarrow f(t) = 2\left[\frac{2t+4}{1-t}\right] + 1 = \frac{3(t)+9}{1-t} \Rightarrow \int \frac{3x+9}{1-x}dx$$

$a = 3, b = 9, c = -1, d = 1$

$$\Rightarrow \frac{3x}{-1} - \left(\frac{3+9}{1}\right)\log_e|1-x| + C$$

12. **(a)** $\boxed{\text{Using T-5}}$ Here, $a = 4, b = 6, c = 9, d = -4$

Multiply numerator and denominator by $e^x \Rightarrow k = 2$

Now, $\int \frac{4e^{2x}+6}{9e^{2x}-4}\,dx = \frac{6x}{-4} + \frac{1}{2}\left(\frac{-16-54}{-36}\right)\log_e|9e^{2x}-4| + C$

$\Rightarrow\ A = \frac{-3}{2},\ B = \frac{35}{36}$

13. **(b)** $\boxed{\text{Using SC-2}}$ $n = 5, m = 2 \Rightarrow$ 'n' is odd

Hence, using case-I, put $\cos x = t \Rightarrow dx = -\frac{1}{\sin x}\frac{dt}{dx}$

$$\Rightarrow \int \sin^5 x \cdot \frac{t^2}{-\sin x}\,dt = -\int t^2(1-t^2)^2\,dt$$

$$= -\int (t^6 - 2t^4 + t^2)\,dt$$

$$= -\frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} - \frac{\cos^3 x}{3}$$

14. **(b)** $\boxed{\text{Using SC-3}}$

$$\int e^x\left\{\underbrace{\frac{x+1}{(x+1)^3}}_{f(x)} + \underbrace{\frac{-2}{(x+1)^3}}_{f'(x)}\right\}dx = \frac{e^x}{(x+1)^2} + C$$

15. **(3)** Since $f(x)$ is decreasing $\Rightarrow f'(x) < 0$

$\Rightarrow\ e^x\,(x-1)(x-2) < 0$

$\Rightarrow\ x \in (1, 2)$

Hence, $a + b = 3$

16. **(1)** $\boxed{\text{Using T-6 (ii)}}$ Here, $m = 3, n = 4$

$$\Rightarrow \int e^{3x}\cos(4x)\,dx = \frac{e^{3x}}{9+16}\left(3\cos 4x + 4\sin 4x\right) + C$$

$$\Rightarrow \frac{e^{3x}}{25}\{4\sin 4x + 3\cos 4x\} + C$$

$$\Rightarrow\ A = \frac{4}{25} \text{ and } B = \frac{3}{25}\ \Rightarrow\ 4A + 3B = 1$$

17. **(1)** $\boxed{\text{Using SC-3}}$ $\because \int e^x \left\{\tan^{-1} x + \frac{1}{1+x^2}\right\} dx$

As we know that $\int e^x \{(f(x)) + f'(x)\}\, dx = e^x f(x) + C$

$\Rightarrow e^x \tan^{-1} x + C$

$\Rightarrow A = 1$

18. **(4)** $\boxed{\text{Using SC-1}}$ $\int \frac{2x+5}{\sqrt{7-6x-x^2}}\, dx = A\sqrt{7-6x-x^2} + B\sin^{-1}\left(\frac{x+3}{4}\right) + C$

Differentiating both sides

$$\Rightarrow \frac{2x+5}{\sqrt{7-6x-x^2}} = \frac{A(-2x-6)}{2\sqrt{7-6x-x^2}} + \frac{B}{\sqrt{1-\left(\frac{x+3}{4}\right)^2}} \times \frac{1}{4}$$

$$\frac{2x+5}{\sqrt{7-6x-x^2}} = \frac{A(-x-3)}{\sqrt{7-6x-x^2}} + \frac{B/2}{\sqrt{7-6x-x^2}}$$

On comparing both sides $\Rightarrow A = -2, B = -2 \Rightarrow |A+B| = 4$

9 Definite Integration

Review of Key Notes and Formulae

1. **Definition:** Let $f(x)$ be a function defined on an interval $[a, b]$ and F (x) be its anti-derivative. Then $\int_a^b f(x)\,dx$ = F (b) – F (a) is defined as the definite integral of $f(x)$ from $x = a$ to $x = b$.

2. **Properties of Definite Integral**

P-1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P-2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P-3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, a < c < b$

P-4 $\int_{-a}^a f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$

Very Important

P-5 **King Property:** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P-6 **Queen Property:** $\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a-x)\}\,dx$

P-7 **Jack Property:** For a periodic function "$f(x)$"

(i) $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx\,; \quad n \in I$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx\,; \quad n \in I$

(iii) $\int_{a+mT}^{a+nT} f(x)\,dx = \int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx; \quad m, n \in I$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx$

P-8 If f is continuous on $[a, b]$, then there exists a number 'c' in $[a, b]$ at which

$$\underset{\downarrow}{f(c)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean value of the function $f(x)$ on the interval $[a, b]$

3. Leibnitz Rule for Differentiation under Integral Sign

If $g(x)$ and $h(x)$ are defined on $[a, b]$ and differentiable at point $x \in (a, b)$ and $f(t)$ is continuous, then

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

4. Summation of Series by Definite Integral

Method: Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and

Limit of the sum by integral,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=r_1}^{rn} f\left(\frac{r}{n}\right)$$

where, Lower limit $= \lim_{n\to\infty}\frac{r_1}{n} = a$ Upper limit $= \lim_{n\to\infty}\frac{r_n}{n} = b$.

5. Some Important Results to Remember

(i) $\int_0^{\pi/2}\sin x\,dx = \int_0^{\pi/2}\cos x\,dx = 1$

(ii) $\int_0^{\pi/2}\sin^2 x\,dx = \int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4}$

(iii) $\int_0^{\pi/2}\sin^3 x\,dx = \int_0^{\pi/2}\cos^3 x\,dx = \frac{2}{3}$

(iv) $\int_0^{\pi/2}\sin^4 x\,dx = \int_0^{\pi/2}\cos^4 x\,dx = \frac{3\pi}{16}$

(v) $\int_0^{\pi/2}\ln\sin x\,dx = \int_0^{\pi/2}\ln\cos x\,dx = \frac{-\pi}{2}\ln 2$

(vi) $\int_0^{\pi/2}\ln\tan x\,dx = \int_0^{\pi/2}\ln\cot x\,dx = 0$

(vii) $\int_0^{\pi/2}\ln\sec x\,dx = \int_0^{\pi/2}\ln\operatorname{cosec} x\,dx = \frac{\pi}{2}\ln 2$

TIPS AND TRICKS: (T-1)

Short trick to solve integration of the form:

$$\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$$

Illustration 1

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx \text{ is equal to}$$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because$ $\cot\left(\frac{\pi}{2}-x\right)=\tan x$, Here, $a=0$ and $b=\frac{\pi}{2}$

$$\Rightarrow \frac{\frac{\pi}{2}-0}{2}=\frac{\pi}{4}$$

Illustration 2

$$\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx \text{ is equal to}$$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because$ $\cos(\pi-x)=-\cos x$, Here, $a=0$ and $b=\pi$

$$\Rightarrow \frac{\pi-0}{2}=\frac{\pi}{2}$$

TIPS AND TRICKS: (T-2)

Short trick to solve integration of the form:

$$\int_0^{\pi} \sin(ax)\cos(bx)\,dx = \begin{cases} 0 & ;\text{if } a-b \text{ is even} \\ \dfrac{2a}{a^2-b^2} & ;\text{if } a-b \text{ is odd} \end{cases}$$

Illustration 3

$$\int_0^{\pi} \sin(4x)\cos(10x)\,dx \text{ is equal to}$$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because$ $a = 4$ and $b = 10$ $\Rightarrow$ $a - b$ is even

Hence, answer = 0.

Illustration 4

$\int_0^{\pi} \sin(60x)\cos(41x)\,dx$ is equal to

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because$ $a = 60,\ b = 41$ $\Rightarrow$ $a - b$ is odd

Hence, answer $= \dfrac{2 \times 60}{3600 - 1681} = \dfrac{120}{1919}$

TIPS AND TRICKS: (T-3)

Short trick to solve integration of the form:

$$\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$$

Illustration 5

$\int_0^{\pi/2} \dfrac{dx}{25\cos^2 x + 16\sin^2 x}$, is equal to

Short-cut solution :

$\boxed{\text{Using T-3}}$ $a = 5$ and $b = 4$

Hence, $\dfrac{\pi}{2ab} = \dfrac{\pi}{2 \times 5 \times 4} = \dfrac{\pi}{40}$

Illustration 6

$\int_0^{\pi/2} \dfrac{dx}{10\sin^2 x + 9\cos^2 x}$, is equal to

Short-cut solution :

$\boxed{\text{Using T-3}}$ $a = 3$ and $b = \sqrt{10}$

Hence, $\dfrac{\pi}{2ab} = \dfrac{\pi}{2 \times 3 \times \sqrt{10}} = \dfrac{\pi}{6\sqrt{10}}$

TIPS AND TRICKS: (T-4)

Short trick to solve integration of the form:

(i) $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi \quad ; \quad b > a$

(ii) $\int_a^b \sqrt{(x-a)(b-x)}\, dx = \frac{\pi}{8}(b-a)^2 \; ; b > a$

Illustration 7

$$\int_2^7 \frac{dx}{\sqrt{(x-2)(7-x)}}, \text{ is equal to}$$

Short-cut solution :

Using T-4 (i) $\because \quad a = 2, \quad b = 7 \quad \text{and} \quad b > a \Rightarrow \pi$

Illustration 8

$$\int_{-5}^{9} \sqrt{(x+5)(9-x)}\, dx, \text{ is equal to}$$

Short-cut solution :

Using T-4 (ii) $\because \quad a = -5, \quad b = 9 \quad \text{and} \quad b > a$

$$\Rightarrow \quad \frac{\pi}{8}(b-a)^2 = \frac{\pi}{8}(9+5)^2 = \frac{\pi}{8} \times 196 = \frac{49\pi}{2}$$

TIPS AND TRICKS: (T-5)

Short trick to solve integration of the form:

$$\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1} \frac{m!\,n!}{(m+n+1)!}$$

Illustration 9

$$\int_1^2 (x-1)^2 (2-x)^3 dx, \text{ is equal to}$$

Short-cut solution :

Using T-5 $\because \quad a = 1, \quad b = 2, \quad m = 2, \quad n = 3$

$$\Rightarrow \quad (b-a)^{m+n+1} \frac{m!\,n!}{(m+n+1)!} = \frac{(1)^6\, 2!\,3!}{6!} = 60$$

TIPS & TRICKS

TIPS AND TRICKS: (T-6)

Short trick to solve integration of the form:

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x\, dx = \left\{\frac{((m-1)(m-3)\ldots 2 \text{ or } 1)((n-1)(n-3)\ldots 2 \text{ or } 1)}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1} \times K\right\}$$

where, $K = \begin{cases} \pi/2; \text{when } m \text{ and } n, \text{ both are even} \\ 1 \quad ; \text{otherwise} \end{cases}$

Illustration 10

$\int_0^{\pi/2} \sin^4 x\, dx$, is equal to

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\because$ $m = 4$, $n = 0$ $\Rightarrow$ both even

Hence, $\dfrac{(4-1)(4-3)}{4(4-2)} \times \dfrac{\pi}{2} = \dfrac{3\pi}{16}$

Illustration 11

$\int_0^{\pi/2} \sin^3 x \cos^2 x\, dx$, is equal to

Short-cut solution :

$\boxed{\text{Using T-6}}$ $\because$ $m = 3$ $n = 2$ $\Rightarrow$ otherwise

Hence, $\dfrac{(3-1)(2-1)}{5(5-2)(5-4)} = \dfrac{2}{15}$

SHORTCUTS: (SC-1)

If $\int_a^b f(x)\, dx = 0$, then the equation $f(x) = 0$ has atleast one root in (a, b),

provided $f(x)$ is continuous in this interval.

Illustration 12

If $2a + 3b + 6c = 0$, then prove that the equation $ax^2 + bx + c = 0$ has a root in $(0, 1)$.

Short-cut solution :

Using SC-1 Let $f(x) = ax^2 + bx + c$

$$\Rightarrow \int_0^1 (ax^2 + bx + c)\,dx = \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_0^1$$

$$0 = \frac{a}{3} + \frac{b}{2} + c \qquad \{\because \text{ It has a root in } (0, 1)\}$$

$\Rightarrow 2a + 3b + 6c = 0$

SHORTCUTS: (SC-2)

If $y = f(x)$ and $x = g(y)$ are inverse of each other and $f(a) = c, f(b) = d$ then,

$$\int_a^b f(x)\,dx + \int_c^d g(y)\,dy = bd - ac$$

Illustration 13

Find: $\displaystyle\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx$

Short-cut solution :

Using SC-2

Since, $2\ln(\ln x)$ is inverse of $e^{\sqrt{e^x}}$ and $a = 0, b = 1, c = e, d = e^{\sqrt{e}}$

$\Rightarrow I = bd - ac = 1 \times e^{\sqrt{e}} - 0 \times e = e^{\sqrt{e}}$

SHORTCUTS: (SC-3)

Sandwich theorem for Definite Integral.

If we have, $g(x) \le f(x) \le h(x), \forall\, x \in [a, b]$

$$\Rightarrow \int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$$

Illustration 14

If $\dfrac{\pi}{A} < \displaystyle\int_{\pi/4}^{\pi/2} (\sin x)^{10}\,dx < \dfrac{\pi}{B}$ then A + B is

Short-cut solution :

Using SC-3 Since, $\frac{1}{\sqrt{2}} \le \sin x \le 1$

Hence, $\int_{\pi/4}^{\pi/2} \frac{1}{32} dx < \int_{\pi/4}^{\pi/2} (\sin x)^{10} dx < \int_{\pi/4}^{\pi/2} 1 \cdot dx$

$$\Rightarrow \quad \frac{\pi}{128} < \int_{\pi/4}^{\pi/2} (\sin x)^{10} dx < \frac{\pi}{4}$$

Hence A + B = 128 + 4 = 132.

SHORTCUTS: (SC-4)

Solving definite integration using graph.

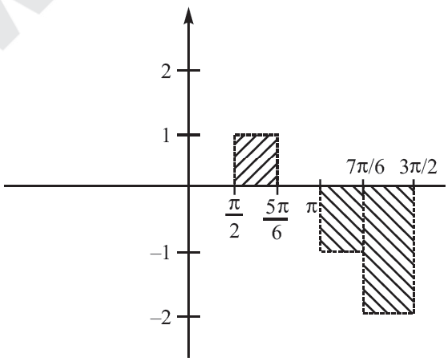
Take area above x-axis be positive and area below x-axis be negative and then find the algebraic sum of the area.

Illustration 15

$\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx$, is equal to (where, $[x]$ is greatest integer function)

Short-cut solution :

Using SC-4 $\because I = \frac{\pi}{3} - \left(\frac{\pi}{6}\right) - \left(\frac{2\pi}{3}\right) = -\frac{\pi}{2}$



TECHNIQUE

Transfer one integration in other by proper substitution.

$$\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x+a)dx$$

Illustration 16

Find the transformation of $3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx$

Short-cut solution :

$\boxed{\text{Using Tech.}}$

$$3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2} dx = 3\left[\left(\frac{2}{3}-\frac{1}{3}\right)\right]\int_0^1 e^{9\left[\left(\frac{2}{3}-\frac{1}{3}\right)x^2+\frac{1}{3}-\frac{2}{3}\right]^2} dx$$

$$= \int_0^1 e^{9\left(\frac{x}{3}-\frac{1}{3}\right)^2} dx$$

$$= \int_0^1 e^{(x-1)^2} dx$$

Concept Booster Exercise

1. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to **[JEE M 2015]**

(a) 2 (b) 4 (c) 1 (d) 6

2. The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to **[JEE M 2013]**

(a) $\pi/6$ (b) $\pi/12$ (c) $\pi/3$ (d) $\pi/2$

3. The value of the integral $\int_0^{\pi} \sin 2x \cdot \cos 20x\, dx$ is equal to

(a) 1 (b) 0 (c) $\frac{2}{99}$ (d) $-\frac{2}{99}$

4. The value of the integral $\int_0^{\pi/2} \frac{dx}{49\sin^2 x + 11\cos^2 x}$ is equal to

(a) $\pi/14\sqrt{11}$ (b) $\pi/\sqrt{11}$ (c) $\pi/14$ (d) $\pi/12\sqrt{11}$

5. The integral $\int_{-2}^{+3} \sqrt{(x+2)(-x+3)}\, dx$ is equal to

(a) $\frac{25\pi}{2}$ (b) $\frac{25\pi}{4}$ (c) $\frac{24\pi}{5}$ (d) $\frac{25\pi}{8}$

6. The integral $\int_3^6 \frac{dx}{\sqrt{(x-3)(6-x)}}$ is equal to

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $-\pi$

7. $\int_3^5 (x-3)^2 (5-x)^3 dx$ is equal to

(a) $\frac{15}{16}$ (b) $\frac{14}{15}$ (c) $\frac{16}{13}$ (d) $\frac{16}{15}$

8. The value of the integral $\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$, is

(a) $\frac{2}{6}$ (b) $\frac{2}{15}$ (c) $\frac{15}{2}$ (d) $\frac{3}{2}$

9. The value of the integral $\int \sin^4 x \cos^6 x\, dx$ is

(a) $\frac{3}{25}\cdot\frac{\pi}{2}$ (b) $\frac{3}{26}\cdot\frac{\pi}{2}$ (c) $\frac{3}{256}\cdot\frac{\pi}{2}$ (d) $\frac{3}{625}\cdot\frac{\pi}{2}$

10. If $2a + 9b + 12c = 0$, then the equation $ax^2 + 3bx + 2c = 0$ has a root in

(a) (0, 2) (b) (0, 3) (c) (0, 4) (d) (0, 1)

11. The value of $\int_1^2 e^x dx + \int_e^{e^2} \ln x\, dx$ is equal to

(a) $e - 2e^2$ (b) $2e^2 - e$ (c) $2e^2$ (d) $2e$

12. Let $f(x) = x - [x]$, for every real number x, where $[x]$ is the integral part of x.

Then $I = \int_{-1}^{1} f(x)\, dx$ is

(a) 1 (b) 2 (c) 0 (d) $-\frac{1}{2}$

13. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\, dx$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

14. The value of $\int_0^{2n\pi} \max.(\sin x, \sin^{-1} \sin x)\, dx$ is equal to

(a) $n\left(\frac{\pi^2}{2}-8\right)$ (b) $n\left(\frac{\pi^2}{4}-8\right)$ (c) $n(\pi^2 - 8)$ (d) $n\left(\frac{\pi^2}{3}-8\right)$

15. Find the value of $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$; $\beta > \alpha$ is equal to

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{8}(\beta-\alpha)^2$ (d) 2π

Numerical Value Problems

16. The value of the integral $\int_{-1}^{3}\left[x+\frac{1}{2}\right]$ dx is equal to (where, $[x]$ is greatest integer function)

17. If $K_1 < \int_0^{\pi/2} \frac{\sin x}{x}\, dx < \frac{\pi}{K_2}$, then the value of $K_1 + K_2$ is equal to

Solutions

1. **(c)** $\boxed{\text{Using T-1}}$ Since, $a = 2$, $b = 4$

$$\Rightarrow \text{I} = \frac{b-a}{2} = \frac{4-2}{2} = 1$$

2. **(b)** $\boxed{\text{Using T-1}}$ $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$

Here, $a = \frac{\pi}{6}$, $b = \frac{\pi}{3}$

$$\Rightarrow \text{I} = \frac{b-a}{2} = \frac{\frac{\pi}{3} - \frac{\pi}{6}}{2} = \frac{\pi}{12}$$

3. **(b)** $\boxed{\text{Using T-2}}$ $\because$ $a = 2$, $b = 20$ $\Rightarrow$ $b - a = 18$ is even

Hence, answer = 0

4. **(a)** $\boxed{\text{Using T-3}}$ $\because$ $a = 7$, $b = \sqrt{11}$

$$\Rightarrow \text{I} = \frac{\pi}{2ab} = \frac{\pi}{14\sqrt{11}}$$

5. **(d)** $\boxed{\text{Using T-4(ii)}}$

$$\Rightarrow a = -2,\quad b = 3 \quad \Rightarrow \text{I} = \frac{\pi}{8}(b-a)^2 = \frac{\pi}{8}(5)^2 = \frac{25\pi}{8}$$

6. **(a)** $\boxed{\text{Using T-4(i)}}$ $\because$ $a = 3$, $b = 6$ $(b > a)$

$\Rightarrow \pi$

7. **(d)** $\boxed{\text{Using T-5}}$ $\because$ $a = 3$, $b = 5$, $m = 2$, $n = 3$

$$\Rightarrow \text{I} = (b-a)^{m+n+1} \cdot \frac{m!\,n!}{(m+n+1)!} = (5-3)^6 \cdot \frac{2!\,3!}{6!} = \frac{16}{15}$$

8. **(b)** $\boxed{\text{Using T-6}}$ $\because$ $m = 2$, $n = 3$ (Both not even)

$$\Rightarrow \text{I} = \frac{(2-1)(3-1)}{5(5-2)(5-4)} = \frac{2}{15}$$

9. **(c)** $\boxed{\text{Using T-6}}$ $\because$ $m = 4$, $n = 6$ (Both even)

$$\Rightarrow \text{I} = \frac{(4-1)(4-3)(6-1)(6-3)(6-5)}{10(10-2)(10-4)(10-6)(10-8)} \times \frac{\pi}{2} = \frac{3}{256} \cdot \frac{\pi}{2}$$

10. **(d)** $\boxed{\text{Using SC-1}}$ Let $f(x) = ax^2 + 3bx + 2c$

$$\Rightarrow \int_0^1 (ax^2 + 3bx + 2c)\,dx = \left[\frac{ax^3}{3} + \frac{3bx^2}{2} + 2cx\right]_0^1 = \frac{6a + 9b + 12c}{6}$$

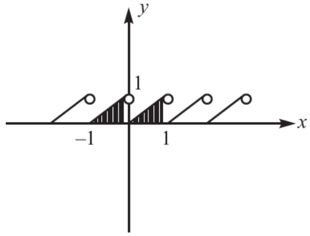
Since, $6a + 9b + 12c = 0$ (given). Hence root will lie in (0, 1).

11. **(b)** $\boxed{\text{Using SC-2}}$ Since $\ln x$ is inverse of e^x

$\Rightarrow \text{I} = bd - ac = 2e^2 - 1 \times e = 2e^2 - e$

12. **(a)** $\boxed{\text{Using SC-4}}$ $\because\ f(x) = x - [x] = \{x\}$

Now, Drawing graph of $y = \{x\}$



$$\Rightarrow \text{I} = \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 1 \times 1$$

$$\Rightarrow \boxed{\text{I} = 1}$$

13. **(c)** $\boxed{\text{Using T-1}}$ $\because\ f(x) = \sqrt{x},\ f(a + b - x) = \sqrt{5 - x}$

$$\Rightarrow \text{I} = \frac{b - a}{2} = \frac{3 - 2}{2} = \frac{1}{2}$$

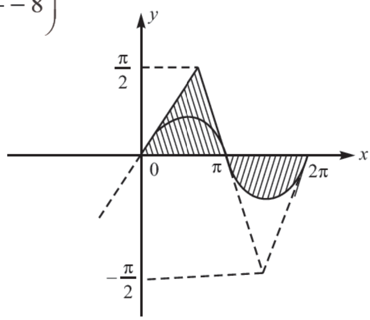
14. **(b)** $\because$ $\boxed{\text{Using P-7(i)}}$

$$\text{I} = n\int_0^{2\pi} \max.\{\sin x, \sin^{-1} \sin x\}\,dx \qquad \left(\begin{array}{l}\because y = \sin x \text{ is} \\ \text{periodic with} \\ \text{period '}2\pi\text{'}\end{array}\right)$$

$$\Rightarrow \text{I} = n\left[\int_0^{\pi} \sin^{-1}(\sin x)\,dx + \int_{\pi}^{2\pi} \sin x\,dx\right]$$

Hence, $\text{I} = n\left[\frac{1}{2} \times \pi \times \frac{\pi}{2} - 2\right]$

$$\Rightarrow \text{I} = n\left(\frac{\pi^2}{4} - 8\right)$$



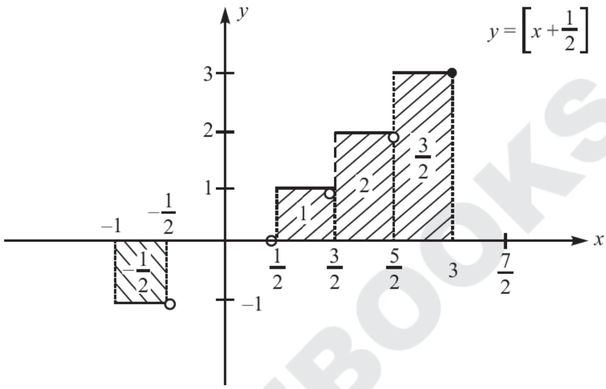
15. **(b)** $\boxed{\text{Using T-4(i)}}$ $a = \alpha$, $b = \beta$ $(\because b > a)$ given

$\Rightarrow \pi$

16. **(4)** $\boxed{\text{Using SC-4}}$ Drawing graph

We will find area of shaded portion

$$\Rightarrow I = \left(1 + 2 + \frac{3}{2} - \frac{1}{2}\right) \Rightarrow I = 4$$



17. **(3)** $\boxed{\text{Using SC-3}}$ For $x \in \left[0, \frac{\pi}{2}\right]$

As we know, $\frac{2}{\pi} < \frac{\sin x}{x} < 1$

$$\Rightarrow \int_0^{\pi/2} \frac{2}{\pi} < \int_0^{\pi/2} \frac{\sin x}{x} dx < \int_0^{\pi/2} dx \quad \Rightarrow \quad 1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$$

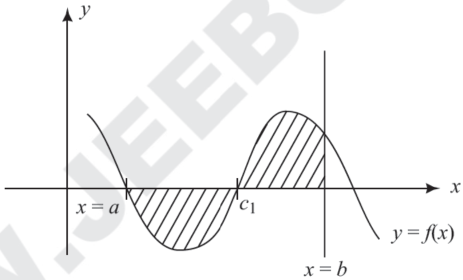
$\Rightarrow K_1 = 1, K_2 = 2 \Rightarrow K_1 + K_2 = 3.$

10 Application of Integrals

Review of Key Notes and Formulae

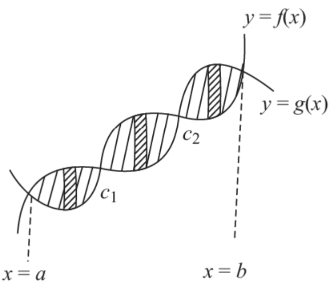
1. **Quadrature :** The process of calculating area under curve is known as Quadrature.

$$\Delta = \left|\int_a^{c_1} f(x)dx\right| + \int_{c_1}^{b} f(x)dx$$



2. **Area between Two Curves :** (Upper curve – Lower curve)

$$\Delta = \int_a^{c_1}\big(g(x) - f(x)\big)dx + \int_{c_1}^{c_2}\big(f(x) - g(x)\big)dx + \int_{c_2}^{b}\big(g(x) - f(x)\big)dx$$



3. **Average Value of the Function:**

The average value of a function $y = f(x)$ w.r.t. x over an interval $x \in [a, b]$ is defined as

$$y_{\text{avg.}} = \frac{\int_a^b f(x)dx}{b-a}$$

★ Average value can be positive, negative or zero.

4. Root Mean Square (rms) Value of a Function

$$y_{rms} = \sqrt{\frac{1}{b-a}\int_a^b f^2(x)dx}$$

TIPS AND TRICKS: (T-1)

Area of the region bounded by $y^2 = 4ax$ & $x^2 = 4by$ is $A = \frac{16ab}{3}$ sq. units

Illustration 1

The area enclosed between the curves $y = ax^2$ and $x = ay^2$ ($a > 0$) is 1 sq. unit. Then value of 'a' is **[AIEEE 2004]**

Short-cut solution :

$\boxed{\text{Using T-1}}$ $y^2 = 4\left(\frac{1}{4a}\right)x$ and $x^2 = 4\left(\frac{1}{4a}\right)y$

Hence, $A = \frac{16}{3} \times \frac{1}{4a} \times \frac{1}{4a} = 1 \Rightarrow a = \frac{1}{\sqrt{3}}$ sq. units

TIPS AND TRICKS: (T-2)

Area of the region bounded by $y^2 = 4ax$ & $y = mx + C$ is $A = \frac{72a^2}{m^3}$ sq. units

Illustration 2

The area of the region bounded by $y^2 = x$ and $y = x - 2$ in sq. units is equal to

Short-cut solution :

$\boxed{\text{Using T-2}}$ $a = \frac{1}{4}$, m = 1

Hence, A $= \frac{72a^2}{m^3} = \frac{72}{(1)^3} \times \left(\frac{1}{4}\right)^2 = \frac{9}{2}$ sq. units

TIPS AND TRICKS: (T-3)

Area of the region bounded by $y^2 = 4ax$ & $y = mx$ is $A = \dfrac{8a^2}{3m^3}$ sq. units

Illustration 3

Find the area of the region bounded by $y^2 = 8x$ and $y = 2x$ in sq. units

Short-cut solution :

Using T-3 $a = 2, m = 2$

Hence, $A = \dfrac{8a^2}{3m^3} = \dfrac{8 \times 4}{3 \times 8} = \dfrac{4}{3}$ sq. units

TIPS AND TRICKS: (T-4)

Area of the region bounded by $x^2 = 4by$ & $y = mx + C$ is $A = 72\, b^2 m^3$ sq. units

Illustration 4

The area of the region bounded by $x^2 = 4y$ and $x = 4y - 2$ in sq. units is equal to **[JEE M 2019]**

Short-cut solution :

Using T-4 $x^2 = 4(1)y$ & $y = \dfrac{x}{4} + \dfrac{1}{2}$

where, $b = 1, m = \dfrac{1}{4}, C = \dfrac{1}{2} \Rightarrow A = 72\, b^2 m^3 = 72 \times 1 \times \dfrac{1}{4^3} = \dfrac{9}{8}$ sq. units

TIPS AND TRICKS: (T-5)

Area of the region bounded by $x^2 = 4by$ & $y = mx$ is $A = \dfrac{8}{3} b^2 m^3$ sq. units

Illustration 5

Find the area of the region bounded by $x^2 = y$ and $y = 3x$ in sq. units.

Short-cut solution :

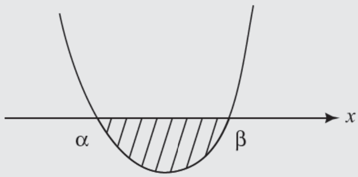
Using T-5 $b = \dfrac{1}{4}, m = 3$

Hence $A = \dfrac{8}{3} b^2 m^3 = \dfrac{8}{3} \times \left(\dfrac{1}{4}\right)^2 \cdot (3)^3 = \dfrac{9}{2}$ sq. units

TIPS AND TRICKS: (T-6)

Area bounded by a parabola between its roots is

$A = \frac{1}{6}|a|\,|\alpha - \beta|^3$ sq. units



★ **Note:** Here, 'a' is coefficient of x^2.

Illustration 6

Find the area of the region between the roots of the quadratic polynomial $y = x^2 - 3x + 2$ and x-axis is sq. units.

Short-cut solution :

$\boxed{\text{Using T-6}}$ Roots of polynomial are $\alpha = 1, \beta = 2$

Here, a (coefficient of x^2) $= 1 \Rightarrow A = \frac{1}{6}|a|\,|\alpha - \beta|^3$

$= \frac{1}{6} \times 1 \times (2-1)^3 = \frac{1}{6}$ sq. units

TIPS AND TRICKS: (T-7)

Area of the region bounded by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $A = \pi ab$ sq. units

Illustration 7

Find the area bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ (in sq. units).

Short-cut solution :

$\boxed{\text{Using T-7}}$ Here, $a = 4, b = \sqrt{7}$

$\Rightarrow A = \pi ab = 4\sqrt{7}\,\pi$ sq. units.

TIPS AND TRICKS: (T-8)

Area bounded by the parabola $y^2 = 4ax$ and its latus rectum $x = a$ is $A = \dfrac{8a^2}{3}$ sq. units

Illustration 8

Find the area bounded by the parabola $y = 8x$ and its latus rectum in sq. units.

Short-cut solution :

$\boxed{\text{Using T-8}}$ Here, $a = 2$,

$$\Rightarrow A = \frac{8a^2}{3} = \frac{8}{3}(2)^2 = \frac{32}{3} \text{ sq. units}$$

TIPS AND TRICKS: (T-9)

Area bounded by the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ is $A = \dfrac{8}{3}\sqrt{ab}\,(a + b)$ sq. units

Illustration 9

Find the area bounded by the curves $y^2 = 4(x + 1)$ and $y^2 = 8(2 - x)$ in sq. units.

Short-cut solution :

$\boxed{\text{Using T-9}}$ Here, $a = 1$, $b = 2$

$$\Rightarrow A = \frac{8}{3}\sqrt{ab}\,(a + b) = \frac{8}{3}\sqrt{2}\,(1 + 2) = 8\sqrt{2} \text{ sq. units}$$

TIPS AND TRICKS: (T-10)

The area of quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $A = \dfrac{2a^2}{e}$ sq. units

Where, 'e' is eccentricity of ellipse.

Illustration 10

The area of the quadrilateral formed by the tangents at the end points of latus rectum to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$ in sq. units. **[AIEEE 2003]**

Short-cut solution :

$\boxed{\text{Using T-10}}$ $e = \sqrt{1-\frac{5}{9}} = \frac{2}{3}$

$$\text{Hence, area} = \frac{2a^2}{e} = \frac{2\times 9}{(2/3)} = 27 \text{ sq. units}$$

TIPS AND TRICKS: (T-11)

The area bounded by the parabola and a line $x = k$, perpendicular to its axis of symmetry is $A = \frac{2}{3}\left(4\sqrt{a}.k^{3/2}\right)$ sq. units

Illustration 11

Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

Short-cut solution :

$\boxed{\text{Using T-11}}$ $a = 1, k = 3.$

$$\text{Hence, A} = \frac{2}{3}\left(4\sqrt{a}\cdot k^{3/2}\right) = \frac{2}{3}\left(4\sqrt{1}\ 3^{3/2}\right) = 8\sqrt{3}$$

SHORTCUTS: (SC-1)

To find the area of convex polygon with n-vertices having coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$, then follow the steps.

Step 1: Roughly arrange the coordinates in the argand plane.

Step 2:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

✶ For up to down take positive

✶ For bottom to up take negative.

$$\Rightarrow \text{Area} = \frac{1}{2}\left|\{(x_1y_2) + (x_2y_3) + (x_3y_4) + \ldots + (x_ny_1)\} - \{x_2y_1 + x_3y_2 + \ldots x_1y_n\}\right|$$

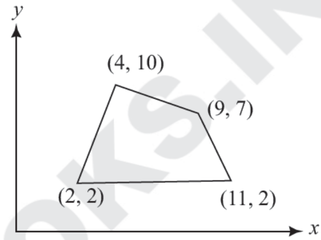
Illustration 12

Find the area of the quadrilateral formed by (4, 10), (11, 2), (2, 2) and (9, 7) in sq. units.

Short-cut solution :

Using SC-1 We take clockwise-direction

$$\Rightarrow A = \frac{1}{2}\begin{vmatrix} 4 & 10 \\ 9 & 7 \\ 11 & 2 \\ 2 & 2 \\ 4 & 10 \end{vmatrix}$$



$$\Rightarrow \text{area} = \left|\frac{1}{2}\{(28+18+22+20)\} - \frac{1}{2}\{(90+77+4+8)\}\right| = \frac{91}{2} \text{ sq. units.}$$

SHORTCUTS: (SC-2)

Area bounded by the inverse of the function. Follow the steps as used in the below example.

Illustration 13

Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is inverse of it. Find the area bounded by $g(x)$, the y-axis and the ordinate at $y = -2$ and $y = 6$.

Short-cut solution :

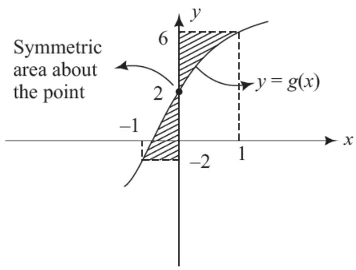
Using SC-2

Area required:

$y = g(x)$

$y = -2$ & $y = 6$

$x = 0$

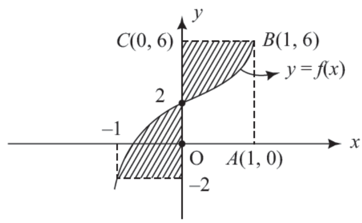


Reflection

$y = f(x)$

$y = -2$ & $y = 6$

$x = 0$



Hence, required area (A)

$A = 2[\text{Area(Rect. O}ABC) - \text{Area of } f(x) \text{ with } x\text{-axis}]$

$$A = 2[6 - \int_0^1 (x^3 + 3x + 2)dx]$$

$$\Rightarrow A = \frac{9}{2}$$

TECHNIQUE

The surface area of a curve y = f(x), a ≤ x ≤ b when rotated about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

The surface area of a curve x = g(y), c ≤ y ≤ d when rotated about y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Illustration 14

The part of straight line y = x + 1 between x = 2 and x = 3 is revolved about x-axis, then find the curved surface area generated by solid.

Short-cut solution :

Using Tech. The given curve y = x + 1, 2 ≤ x ≤ 3 and $\frac{dy}{dx} = 1 + 0 = 1$

$$\therefore \text{ Curved surface area } = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_2^3 2\pi (x+1)\sqrt{1 + (1)^2} dx = 2\sqrt{2}\pi \int_2^3 (x+1) dx$$

$$= 2\sqrt{2}\pi \left[\frac{(x+1)^2}{2}\right]_2^3 = 7\sqrt{2}\pi \text{ square unit}$$

Concept Booster Exercise

1. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ in sq. units is equal to **[AIEEE 2001]**

   (a) $\frac{32}{3}$ (b) $\frac{16}{3}$ (c) $\frac{8}{3}$ (d) 0

2. The area enclosed between $y^2 = x$ and $y = |x|$ in sq. units is: **[AIEEE 2007]**

   (a) 2 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) 3

3. The area (in sq. units) of the region described by $\{(x, y); y^2 \le 2x$ and $y \ge 4x - 1\}$ is **[JEE M 2015]**

   (a) $\frac{7}{32}$ (b) $\frac{5}{64}$ (c) $\frac{15}{64}$ (d) $\frac{9}{32}$

4. The area (in sq. units) of the region $A = \left\{(x, y): \frac{y^2}{2} \le x \le y + 4\right\}$ is: **[JEE M 2019]**

   (a) $\frac{53}{3}$ (b) 18 (c) 30 (d) 16

5. The area (in sq. units) of the region $A = \{(x, y) : x^2 \le y \le x + 2\}$ is: **[JEE M 2019]**

   (a) $\frac{31}{6}$ (b) $\frac{13}{6}$ (c) $\frac{9}{2}$ (d) $\frac{10}{3}$

6. The area (in sq. units) of the region $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in the first quadrant. **[JEE M 2013]**

   (a) 36 (b) 18 (c) $\frac{27}{4}$ (d) 9

7. Find the area of the quadrilateral formed by (1, 1), (1, −4), (3, −5), (3, −2) in sq. units is

   (a) 8 (b) 10 (c) 6 (d) 12

8. If the area bounded by the curves $y^2 = 4ax$ and $y = mx$ is $\frac{a^2}{3}$. Find 'm'.

   (a) 2 (b) 1 (c) 3 (d) 4

9. The area bounded by the ellipse

$\frac{x^2}{6}+\frac{y^2}{9}=1$ in sq. units.

(a) $2\sqrt{6}$ (b) $3\sqrt{6}$ (c) $\sqrt{6}$ (d) $\sqrt{2}$

10. Area bounded by the curves $y^2 = 8(x+2)$ and $y^2 = 32(8-x)$ is equal to (in sq. units)

(a) $\frac{300}{3}$ (b) $\frac{310}{3}$ (c) $\frac{300}{7}$ (d) $\frac{320}{3}$

Numerical Value Problems

11. The area of the region described by $y^2 \le 4x$ and $y \ge 2x - 1$ is equal to

12. The area of the region between the roots of the quadratic polynomial $y = x^2 - 8x + 7$ and x-axis is equal to _____ sq. units.

13. The area bounded by the parabola $y^2 = 8x$ and $x = 2$ line is equal to 'P' sq. units then $[P]$ is equal to (where $[P]$ is G.I.F.)

14. The area bounded by $x^2 = \frac{2}{3}y$ and $2y = 4x + 8$ is equal to _____

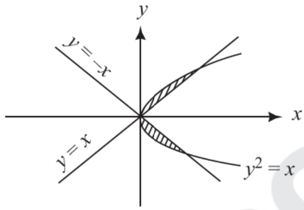
15. The area of a quadrilateral formed by (1, 1), (1, –4), (–3, –5) and (–3, –2) is equal to _____ sq. units.

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ $\because a = 1, b = 1$

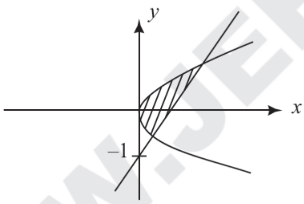
$$\Rightarrow A = \frac{16ab}{3} = \frac{16}{3} \text{ sq. units}$$

2. **(c)** $\boxed{\text{Using T-3}}$ $\because a = \frac{1}{4}, m = 1$



$$\Rightarrow A = \frac{2.8a^2}{3m^3} = \frac{2.8}{3} \times \frac{1}{16 \times 1} = \frac{1}{3} \text{ sq. units}$$

3. **(d)** $\boxed{\text{Using T-2}}$ $\because a = \frac{1}{2}, m = 4$



$$\Rightarrow A = \frac{72a^2}{m^3} = 72 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{4^3} = \frac{9}{32} \text{ sq. units}$$

4. **(b)** $\boxed{\text{Using T-2}}$ Equations are $y^2 \le 2x$ and $x \le y + 4$

Since, $a = \frac{1}{2}, m = 1$

$$\Rightarrow A = \frac{72a^2}{m^3} = 72\left(\frac{1}{4}\right) = 18 \text{ sq. units}$$

5. **(c)** $\boxed{\text{Using T-4}}$ Equations are $x^2 \le y$ and $y \le x + 2$

Since, $b = \frac{1}{4}, m = 1$

$$\Rightarrow A = 72b^2m^2 = 72\left(\frac{1}{4}\right)^2 1^2 = \frac{9}{2} \text{ sq. units}$$

6. **(d)** $\boxed{\text{Using T-2}}$ Given: $y = \sqrt{x}, y = \frac{x}{2} - \frac{3}{2}$

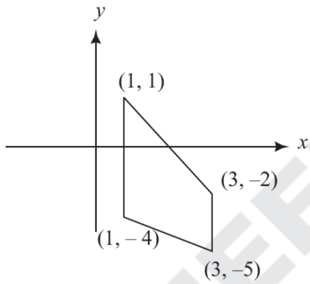
$\Rightarrow y^2 = x$, on squaring $y = \frac{x}{2} - \frac{3}{2}$ (Remember to divide the area by 2^2)

$\because a = \frac{1}{4}, m = \frac{1}{2}$

$\Rightarrow A = \frac{72a^2}{m^3} = 72\left(\frac{1}{4}\right)^2 \times 2^3 = 36$

Now, dividing by $2^2 \Rightarrow \frac{36}{4} = 9$

7. **(a)** $\boxed{\text{Using SC-1}}$ We take anti-clockwise direction



$$\Rightarrow A = \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 1 & -4 \\ 3 & -5 \\ 3 & -2 \\ 1 & 1 \end{vmatrix}$$

$= \frac{1}{2}|\{-4 - 5 - 6 + 3) - (1 - 12 - 15 - 2)\}| = 8$ sq. units.

8. **(a)** $\boxed{\text{Using T-3}}$ Here, $A = \frac{a^2}{3}$

$\because A = \frac{8a^2}{3m^3} = \frac{a^2}{3} \Rightarrow m = 2$

9. **(b)** $\boxed{\text{Using T-7}}$ Here, $a = \sqrt{6}$, b = 3

$\because A = \pi ab \Rightarrow A = 3\sqrt{6}\,\pi$ sq. units.

10. **(d)** $\boxed{\text{Using T-9}}$ Here, $a = 2$, $b = 8$

$\because\ a = \frac{8}{3}\sqrt{ab}(a+b) = \frac{8}{3}\sqrt{16}(8+2) = \frac{320}{3}$

11. **(9)** $\boxed{\text{Using T-2}}$ Equations are $y^2 \le 4x$ and $y \ge 2x - 1$

$\because = a = 1, m = 2$

$\Rightarrow A = \frac{72a^2}{m^3} = \frac{72\times1}{2\times2\times2} = 9$ sq. units

12. **(36)** $\boxed{\text{Using T-6}}$ Roots of $x^2 - 8x + 7 = 0$, $\alpha = 1$, $\beta = 7$ and $a = 1$

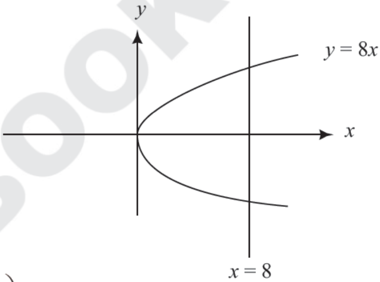
Hence, $A = \frac{1}{6}|a|\,|\alpha - \beta|^3 = \frac{1}{6} \times 1 \times 6 \times 6 \times 6 = 36$ sq. units

13. **(10)** $\boxed{\text{Using T-11}}$ $\because\ a = 2, k = 2$

$\Rightarrow A = \frac{2}{3}\left(4\sqrt{a}.k^{3/2}\right)$

$= \frac{2}{3}\left(4\sqrt{2}.2^{3/2}\right) = \frac{32}{3}$

$\Rightarrow \left[\frac{32}{3}\right] = 10$

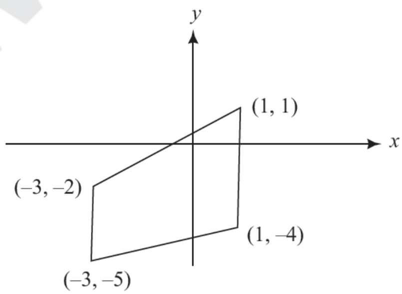


14. **(16)** $\boxed{\text{Using T-4}}$ Equations are $x^2 = 4\left(\frac{1}{6}\right)y$ and $y = 2x + 4$

Here, $b = \frac{1}{6}$ and $m = 2$

$\Rightarrow A = 72b^2m^3 = 72\left(\frac{1}{6}\right)^2(2)^3 = 16$

15. **(16)** $\boxed{\text{Using SC-1}}$ We will take clockwise direction



$$\Rightarrow A = \frac{1}{2}\left|\begin{array}{cc} 1 & 1 \\ 1 & -4 \\ -3 & -5 \\ -3 & -2 \\ 1 & 1 \end{array}\right|$$

$$\Rightarrow A = \frac{1}{2}\left|\{(-4 - 5 + 6 - 3) - (1 + 12 + 15 - 2)\}\right| = 16 \text{ sq. units}$$

11 Differential Equation

Review of Key Notes and Formulae

1. **Definition:** An equation involving independent variable (x), dependent variable (y) and derivative of dependent variable with respect to independent variable $\left(\frac{dy}{dx}\right)$ is called differential equation.

   Order of differential equation : Order of highest derivative occuring in the equation.

   Degree: The degree of a differential equation is the exponent of the derivative of the highest order in the equation, where it is in polynomial in derivatives.

2. **Formation of Differential Equation :** Suppose we have an equation $f(x, y, c_1, c_2, \ldots c_n) = 0$

   where, $c_1, c_2, c_3, \ldots c_n$ are arbitrary constants, then to form differential equation.

   Step 1: Differentiate the equation as many times as number of arbitrary constants.

   Step 2: Eliminate the arbitary constants in the given equation, which leads to the required differential equation.

3. **Solution of Differential Equation:**

   (i) *Variable separable form:* Separate the variables and then integrate to obtain the solution.

   $$\frac{f(x)}{g(y)} = \frac{dy}{dx} \quad \Rightarrow \quad \int f(x)\,dx = \int g(y)\,dy$$

   (ii) *Homogeneous differential equation:* A differential equation of the form:

   $$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where, $f(x, y)$ and $g(x, y)$ are homogeneous functions of same degree.

To solve $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then the given equation reduces to

$$x\frac{dv}{dx} = F(v) - v$$

(Now, separate the variables and integrate)

(iii) *Linear differential equation:* A linear differential equation of first order can be of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where, $P(x)$ and $Q(x)$ are functions of 'x'.

Steps to follow:

Step 1: Consider the equation $\frac{dy}{dx} + P(x).y = Q(x)$

Step 2: Find integrating factor (I.F.) = $e^{\int P(x)dx}$

Step 3: Solution is: y(I.F.) = $\int Q(x)\,(\text{I.F.})dx$

4. Some Important Results to Remember:

(i) $xdy + ydx = d(xy)$

(ii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{xdy + ydx}{xy} = d(\log xy)$

(iv) $\frac{xdy - ydx}{xy} = d\left(\log\frac{y}{x}\right)$

(v) $\frac{dx + dy}{x + y} = d(\log(x + y))$

TIPS AND TRICKS: (T-1)

Short trick to form differential equation for the type:

$$y = ae^{mx} \pm be^{-mx} \xrightarrow{\text{D.E.}} \boxed{y_2 - m^2 y = 0}$$

where, y_2 is $\frac{d^2y}{dx^2}$, y_1 is $\frac{dy}{dx}$

Illustration 1

Find the differential equation of the family of curves $y = Ae^{4x} + Be^{-4x}$ for different values of A and B.

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because$ $m = 4$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 16y = 0$$

Illustration 2

Find the differential equation of the curve $y = 2e^{x/2} + 3e^{-x/2}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because$ $m = \frac{1}{2}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{4}.y = 0$$

TIPS AND TRICKS: (T-2)

Short trick to form differential equation of the type:

$$y = a.e^{mx} + b.e^{nx} \xrightarrow{\text{D.E.}} \boxed{y_2 - (m+n)y_1 + mny = 0}$$

where, y_2 is $\frac{d^2y}{dx^2}$, y_1 is $\frac{dy}{dx}$

Illustration 3

Find the differential equation of the curve:

$y = a.e^{3x} + b.e^{5x}$ for different 'a' and 'b'

Short-cut solution :

Using T-2 $\because\ m = 3, n = 5$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

Illustration 4

Find the differential equation of the curve:

$y = 2.e^{x} - 3.e^{2x}$

Short-cut solution :

Using T-2 $\because\ m = 1, n = 2$

$$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

TIPS & TRICKS!

TIPS AND TRICKS: (T-3)

Short trick to form differential equation of the type:

$$y = (a + bx)e^{mx} \xrightarrow{\text{D.E.}} \boxed{y_2 - 2my_1 + m^2y = 0}$$

Illustration 5

Find the differential equation of the curve

$y = (2 + 3x)e^{5x}$

Short-cut solution :

Using T-3 $\because\ m = 5$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\times 5\frac{dy}{dx} + 25y = 0$$

TIPS AND TRICKS: (T-4)

Short trick to form differential equation of the type:

$$y = e^{mx}[a\sin(nx) \pm b\cos(nx)] \xrightarrow{\text{D.E.}} \boxed{y_2 - 2my_1 + (m^2 + n^2)y = 0}$$

Illustration 6

Find the differential equation of the curve

$y = e^{2x}[3\sin(3x) + 4\cos(3x)]$

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\because$ $m = 2, n = 3$

$$\Rightarrow \frac{d^2y}{dx^2} - 2 \times 2\frac{dy}{dx} + (4+9)y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Illustration 7

Find the differential equation of the curve

$y = e^{x/2}[2\sin 3x - 12\cos 3x]$

Short-cut solution :

Solution. $\boxed{\text{Using T-4}}$ $\because$ $m = \frac{1}{2}, n = 3$

$$\Rightarrow \frac{d^2y}{dx^2} - 2 \times \frac{1}{2}\frac{dy}{dx} + \frac{37}{4}y = 0$$

TIPS AND TRICKS: (T-5)

Short trick to solve differential equation of the type:
(Solution of differential equation)

$$\frac{dy}{dx} = \frac{m(ax \pm by) + c}{n(ax \pm by) + d}$$

In other words:

For, $\dfrac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} = \dfrac{dy}{dx}$

★ If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = k$ then use this method.

x	y	Put these values one by one	$ax \pm by$
n	m		$an \pm bm = p$(say)
d	c		$ad \pm bc = q$(say)

$$\frac{mx - ny}{nc - md}(an \pm bm) + \log\left|\,(ax \pm by)(an \pm bm) + (ad \pm bc)\,\right| = c$$

OR

$$\frac{mx - ny}{nc - md}(p) + \log\left|\,(ax \pm by)p + q\,\right| = c$$

★**Note:** Don't learn the above formula but only understand the procedure. (In order to explain the procedure may be long but once you understand it, then it will a magical tool for you.)

Illustration 8

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{-x - y + 2}{x - y + 2}$

Short-cut solution :

$\boxed{\text{Using T-5}}$

x	y	$x - y$
1	-1	2
2	$+2$	0

Now, $\dfrac{-x + y}{4}(2) + \log\left|\,(x - y)2 + 0\,\right| = c$

Illustration 9

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2(2x-5y)+6}{3(2x-5y)-12}$

Short-cut solution :

$\boxed{\text{Using T-5}}$

$$\begin{array}{cc} x & y \\ 3 & 2 \\ -12 & 6 \end{array}$$

$2x - 5y$

$6 - 10 = -4$

$-24 - 30 = -54$

Now, $\dfrac{2x-3y}{(18+24)}(-4) + \log\left|\,(2x-5y)(-4) - 54\,\right| = c$

SHORTCUTS: (SC-1)

To find solution of differential equation of the type:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

★ If $b_1 + a_2 = 0 \quad \Rightarrow \quad$ Cross multiply and integrate.

Illustration 10

Solve: $\dfrac{dy}{dx} = \dfrac{4x-3y}{3x-2y}$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $b_1 + a_2 = -3 + 3 = 0$ (Cross multiply)

$\Rightarrow 3x\,dy - 2y\,dy = 4x\,dx - 3y\,dx$

$\Rightarrow \int 3d(xy) = \int 4x\,dx + \int 2y\,dy$

$\Rightarrow 3xy = 2x^2 + y^2 + C$

Illustration 11

Solve: $\frac{dy}{dx}=\frac{ax-by}{bx+cy}$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $b_1 + a_2 = -b + b = 0$ (Cross multiply)

$\Rightarrow bx\,dy + cy\,dy = ax\,dx - by\,dx$

$\Rightarrow \int b(x\,dy + y\,dx) = \int ax\,dx - \int cy\,dy$

$\Rightarrow bxy = \frac{ax^2}{2} - \frac{cy^2}{2} + c$

SHORTCUTS: (SC-2)

Substitution Method: Substitute the options to check the correct answer.

Illustration 12

A solution of the differential equation, $\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0$

(a) $y = 2$ (b) $y = 2x$

(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Short-cut solution :

$\boxed{\text{Using SC-2}}$

(a) $\frac{dy}{dx} = 0$, on substituting $\Rightarrow$ $0 - 0 + y \neq 0$

(b) $\frac{dy}{dx} = 2$, on substituting $\Rightarrow$ $4 - 2x + y \neq 0$

(c) $\frac{dy}{dx} = 2$, on substituting $\Rightarrow$ $4 - 2x + y = 0$ **Ans. (c)**

Illustration 13

The solution of the differential equation,

$y\,dx + (x + x^2y)dy = 0$ is **[AIEEE 2004]**

(a) $\log y = Cx$

(b) $-\frac{1}{xy} + \log y = C$

(c) $\frac{1}{xy} + \log y = C$

(d) $xy = \log y = C$

Short-cut solution :

Using SC-2

(a) $\log y = C$

$\Rightarrow \quad \frac{dy}{dx} = Cy$ does not satisfy

(b) $-\frac{1}{xy} + \log y = C$

$$\Rightarrow \quad -\left[\frac{1}{x}\left(-\frac{1}{y^2}\right)\frac{dy}{dx} + \frac{1}{y}\left(-\frac{1}{x^2}\right)\right] + \frac{1}{y} = 0$$

Ans. (b)

TECHNIQUE

Clairaut's Equation : The differential equation of the form $y = px + f(p)$ where $p = \frac{dy}{dx}$, is known as Clairaut's differential equation.

The solution of this Clairaut's equation is obtained by replacing p by constant c, that is $y = cx + f(c)$

Illustration 14

Find the solution of differential equation $y = \frac{dy}{dx}x + \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$

Short-cut solution :

Using Tech. $y = px + \dfrac{p}{\sqrt{1+p^2}}$ where $p = \dfrac{dy}{dx}$, then solution is

$$y = cx + \frac{c}{\sqrt{1+c^2}}$$

Illustration 15

Solve $y^2 \log y = pxy + p^2$

Short-cut solution :

Using Tech.

Let $\log y = t$. Then $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

So, if $\dfrac{dt}{dx} = p$,

So the given equation can be written as

$y^2 t = y.p\, xy + p^2 y^2$ or, $t = px + p^2$

Which is in Clairaut's form

Thus, the required solution is

$t = cx + c^2$ or $\log y = cx + c^2$

(c being an arbitrary constant)

Concept Booster Exercise

1. Find the differential equation of the curve $y = 4e^{2x} - 7e^{-2x}$

(a) $y'' - 2y' = 0$ (b) $y'' - y' = 0$

(c) $y'' - 4y' = 0$ (d) $y'' - \frac{y'}{2} = 0$

2. Find the differential equation of the curve

$y = 8e^{3x} + 5e^{4x}$

(a) $y'' - 7y' + 12y = 0$ (b) $y'' - 6y' + 12y = 0$

(c) $y'' - 6y' - 12y = 0$ (d) $y'' - y = 0$

3. Find the differential equation of the curve

$y = (4 + 7x)\, e^{2x}$

(a) $y'' - 4y' + y = 0$ (b) $y'' - 2y' + 2y = 0$

(c) $y'' - 4y' + 4y = 0$ (d) $y'' - 4y' = 0$

4. Find the differential equation of the curve

$y = e^{4x}\, [5 \sin 6x - 8 \cos 6x]$

(a) $y'' - y' + 52y = 0$ (b) $y'' - 8y' + 52y = 0$

(c) $y'' - y' + y = 0$ (d) $y'' - y' + 2y = 0$

5. The solution of differential equation, $\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$ is

(a) $x - y + \log |x - y + z| = c$ (b) $x - 3y + \log |x - y + 2| = c$

(c) $x - 4y + \log |x - 4y + z| = c$ (d) $x - 2y + \log |x - y + 2| = c$

6. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is **[AIEEE 2008]**

(a) $y = \log x + x$ (b) $y = x \log x + x^2$

(c) $y = xe^{(x-1)}$ (d) $y = x \log x + x$

7. A curve passes through that point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right); x>0$. Then the equation of the curve is

[JEE M 2013]

(a) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(c) $\sec\left(\frac{2y}{x}\right)=\log x+2$ (d) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

8. The differential equation which represents the family of curves $y=c_1e^{c_2x}$, where c_1 and c_2 are arbitary constants is

(a) $y'=y^2$ (b) $y''=y'y$

(c) $y.y''=y'$ (d) $y.y''=(y')^2$

9. The differential equation of the family of curves, $x^2=4b(y+b)$, $b \in \mathrm{R}$ is

(a) $x(y')^2=x-2yy'$ (b) $xy''=y'$

(c) $x(y')^2=x+2yy'$ (d) $x(y')^2=2yy'$

10. Let 'I' be the purchased value of an equipment and $v(t)$ be the value after it has been used for 't' years. The value $v(t)$ depreciates at a rate given by differential equation $\frac{dv(t)}{dt}=-k(\mathrm{T}-t)$, where $k>0$ is a constant and 'T' is the total life in years of the equipment. Then the scrap value of $v(\mathrm{T})$ of the equipment is:

[AIEEE 2011]

(a) $\mathrm{T}^2-\frac{1}{k}$ (b) $\mathrm{I}-\frac{k\mathrm{T}^2}{2}$

(c) $\mathrm{I}-\frac{k(\mathrm{T}-t)^2}{2}$ (d) $e^{-k\mathrm{T}}$

NUMERICAL VALUE PROBLEMS

11. Let $f : \mathrm{R} \rightarrow \mathrm{R}$ be a continuous function, which satisfies $f(x) = \int_0^x f(t)dt$.

Then the value of $f(\ln 5)$ is _______ **[AIEEE 2009]**

12. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ___________

Solutions

1. **(c)** $\boxed{\text{Using T-1}}$ $\because m = 2,\ y'' - m^2y = 0 \Rightarrow y'' - 4y = 0$

2. **(a)** $\boxed{\text{Using T-2}}$ $\because m = 3,\ n = 4$

$$\frac{d^2y}{dx^2} - (3+4)\frac{dy}{dx} + 3 \times 4y = 0$$

$y'' - 7y' + 12y = 0$

3. **(c)** $\boxed{\text{Using T-3}}$ $\because m = 2$

$y'' - 2 \times 2y' + 2^2y = 0$

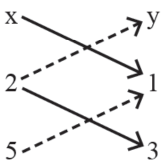
$\Rightarrow y'' - 4y' + 4y = 0$

4. **(b)** $\boxed{\text{Using T-4}}$ $\because m = 4,\ n = 6$

$y'' - 2 \times 4y' + (16 + 36)y = 0$

$y'' - 8y' + 52y = 0$

5. **(d)** $\boxed{\text{Using T-5}}$



$$\frac{x-2y}{6-5}(2-1) + \log\left|(x-y)(2-1) + (5-3)\right| = C$$

$$\Rightarrow (x-2y) + \log\left|(x-y) + 2\right| = C$$

6. **(d)** $\boxed{\text{Using SC-1}}$ $x\,dy = x\,dx + y\,dx$

$$\frac{x\,dy - y\,dx}{x} = dx$$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x}dx$$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int \frac{1}{x}dx \Rightarrow \frac{y}{x} = \log x + C$$

$\because y(1) = 1$

$\Rightarrow C = 1$

$$\frac{y}{x} = \log x + 1 \Rightarrow y = x\log x + x$$

7. **(a)** Using SC-1

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \log|x| + C$$

$$\because y(1) = \frac{\pi}{6} \Rightarrow C = \frac{1}{2}$$

$$\therefore \sin\left(\frac{y}{x}\right) = \log|x| + \frac{1}{2}$$

8. **(d)** $y = c_1 e^{c_2 x}$

$y' = c_2 y$

$y'' = c_2 y'$

$$\therefore \frac{y''}{y'} = \frac{c_2 y'}{c_2 y}$$

$$\Rightarrow yy'' = (y')^2$$

9. **(c)** $x^2 = 4b(y + b)$

$$2x = 4b\frac{dy}{dx} \Rightarrow b = \frac{x}{2\frac{dy}{dx}} = \frac{x}{2y'}$$

$$x^2 = 4\frac{x}{2y'}y + 4\left(\frac{x}{2y'}\right)^2$$

$$\Rightarrow x(y')^2 = 2yy' + x$$

10. **(b)** Using SC-1 $\frac{dv(t)}{dt} = -k(T-t)$

$$\Rightarrow v(t) = \frac{k(T-t)^2}{2} + C$$

$$\because v(0) = I$$

$$v(t) = I$$

$$\therefore v(t) = I + \frac{k(t^2 - 2tT)}{2}$$

$$v(T) = I - \frac{k}{2}T^2$$

11. **(0)** $f(x) = \int_0^x f(t)\,dt \Rightarrow f'(x) = f(x)$

$$\Rightarrow \int \frac{f'(x)}{f(x)}dx = \int dx \Rightarrow \log f(x) = x + k$$

$$\Rightarrow f(x) = e^{x+k} = e^x e^k \Rightarrow f(x) = Ce^x \left(\text{say } e^k = C\right)$$

$$\therefore f(0) = 0 \Rightarrow Ce^0 = 0 \Rightarrow C = 0$$

$$\therefore f(x) = 0 \Rightarrow f(\ln 5) = 0$$

12. **(2)** $f(x) = x^3 + e^{x/2} \Rightarrow f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$$\therefore f(0) = 1, f'(0) = \frac{1}{2}$$

$$\because\ g(x) = f^{-1}(x) \Rightarrow g(f(x)) = x$$

$$\Rightarrow\ g'[f(x)]f'(x) = 1 \Rightarrow g'[f(x)] = \frac{1}{f'(x)}$$

$$g'(1) = \frac{1}{f'(0)} = 2$$

12 Complex Numbers

Review of Key Notes and Formulae

1. **Complex Number:** A Number of the from $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number where 'x' is called as real part and y is called imaginary part of complex number.

   i.e. $\text{Re}(z) = x$, $\text{Im}(z) = y$,

   ★ If $\text{Re}(z) = 0 \Rightarrow$ Purely imaginary complex number.

   ★ If $\text{Im}(z) = 0 \Rightarrow$ Purely real complex number.

2. **Algebra of Complex Number:**

   If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be any two complex numbers

   (i) Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

   (ii) Subtraction: $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

   (iii) Multiplication: $z_1 . z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

   (iv) Division: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$ [On rationalisation]

   $$\frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

   ★ **Note:** Additive inverse of z is $-z$ and multiplicative inverse of z is —.

3. **Conjugate of Complex Numbers:** If $z = x + iy$ is a complex number, the conjugate of z is denoted by $\overline{z} = x - iy$.

   Properties

   (i) $(\overline{\overline{z}}) = z$

   (ii) $z + \overline{z} = 2\,\text{Re}(z) =$ Purely real

   (iii) $z - \overline{z} = 2i\,\text{Im}(z) =$ Purely imaginary

   (iv) $z . \overline{z} = x^2 + y^2 = |z|^2$

   (v) $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$

   (vi) $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$

(vii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$; $z_2 \neq 0$

(viii) $\overline{z^n} = (\overline{z})^n$

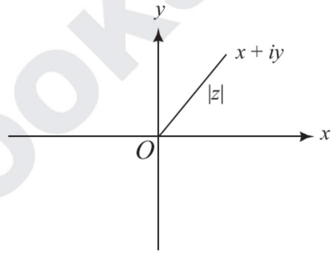
(ix) if $z = f(z_1)$, then $\overline{z} = f(\overline{z_1})$

(x) $z_1\overline{z}_2 + z_1 z_2 = 2\,\text{Re}(\overline{z}_1 z_2) = 2\,\text{Re}(z_1\overline{z}_2)$

(xi) $z_1\overline{z}_2 - \overline{z}_1 z_2 = 2\,\text{Im}(\overline{z}_1 z_2) = 2\,\text{Im}(z_1\overline{z}_2)$.

4. **Modulus of a Complex Number:** If $z = x + iy$, then modulus or magnitude of z is denoted by $|z| = \sqrt{x^2 + y^2}$

Properties:



(i) $|z| \geq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$

(iii) $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\overline{z}| = |-z| = |-\overline{z}|$

(v) $z\overline{z} = |z|^2$

(vi) $|z_1 \cdot z_2| = |z_1|\,|z_2|$

(vii) $z^{-1} = \dfrac{\overline{z}}{|z|^2}$

(viii) $|z^n| = |z|^n$

(ix) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $|z_2| \neq 0$

(x) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}\,(z_1\overline{z}_2)$

(xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(xii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.

(xiii) Triangle inequility:

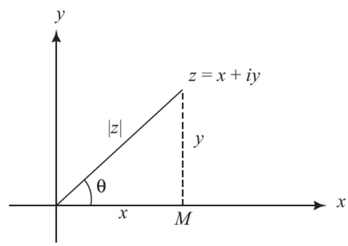
(a) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ (b) $|z_1 + z_2| \geq |z_1| - |z_2|$

(c) $|z_1 - z_2| \leq ||z_1| - |z_2||$

(xiv) z is unimodulus, if $|z| = 1$

5. **Amplitude/Argument of a Complex Number:**

The argument of a Complex Number z is the inclination of the directed line segment representing z, with real axis.



Amp(z) = Arg(z) = $\tan^{-1}\left(\dfrac{y}{x}\right)$

★ **Note:** Principle value of argument of a complex number lies between $-\pi < \theta \le \pi$.

Properties:

(i) $\text{amp}(k) = \begin{cases} 0; & \text{If } k \in R^+ \\ \pi; & \text{If } k \in R^- \end{cases}$

(ii) $\text{amp}(z_1 . z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2k\pi$, ($k$ is any integer)

(iii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2k\pi$, ($k = 0$, 1 or -1)

(iv) $\text{amp}(\overline{z}) = -\text{amp}(z) = \text{amp}\left(\dfrac{1}{z}\right)$

(v) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$ ($k = 0$, 1 or -1)

(vi) $\text{amp}(-z) = \text{amp}(z) \pm \pi$

(vii) $\text{amp}(z) + \text{amp}(\overline{z}) = 2k\pi$; $(2k\pi \in (-\pi, \pi])$

6. **Polar Form of a Complex Number :** Polar form of $z = x + iy$ is

$z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$

and Arg (z) = θ.

7. **Eulerian Representation of Complex Number:**

$z = re^{i\theta}$ where, $|z| = r$, $\arg(z) = \theta$

$\overline{z} = re^{-i\theta}$ and $e^{i\theta} = \cos\theta + i\sin\theta$

8. **De-moivre's Theorem:**

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$; $n \in I$

Then $\cos\theta + i\sin\theta$ is one of the value of $(\cos\theta + i\sin\theta)^n$.

9. **Cube Roots of Unity:**

Cube roots of unity are 1, w, w^2, where $w = \dfrac{-1}{2} + \dfrac{i\sqrt{3}}{2}$ and $w^2 = \dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}$

and $\arg(w) = \dfrac{2\pi}{3}$ and $\arg(w^2) = \dfrac{4\pi}{3}$

Properties:

(i) $w^3 = 1$ or $w^{3r} = 1$, $w^{3r+1} = w$ and $w^{3r+2} = w^2$; $r \in I$

(ii) $1 + w + w^2 = 0$

(iii) Cube root of unity lie on the unit circle $|z| = 1$ and divide its circumference into 3 equal parts.

(iv) 1, w and w^2 always from equilateral triangle.

(v) Cube roots of -1 are -1, $-w$, $-w^2$.

10. n^{th} Roots of Unity:

It means any complex number z, which satisfies the equation $z^n = 1$ or $z = (1)^{1/n}$ or $z = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$, where $k = 0, 1, 2, ..., (n-1)$

Properties:

(i) n^{th} root of unity form a GP with common ration $e^{i2\pi/n}$

(ii) Sum of n^{th} root of unity is always zero.

(iii) Sum of p^{th} powers of n^{th} root of unity is

'n' — If 'p' is a multiple of 'n'.

'o' — If 'p' is not a multiple of 'n'.

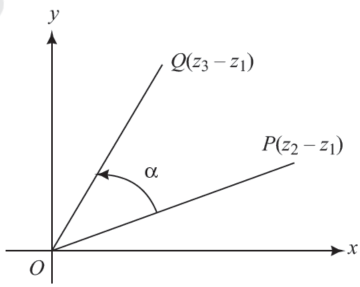
(iv) Product of n^{th} root of unity is $(-1)^{n-1}$

(v) The nth roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into 'n' equal parts.

11. Concept of Rotation:

If z_1, z_2, z_3 be the three vertices of a ΔABC described in the anti-clockwise direction. Draw OP and OQ parallel and equal to AB and AC respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$ then,

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}e^{i\alpha}$$



12. Some Important Results to Remember:

If $z = \cos\theta + i\sin\theta$

(i) $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$

(ii) $z^n + \dfrac{1}{z^n} = 2\cos(n\theta)$ and $z^n - \dfrac{1}{z^n} = 2i\sin(n\theta)$

(iii) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and given, $x + y + z = 0$, then

(a) $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} = 0$ (b) $xy + yz + zx = 0$

(c) $x^2 + y^2 + z^2 = 0$ (d) $x^3 + y^3 + z^3 = 3xyz$

TIPS AND TRICKS: (T-1)

To evaluate complex expression of the form:

$\boxed{(1 \pm i)^n = (-4)^q\ (1 \pm i)^r}$

where, q and r are quotient and remainder when 'n' is divided by 4.

Illustration 1

Evaluate: $(1 + i)^{12} + (1 - i)^{12}$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\therefore$ When 12 is divided by 4 then $q = 3$, $r = 0$

$\Rightarrow$ Expression $= (-4)^3(1 + i)^0 + (-4)^3\ (1 - i)^0 = -128$

Illustration 2

The expression $(1 + i)^{58} + (1 - i)^{58}$ is equal to

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\therefore$ When 58 is divided by 4 then $q = 14$, $r = 2$

$\Rightarrow$ Expression $= (-4)^{14}\ (1 + i)^2 + (-4)^{14}\ (1 - i)^2$

$= (-4)^{14}(2i) + (-4)^{14}(-2i) = 0$

TIPS AND TRICKS: (T-2)

To find square root of $a + ib$, first find the number $\frac{|b|}{2}$ and then factorized this number in such a way that difference of square of these factors is equal to real number 'a'

Illustration 3

Square root of $5 + 12i$ is equal to:

Short-cut solution :

$\boxed{\text{Using T-2}}$ Let $\sqrt{5+12i} = \sqrt{a+ib}$

Here, $b = 12 \xrightarrow{b/2} 6 \begin{array}{l} \nearrow 3 \\ \searrow 2 \end{array} \Rightarrow \sqrt{5+12i} = \pm(3 + 2i)$

Illustration 4

Square root of $3 - 4i$ is equal to:

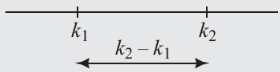
Short-cut solution :

$\boxed{\text{Using T-2}}$ Here, $|b| = 4 \xrightarrow{b/2} 2 \begin{array}{l} \nearrow 2 \\ \searrow 1 \end{array}$

$\Rightarrow \sqrt{3-4i} = \pm(2 - i)$

TIPS AND TRICKS: (T-3)

Minimum value of the expression of the form:
$|z - k_1| + |z - k_2|$ where, $k_1, k_2 \in R$
Step 1: Mark k_1 and k_2 on number line.



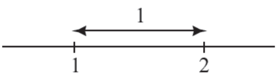
Step 2: The difference of k_1 and k_2 will give minimum value.
$\Rightarrow$ Minimum value $= k_2 - k_1$

Illustration 5

Find the minimum value of $|z - 1| + |z - 2|$

Short-cut solution :

Using T-3



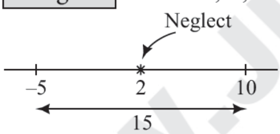
$\Rightarrow$ Minimum value = 1

Illustration 6

The minimum value of $|z - 2| + |z + 5| + |z - 10|$

Short-cut solution :

Using T-3 Mark –5, 2, 10 on number line



$\Rightarrow$ Minimum value = 15

TIPS AND TRICKS: (T-4)

If $\left|z + \frac{1}{z}\right| = a$, then the greatest and least value of $|z|$ are respectively $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$

Illustration 7

The maximum and minimum value of $|z|$ if $\left|z + \frac{1}{z}\right| = 5$, is equal to

Short-cut solution :

Using T-4 $\because a = 5$

$$\Rightarrow |z|_{\max} = \frac{5 + \sqrt{29}}{2} \text{ and } |z|_{\min} = \frac{-5 + \sqrt{29}}{2}$$

Illustration 8

If $\left|z - \frac{1}{z}\right| = 1$, then

(a) $|z|_{\max} = \frac{-1+\sqrt{5}}{2}$

(b) $|z|_{\max} = \frac{1+\sqrt{5}}{2}$

(c) $\sqrt{5}$

(d) 2

Short-cut solution :

$\boxed{\text{Using T-4}}$ $\because a = 1 \Rightarrow |z|_{\max} = \frac{1+\sqrt{1+4}}{2} = \frac{\sqrt{5}+1}{2}$ **Ans. (b)**

TIPS AND TRICKS: (T-5)

We can use substitution method to solve complex number problems.

➔ Assume $z = 1$ or $z = -1$ or $z = i$ or $z = -i$

★**Note:**

(i) In case when it is given that $|z| \neq 1$ then we can assume $z = \pm 2$ or $z = \pm 2i$, etc.

(ii) If more than one option matches, then change the substitution of z.

Illustration 9

If z is a complex number of unit modulus and argument 'θ', then the real part of $\frac{z(1-\overline{z})}{\overline{z}(1+z)}$ is **[JEE M 2014]**

(a) $2\cos^2\frac{\theta}{2}$

(b) $1-\cos\frac{\theta}{2}$

(c) $1+\sin\frac{\theta}{2}$

(d) $-2\sin^2\frac{\theta}{2}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ Let $z = i \Rightarrow \text{Arg}(z) = \theta = \frac{\pi}{2}$

and $\frac{z(1-\overline{z})}{\overline{z}(1+z)} = \frac{i(1+i)}{-i(1+i)} = -1$

Now, check options (a), (b), (c), (d) for $\theta = \pi/2$

$\Rightarrow$ option (d) $= -2\sin^2\frac{\theta}{2} = -1$ **Ans (d)**

Illustration 10

If $|z| = 1$, $z \neq -1$ and $\alpha = \dfrac{z+1}{z-1}$ then real part of 'α' is equal to: **[AIEEE 2003]**

Short-cut solution :

$\boxed{\text{Using T-5}}$ Let $z = i \Rightarrow \alpha = \dfrac{i+1}{i-1} \times \dfrac{i+1}{i+1} = -i$

$\Rightarrow \text{Re}(\alpha) = 0$

Illustration 11

The set of all $\alpha \in R$, for which $w = \dfrac{1+(1-8\alpha)z}{1-z}$ is a purely imaginary number for all $z \in C$ satisfying $|z| = 1$ and $\text{Re}(z) \neq 1$ is **[JEE M 2018]**

Short-cut solution :

$\boxed{\text{Using T-5}}$ Let $z = i \Rightarrow w = \dfrac{1+(1-8\alpha)i}{1-i} \times \dfrac{1+i}{1-i}$

$$\Rightarrow \quad w = \frac{8\alpha + i(2-8\alpha)}{2}$$

Since, purely imaginary $\Rightarrow \text{Re}(w) = 0 \Rightarrow 8\alpha = 0$

$$\Rightarrow \boxed{\alpha = 0}$$

Illustration 12

If $z \in C$ of unit modulus and arg θ, then $\arg\left(\dfrac{1+z}{1+\overline{z}}\right)$ is equal to: **[JEE M 2013]**

(a) $\dfrac{\pi}{2} - \theta$ (b) θ

(c) $\pi - \theta$ (d) $-\theta$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\because |z| = 1$, Let $z = 1$

$\Rightarrow \quad \arg\left(\dfrac{1 \quad 1}{1 \quad 1}\right) = \arg(z) = \theta$ **Ans. (b)**

Illustration 13

If $a^2 + b^2 = 1$, then $\dfrac{1+a+ib}{1+a-ib}$ is equal to

(a) i (b) $-i$

(c) $1+i$ (d) $1-i$

Short-cut solution :

$\boxed{\text{Using T-5}}$ $\because a^2 + b^2 = 1$, Let $a + ib = i \Rightarrow a = 0, b = 1$

$$\Rightarrow \quad \frac{1+a+ib}{1+a-ib} = \frac{1+i}{1-i} = i$$

Ans. (a)

SHORTCUTS: (SC-1)

Locus of a Point

Let $A(z_1)$ and $B(z_2)$ are two fixed points and a point $P(z)$ moves in a plane such that:

(i) $|z - z_1| + |z - z_2| = k$
- If $k = |z_1 - z_2| \Rightarrow$ Locus is line joining A and B
- If $k \neq |z_1 - z_2| \Rightarrow$ Locus is ellipse

(ii) $|z - z_1| - |z - z_2| = k$
- If $k = |z_1 - z_2| \Rightarrow$ Locus lies on ray BA
- If $k \neq |z_1 - z_2| \Rightarrow$ Locus is hyperbola

(iii) $\left|\dfrac{z - z_1}{z - z_2}\right| = k$
- If $k = 1 \Rightarrow$ Perpendicular bisector of AB
- If $k \neq 1 \Rightarrow$ Circle

(iv) $|z + \bar{z}| + |z - \bar{z}| = k \Rightarrow$ Locus is a square.

Illustration 14

The locus of $|z - 1| + |z + 1| = 4$ is:

Short-cut solution :

$\boxed{\text{Using SC-1 (i)}}$ $\because z_1 = 1, z_2 = -1$ and $|z_1 - z_2| = |1 + 1| = 2 \neq 4$

Hence, locus is ellipse.

Illustration 15

The locus of $|z + 2| - |z - 2| = \pm 3$ is:

Short-cut solution :

$\boxed{\text{Using SC-1 (ii)}}$ $\because z_1 = -2, z_2 = 2, |z_1 - z_2| = |-2 - 2| = 4 \neq \pm 3$

Hence, locus is hyperbola.

Illustration 16

Find the locus of $\left|\dfrac{z - 5i}{z + 5i}\right| = 1$

Short-cut solution :

$\boxed{\text{Using SC-1 (iii)}}$ $\because k = 1 \Rightarrow$ Locus is perpendicular bisector line joining z_1 and z_2.

Illustration 17

The locus of $|z+\overline{z}|+|z-\overline{z}| = 10$ is:

Short-cut solution :

$\boxed{\text{Using SC-1 (iv)}}$ Since $\overline{z}$ is a conjugate of $\overline{z}$

$\Rightarrow$ Locus is a square.

SHORTCUTS: (SC-2)

Use of AM $\geq$ GM in complex numbers.

Illustration 18

If z is a complex number satisfying $|z^3 + z^{-3}| \leq 2$, then the maximum possible value of $|z + z^{-1}|$ is

(a) 2 (b) $\sqrt[3]{2}$

(c) $2\sqrt{2}$ (d) 1

Short-cut solution :

$\boxed{\text{Using SC-2}}$ As we know that $\left|z^3+\dfrac{1}{z^3}\right| \leq |z|^3+\dfrac{1}{|z|^3}$

AM $\geq$ GM

So when $|z| = 1 \Rightarrow |z|^3 + \dfrac{1}{|z|^3} = 2$

Now, $\left|z+\dfrac{1}{z}\right| \leq |z|+\dfrac{1}{|z|} = 2$

Hence, maximum value = 2 **Ans. (a)**

SHORTCUTS: (SC-3)

Square root of any complex number:

(i) $\sqrt{z} = \sqrt{a+ib} = \pm\left[\sqrt{\dfrac{|z|+\text{Re}(z)}{2}}+i\sqrt{\dfrac{|z|-\text{Re}(z)}{2}}\right]$

(ii) $\sqrt{z} = \sqrt{a-ib} = \pm\left[\sqrt{\dfrac{|z|+\text{Re}(z)}{2}}-i\sqrt{\dfrac{|z|-\text{Re}(z)}{2}}\right]$

Illustration 19

Square root of $2 + \dfrac{5}{2}i$ is equal to:

Short-cut solution :

$\boxed{\text{Using SC-3 (i)}}$ $\sqrt{2+\frac{5i}{2}} = \pm\left[\sqrt{\frac{\sqrt{41}+4}{4}} + i\sqrt{\frac{\sqrt{41}-4}{4}}\right]$

Illustration 20

Square root of $z = 3 - \frac{2i}{5}$ is equal to

Short-cut solution :

$\boxed{\text{Using SC-3 (iv)}}$ $\sqrt{z} = \pm\left[\sqrt{\frac{\sqrt{229}+15}{10}} - i\sqrt{\frac{\sqrt{229}-15}{10}}\right]$

SHORTCUTS: (SC-4)

The equation $|z - z_1|^2 + |z - z_2|^2 = k$, where $k \in R$ will represent a circle with centre $\frac{1}{2}(z_1 + z_2)$ and radius $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2}$ when $k \geq \frac{1}{2}|z_1 - z_2|^2$

Illustration 21

Find the centre and radius of the circle $|z - 2|^2 + |z - 4i|^2 = 20$.

Short-cut solution :

$\boxed{\text{Using SC-4}}$ Here $z_1 = 2$ and $z_2 = 4i$

Centre = $\frac{1}{2}(z_1 + z_2) = \frac{1}{2}(2 + 4i) = 1 + 2i$

Radius = $\frac{1}{2}\sqrt{2k - |z_1 - z_2|^2} = \frac{1}{2}\sqrt{40 - 20} = \sqrt{5}$

TECHNIQUE

Trigonometry of Complex Number:

Substitute $x = \sin\theta + i\cos\theta$ or $\sin x = \frac{e^{ix} + e^{-ix}}{2}$ and $\cos x = \frac{e^{ix} - e^{-ix}}{2}$ to simplify trigonometric function or find expansion of $\sin^n\theta$ or $\cos^n\theta$.

Illustration 22

If $\sin\alpha + \sin\beta + \sin\gamma = \cos\alpha + \cos\beta + \cos\gamma = 0$, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.

Short-cut solution :

Using Tech.

Let $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$.

$$\therefore \frac{1}{a} = \cos\alpha - i\sin\alpha, \frac{1}{b} = \cos\beta - i\sin\beta \text{ and } \frac{1}{c} = \cos\gamma - i\sin\gamma$$

Now, $a + b + c = 0$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0 \Rightarrow ab + bc + ca = 0$

$\therefore (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow a^2 + b^2 + c^2 = 0$

$\Rightarrow (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

Illustration 23

If $\sin^6 x - \cos^6 x = A + B\cos 4x$, then find A and B

Short-cut solution :

Using Tech.

We have A + B cos $4x$

$$= \frac{1}{64}\left[\left(e^{ix} + e^{-ix}\right)^6 - \left(e^{ix} - e^{-ix}\right)^6\right]$$

$$= \frac{1}{32}[{}^6C_1(e^{ix})^5(e^{-ix}) + {}^6C_3(e^{ix})^3(e^{-ix})^3 + {}^6C_5(e^{ix})(e^{-ix})^5]$$

$$= \frac{1}{32}[6(e^{4ix} + e^{-4ix}) + 20]$$

$$= \frac{1}{32}[(6)(2\cos 4x) + 20] = \frac{3}{8}\cos 4x + \frac{5}{8}$$

Thus, A = 5 / 8 and B = 3 / 8

Concept Booster Exercise

1. Square root of $8 - 6i$ is equal to

(a) $\pm(2 - i)$ (b) $\pm(3 - i)$ (c) $\pm(3 + i)$ (d) None of these

2. The simplest form of $z = (1 + i)^{13}$ is equal to

(a) $-8 - 8i$ (b) $8 + 8i$ (c) $-64 - 64i$ (d) $64 + 64i$

3. Simplest form of $z = (1 + i)^{13} + (1 - i)^{13}$ is equal to:

(a) 128 (b) 64 (c) -64 (d) -128

4. Minimum value of the expression $|z - 4| + |z - 6| + |z - 1|$ is equal to:

(a) 5 (b) 10 (c) -5 (d) -10

5. Minimum value of the expression $|z + 2| + |z - 4|$ is equal to

(a) 2 (b) 4 (c) -2 (d) 6

6. If $\left|z + \frac{1}{z}\right| = 2$, then

(a) $z_{max} = -1 + \sqrt{2}$ (b) $z_{max} = 1 = \sqrt{2}$

(c) $z_{min} = -1 - \sqrt{2}$ (d) $z_{min} = 1 + \sqrt{2}$

7. If $|z| = 1$ and $z \neq \pm 1$ then all values of $\frac{z}{1 - z^2}$ lies on: **[AIEEE 2007]**

(a) A line which does not pass through origin

(b) $|z| = \sqrt{2}$

(c) x-axis

(d) y-axis

8. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then $\left|\frac{z_1 + z_2 + \ldots + z_n}{z_1^{-1} + z_2^{-1} + \ldots + z_n^{-1}}\right|$ equals to

(a) n (b) $1/n$ (c) $|z_1 + z_2|$ (d) 1

9. The complex numbers $z = x + iy$ which satisfy the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$, lie on

(a) the x-axis

(b) the straight line $y = 5$

(c) a circle passing through origin

(d) None of these

10. The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, for

(a) $x = n\pi$ (b) $x = 0$ (c) $x = \left(n+\frac{1}{2}\right)^{\pi}$ (d) No value of x

11. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$:

[JEE M 2014]

(a) is strictly greater than 5/2
(b) is strictly greater than 3/2 but less than 5/2
(c) is equal to 5/2
(d) lie in the interval (1, 2)

12. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is: **[JEE M 2016]**

(a) $\pi/6$ (b) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (c) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $\pi/3$

13. The value of the expression

$$\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)+\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right)+\ldots+\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$$

where ω is cube root of unity

(a) $\frac{n(n^2+2)}{3}$ (b) $\frac{n(n^2+3)}{3}$ (c) $\frac{n(n^2+1)}{3}$ (d) $\frac{n(n^2+2)}{2}$

14. The locus of $|z - 3 - 4i| - |z - 2 - 3i| = \sqrt{2}$ is

(a) ellipse (b) hyperbola (c) lies on a ray (d) circle

15. The locus of $\left|\frac{z-3-4i}{z-1+i}\right| = 2$ is:

(a) ellipse (b) circle (c) hyperbola (d) None of these

16. If $Z_1 \neq 0$ and Z_2 be two complex numbers such that $\frac{Z_2}{Z_1}$ is a purely imaginary number, then $\left|\frac{2Z_1+3Z_2}{2Z_1-3Z_2}\right|$ is equal to: **[JEE M 2013]**

(a) 2 (b) 5 (c) 3 (d) 1

17. The radius of the circle $|z - 2i|^2 = 25 - |z - 3 + i|^2$ is

(a) $\frac{5}{2}$ (b) 5 (c) $\frac{3}{2}$ (d) 2

NUMERICAL VALUE PROBLEMS

18. Let $w = e^{i\pi/3}$ and a, b, c, x, y, z be non-zero complex numbers such that $a + b + c = x$, $a + bw + cw^2 = y$, $a + bw^2 + cw = z$. Then value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is _______ **[AIEEE 2011]**

19. If $|z + 4| \le 3$ then minimum value of $|z + 1|$ is _____

20. If $|z - 3 + 2i| \le 4$ then the difference between the greatest and least value of $|z|$ is $2\sqrt{p}$ then 'p' is _____

21. The value of $(1 + i)^{24}$ is equal to _____

22. If $\left(\dfrac{1+i}{1-i}\right)^{m/2} = \left(\dfrac{1+i}{i-1}\right)^{n/3} = 1$, $(m, n \in \mathbf{N})$, then the greatest common divisor of the least values of m and n is _________. **[JEE M 2020]**

Solutions

1. **(b)** $\boxed{\text{Using T-2}}$ Let $\sqrt{8-6i} = \sqrt{a-ib}$

Here, $|b| = 6 \xrightarrow{|b|/2} 3 \begin{smallmatrix}\nearrow 3\\ \searrow 1\end{smallmatrix} \Rightarrow \sqrt{8-6i} = \pm(3 - i)$

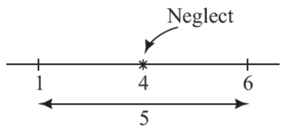
2. **(c)** $\boxed{\text{Using T-1}}$ $\therefore$ When 13 is divided by 4 then $q = 3, r = 1$

$\Rightarrow$ Expression $= (-4)^3(1 + i) = -64 - 64i$

3. **(d)** $\boxed{\text{Using T-1}}$ $\because q = 3, r = 1$

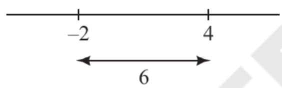
$\Rightarrow$ Expression $= (-4)^3(1 + i) + (-4)^3(1 - i) = -128$

4. **(a)** $\boxed{\text{Using T-3}}$ Mark 1, 4 and 6 on number line



$\Rightarrow$ Minimum value = 5

5. **(d)** $\boxed{\text{Using T-3}}$ Mark –2 and 4 on number line.



$\Rightarrow$ Minimum value = 6

6. **(b)** $\boxed{\text{Using T-4}}$ $\because a = 2 \Rightarrow z_{max} = \dfrac{2+\sqrt{4+4}}{2} = 1 + \sqrt{2}, z_{min} = -1 + \sqrt{2}$

7. **(d)** $\boxed{\text{Using T-5}}$ $\because |z| = 1$, Let $z = i$

$\Rightarrow \dfrac{z}{1-z^2} = \dfrac{i}{2}$ (lie on imaginary axis) $\Rightarrow y$-axis

8. **(d)** $\boxed{\text{Using T-5}}$ Let $n = 2$ and $z_1 = 1, z_2 = i$

$\Rightarrow \left|\dfrac{z_1+z_2}{z_1^{-1}+z_2^{-1}}\right| = \left|\dfrac{1+i}{1-i}\right| = 1$

9. **(a)** $\boxed{\text{Using T-5}}$ Let $z = 5$

$\Rightarrow \left|\dfrac{z-5i}{z+5i}\right| = \left|\dfrac{5-5i}{5+5i}\right| = 1 \Rightarrow x$-axis

10. **(d)** $\boxed{\text{Using T-5}}$ Check for the options (a), (b), (c).

Since, no value of 'x' satisfies $\Rightarrow$ Correct option is (d).

11. **(d)** $\boxed{\text{Using SC-2}}$ $\because \left|z+\dfrac{1}{2}\right| \le |z| + \dfrac{1}{2}$ (Triangle Inequality)

By AM $\ge$ GM $\Rightarrow \dfrac{|z|+\dfrac{1}{2}}{2} \ge \left(\dfrac{|z|}{2}\right)^{1/2}$ $(|z|_{min} = 2)$

$\Rightarrow |z| + \frac{1}{2} \geq 2$, Hence, $\left|z+\frac{1}{2}\right| \leq 2$

We neglect the equality, since $|z| \neq \frac{1}{2}$

Therefore, correct option is (d).

12. **(c)** $\boxed{\text{Using T-5}}$ Checking options for (a), (b), (c), (d)

Therefore, on substituting $\theta = \sin^{-1}\frac{1}{\sqrt{3}} \Rightarrow$ Expression $= \dfrac{2+i\sqrt{3}}{1-\dfrac{2i}{\sqrt{3}}}$

$= \dfrac{\sqrt{3}(2+i\sqrt{3})(\sqrt{3}+2i)}{(\sqrt{3}-2i)(\sqrt{3}+2i)} = \dfrac{\sqrt{3}}{7}(2\sqrt{3}+3i+4i-2\sqrt{3})$ is purely imaginary.

13. **(a)** $\boxed{\text{Using T-5}}$ Let $n = 1 \Rightarrow \left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right) = (1+\omega^2)(1+\omega)$

$= (-\omega)(-\omega^2) = 1$

Now, check options (a), (b), (c), (d) $\Rightarrow$ option (a) $= \dfrac{n(n^2+2)}{3} = 1$

Hence, correct option is (a).

14. **(c)** $\boxed{\text{Using SC-1 (ii)}}$ Comparing with $|z - z_1| - |z - z_2| = k$

Since, $k = |z_1 - z_2| = |(3+4i) - (2+3i)| = \sqrt{2}$

Hence, locus lies on ray joining points.

15. **(b)** $\boxed{\text{Using SC-1 (iii)}}$ Comparing with $\left|\dfrac{z-z_1}{z-z_2}\right| = k$

Since, $k \neq 1 \Rightarrow$ Locus is circle.

16. **(d)** $\boxed{\text{Using T-5}}$ Let $z_1 = 1 + i$ and $z_2 = 1 - i$

$$\frac{z_2}{z_1} = \frac{1-i}{1+i} = \frac{(1-i)\,(1-i)}{(1+i)\,(1-i)} = -i$$

$$\frac{2z_1+3z_2}{2z_1-3z_2} = \frac{2+3\left(\dfrac{z_2}{z_1}\right)}{2-3\left(\dfrac{z_2}{z_1}\right)} = \frac{2-3i}{2+3i}$$

$$\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right| = \left|\frac{2-3i}{2+3i}\right| = \left|\frac{2-3i}{2+3i}\right| \qquad \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$= \frac{\sqrt{4+9}}{\sqrt{4+9}} = 1$$

17. **(a)** $\boxed{\text{Using SC-4}}$ Here $z_1 = 2i$ and $z_2 = 3 - i$

$$\text{Centre } = \frac{1}{2}(z_1 + z_2) = \frac{3}{2} + \frac{1}{2}i$$

$$\text{Radius} = \frac{1}{2}\sqrt{2k - |z_1 - z_2|^2} = \frac{1}{2}\sqrt{50 - 25} = \frac{5}{2}$$

18. **(3)** $\boxed{\text{Using T-5}}$ Let $a = b = c = 1 \Rightarrow x = 3, y = 0, z = 0$

Therefore, expression $= \dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \dfrac{9}{1+1+1} = 3$

19. **(0)** $\boxed{\text{Using Key Note}}$ $\because$ $|z + 1| = |(z + 4) - 3|$ Triangle inequality

As we know that $|z_1 - z_2| \le ||z_1| - |z_2||$ (equality holds when $z_1 = z_2$)

$\Rightarrow |z + 1|_{\text{Min}} = |3| - |-3| = 0$

20 **(13)** $\boxed{\text{Using T-4}}$ $\because$ $|z| = |z - 3 + 2i + (3 - 2i)|$

$\Rightarrow z_{\max} = 4 + \sqrt{13}$ and $z_{\min} = 4 - \sqrt{13}$

Then difference $= 2\sqrt{13} \Rightarrow P = 13$

21. **(256)** $\boxed{\text{Using T-1}}$ $\therefore$ When 24 divided by 4 then $q = 6, r = 0$

Hence, $(1 + i)^{24} = (-4)^4(1 + i)^0 = 256$

22. **(4)** $\boxed{\text{Using Key Note}}$ Given that $\left(\dfrac{1+i}{1-i}\right)^{m/2} = \left(\dfrac{1+i}{i-1}\right)^{n/3} = 1$

$$\Rightarrow \left(\frac{(1+i)^2}{2}\right)^{m/2} = \left(\frac{(1+i)^2}{-2}\right)^{n/3} = 1$$

$$\Rightarrow i^{m/2} = (-i)^{n/3} = 1$$

m (least) $= 8$, n (least) $= 12$

GCD $(8, 12) = 4$.

13 Quadratic Equations

Review of Key Notes and Formulae

1. **Definition:** Quadratic equations are the polynomial equations of degree 2 in one variable of type $f(x) = ax^2 + bx + c$; $a, b, c \in$ R and $a \neq 0$

2. **Nature of Roots of Quadratic Equation:**

   If $D = b^2 - 4ac > 0 \Rightarrow$ Two distinct real roots

   If $D = b^2 - 4ac = 0 \Rightarrow$ Two equal real roots

   If $D = b^2 - 4ac < 0 \Rightarrow$ Two imaginary roots

   If $D = b^2 - 4ac$ is a perfect square $\Rightarrow$ Rational roots

   If 'D' is a perfect square, $a = 1$ & b and c are integers $\Rightarrow$ Integral roots.

3. **Quadratic Equations having Common Roots:**

   (i) *If one root is common*. ('α' is common root)

   $a_1\alpha^2 + b_1\alpha + c_1 = 0$

   $a_2\alpha^2 + b_2\alpha + c_2 = 0$

   $$\Rightarrow \frac{\alpha^2}{b_1c_2 - c_1b_2} = \frac{\alpha}{-a_1c_2 + c_1a_2} = \frac{+1}{a_1b_2 - b_1a_2}$$

   (ii) *If both roots are common*

   $$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

4. **Relationships between Coefficient and Roots of Quadratic Equation:** If α and β are roots of quadratic equation $ax^2 + bx + c = 0$

   • $\alpha + \beta = \dfrac{-b}{a}$ • $\alpha\beta = \dfrac{c}{a}$

   ★ **Note:** Let $f(x) = ax^2 + bx + c$; $a, b, c \in$ R, $a \neq 0$

   (i) If $f(x) = 0$ has more than two roots then it becomes identify

   $\Rightarrow a = b = c = 0$

(ii) If $f(x) > 0 \ \forall \ x \in R \Rightarrow a > 0, D < 0$

(iii) If $f(x) \geq 0 \ \forall \ x \in R \Rightarrow a > 0, D \leq 0$

(iv) If $f(x) < 0 \ \forall \ x \in R \Rightarrow a < 0, D < 0$

(v) If $f(x) \leq 0 \ \forall \ x \in R \Rightarrow a < 0, D \leq 0$

5. Location of Roots: Condition for $ax^2 + bx + c = 0, a \neq 0; a, b, c \in R$ to have

(i) Both roots greater than 'k', ($k \in R$) then

(a) $D \geq 0$ (b) $\frac{-b}{2a} > k$ (c) $af(k) > 0$

(ii) Both roots less than 'k' then

(a) $D \geq 0$ (b) $\frac{-b}{2a} < k$ (c) $af(k) > 0$

(iii) Both roots lie on either side of 'k' then

(a) $D > 0$ (b) $af(k) < 0$

(iv) Exactly one root lying in k_1 and k_2 ($k_1, k_2 \in R$)

$\boxed{f(k_1).f(k_2) < 0}$ ★ **Note:** Check for $f(k_1) = 0$ or $f(k_2) = 0$

(v) Both roots lie in k_1 and k_2

(a) $D \geq 0$ (b) $k_1 < \frac{-b}{2a} < k_2$

(c) $af(k_1) > 0$ and $af(k_2) > 0$

(vi) One root is less than k_1 and other root greater than k_2

$\boxed{af(k_1) < 0 \text{ and } af(k_2) < 0}$

TIPS AND TRICKS: (T-1)

Use of method of substitution

For $ax^2 + bx + c = 0, a \neq 0 \rightarrow$ Choose any two roots of this equation and further manipulate the equation.

Illustration 1

Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 = -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is: **[AIEEE 2010]**

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$

(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$

(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Short-cut solution :

Using T-1 Assume 'α' and 'β' be '1' and '2' respectively.

$\therefore$ Sum of roots $= \alpha + \beta = 3 = -p$

$\Rightarrow \quad p = -3$ and $q = 9$

Now, $\dfrac{\alpha}{\beta} = \dfrac{1}{2}$ and $\dfrac{\beta}{\alpha} = 2$

$\Rightarrow \quad$ Equation is $x^2 - \dfrac{5x}{2} + 1 = 0$

Hence, sum of roots in $= \dfrac{p^3 - 2q}{p^3 + q} = \dfrac{5}{2}$ **Ans. (b)**

Illustration 2

If α and β are the roots of the equation $x^2 - m(x + 1) - n = 0$, then value of $\dfrac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + m} + \dfrac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + n}$ is equal to

(a) 2 (b) 0

(c) 1 (d) 4

Short-cut solution :

Using T-1 Assuming $\alpha = 1$ and $\beta = 2$

$\therefore$ Equation is $x^2 - 3x + 2 = 0$

On comparing with $x^2 - m(x + 1) + n = 0 \Rightarrow m = 3, n = -5$

Now, expression $= \dfrac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + m} + \dfrac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + n} = 1$ **Ans. (c)**

Illustration 3

If a, b and c are distinct positive numbers, the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is

(a) Positive (b) Negative

(c) Non-positive (d) Non-negative

Short-cut solution :

Using T-1 Let $a = 1$, $b = 2$ and $c = 3$

$\Rightarrow$ Expression $= 4 \times 2 \times 0 - 1 \times 2 \times 3 = -6 < 0$ (Negative) **Ans. (b)**

Illustration 4

If (α, β) are roots of $px^2 + 2qx - p = 0$ and quadratic equation whose roots are $\left(3\alpha - \frac{1}{\beta}\right)$ and $\left(3\beta - \frac{1}{\alpha}\right)$ is $ax^2 + bx + c = 0$, then $a + b + c$ is equal to

(a) $2p$ (b) $13p - 15q$

(c) $13q - 15p$ (d) 0

Short-cut solution :

$\boxed{\text{Using T-1}}$ Let $\alpha = 1, \beta = -1 \Rightarrow$ Equation is $x^2 - 1 = 0$

Comparing with $px^2 + 2qx - p = 0 \Rightarrow p = 1, q = 0$

Now, roots of other equation are $(4, -4) \Rightarrow$ Equation is $x^2 - 16 = 0$

Hence, on comparing with $ax^2 + bx + c = 0 \Rightarrow a + b + c = -15$ **Ans. (c)**

Illustration 5

If α and β $(\alpha < \beta)$ are roots of the equation $x^2 + bx + c = 0$ where $c < 0 < b$, then **[AIEEE 2000]**

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$

(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Let $b = 3, c = -4$

$\Rightarrow x^2 + 3x - 4 = 0 \begin{cases} \nearrow \alpha \\ \searrow \beta \end{cases}$

$\therefore \alpha = -4, \beta = 1$

Now checking options **Ans. (b)**

Illustration 6

Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then, the value of 'r' is

[AIEEE 2007]

(a) $\frac{2}{9}(p-q)(2q-p)$ (b) $\frac{2}{9}(q-p)(2p-q)$

(c) $\frac{2}{9}(q-2p)(2q-p)$ (d) $\frac{2}{9}(2p-q)(2q-p)$

Short-cut solution :

$\boxed{\text{Using T-1}}$ Let $\alpha = 2, \beta = 1$

$\Rightarrow$ Equation is $x^2 - 3x + 2 = 0$

Compare with $x^2 - px + r = 0 \Rightarrow p = +3$ and r = 2

Equation with roots $\frac{\alpha}{2} = 1, 2\beta = 2$ is $x^2 - 3x + 2 = 0$

Compare with $x^2 - qx + r = 0 \Rightarrow q = 3$ and $r = 2$

Substituting p and q in options (a), (b), (c), (d).

$= \frac{2}{9}(2p - q)(2q - p) = 2 = r$ **Ans. (d)**

TIPS AND TRICKS: (T-2)

If the roots of the equation $Px^2 + Qx + R = 0$ are real and equal and $P + Q + R = 0$ then $P = R$

Illustration 7

If the roots of the equation $(2a - 3b)x^2 + (c - 2a)x + (3b - c) = 0$ are equal then,

(a) $2a - c = 6b$ (b) $2a + c = 6b$

(c) $c = \frac{a}{2}$ (d) $a = 0$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because$ $P + Q + R = 0$

$\Rightarrow \quad P = R$

$\Rightarrow \quad 2a - 3b = 3b - c$

$\Rightarrow \quad 2a + c = 6b$ **Ans. (b)**

Illustration 8

If the roots of the equation $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2) = 0$ are equal then

(a) $b = \sqrt{\frac{2c}{c^2 + a^2}}$ (b) $b = \sqrt{\frac{2a}{c^2 + a^2}}$

(c) $b = ac\sqrt{\frac{2}{a^2 + c^2}}$ (d) None

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because$ P + Q + R = 0

$$\Rightarrow \quad P = R$$

$$\Rightarrow a^2(b^2 - c^2) = c^2(a^2 - b^2)$$

$$\Rightarrow b = \sqrt{\frac{2a^2c^2}{a^2 + c^2}}$$

Ans. (c)

TIPS AND TRICKS: (T-3)

For positive values $a_1, a_2, a_3, ..., a_n$

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

Arithmetic Mean Geometric Mean Harmonic Mean

Illustration 9

If $a > 0$, $b > 0$ and $c > 0$ then prove that:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Short-cut solution :

$\boxed{\text{Using T-3}}$ $\because$ AM $\geq$ HM (for a, b, c)

$$\Rightarrow \quad \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Hence proved.

Illustration 10

The least value of the expression $3\log_{10} x + \log_x 1000$ is

(a) 2 (b) 4

(c) 1 (d) 6

Short-cut solution :

$\boxed{\text{Using T-3}}$ Applying AM ≥ GM

$$\frac{3\log_{10}x+\dfrac{3}{\log_{10}x}}{2} \geq \left(3\log_{10}x\frac{3}{\log_{10}x}\right)^{1/2} \qquad \left\{\because \log_x 10=\frac{1}{\log_{10}x}\right\}$$

$\Rightarrow \quad 3\log_{10}x + \log x\ 1000 \geq 6$

$\Rightarrow$ Least value of the expression is 6. **Ans. (d)**

Concept Booster Exercise

1. Let $a > 0, b > 0, c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$

(a) are real and negative (b) have negative real parts

(c) have positive real parts (d) None of these

2. Let α and β be the roots of the equation $ax^2 + bx + c = 0$ then equation whose roots are $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$ is

(a) $abx^2 + b(c + a)x + (c + a)^2 = 0$ (b) $(c + a)^2x^2 + b(c + a)x + ac = 0$

(c) $cax^2 + b(c + a)x + (c + a)^2 = 0$ (d) $cax^2 + b(c + a)x + c(c + a)^2 = 0$

3. If one root is square of the other root of the equation $x^2 + px + q = 0$, then relation between p & q is **[AIEEE 2004]**

(a) $p^3 - (3p - 1)q + q^2 = 0$ (b) $p^3 - q(3p + 1) + q^2 = 0$

(c) $p^3 + q(3p - 1) + q^2 = 0$ (d) $p^3 + q(3p + 1) + q^2 = 0$

4. If p and q are the roots of the equation $x^2 + mx + m^2 + a = 0$ then the value of $p^2 + q^2 + pq$ is equal to

(a) 0 (b) a

(c) $-a$ (d) $\pm m^2$

5. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, then $aS_{n+1} + cS_{n-1}$ is equal to

(a) b^2S_n (b) bS_n

(c) $-bS_n$ (d) $2bS_n$

6. If $b > a$, then the equation $(x - a)(x - b) = 1$ has **[AIEEE 2000]**

(a) Both roots in (a, b)

(b) Both roots in $(-\infty, a)$

(c) Both roots in (b, ∞)

(d) One root in $(-\infty, a)$ and the other in (b, ∞)

7. If the roots of $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is

(a) 1 (b) 2

(c) 3 (d) 4

8. If $k \in (-\infty, -2) \cup (2, \infty)$ then roots of the equation $x^2 + 2kx + 4 = 0$ are

(a) one real and one imaginary (b) real and unequal

(c) real and equal (d) complex

9. If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then a, b, c are in

(a) HP (b) GP

(c) AP (d) None of these

10. The least value of the expression $2\log_{10} x - \log_x(0.01)$, for $x > 1$, is

(a) 10 (b) 2

(c) -0.01 (d) None of these

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ Let $a = 1, b = 4, c = 3$

Then, $x^2 + 4x + 3 = 0 \Rightarrow x = -1, -3$

Again, $a = b = c = 1$, then

$x^2 + x + 1 = 0 \quad \Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

2. **(c)** $\boxed{\text{Using T-1}}$ Let $\alpha = 1, \beta = 2$

Then equation is $x^2 - 3x + 2 = 0$ comparing with

$ax^2 + bx + c = 0 \Rightarrow a = 1, b = -3, c = 2$

Equation with root $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$ is

$2x^2 - 9x + 9 = 0$

$cax^2 + b(c + a)x + (c + a)^2 = 0 \Rightarrow 2x^2 - 9x + 9 = 0$

3. **(a)** $\boxed{\text{Using T-1}}$ Let $\alpha = 2, \beta = 4$

Then, the equation is $x^2 - 6x + 8 = 0$ compare with $x^2 + px + q = 0$

$\Rightarrow p = -6, q = 8$

$\therefore p^3 - [3p - 1]q + q^2 = (-6)^3 - [3(-6) - 1] \times 8 + 8^2$

$= -216 + 19 \times 8 + 64 = -216 + 216 = 0$

4. **(c)** $\boxed{\text{Using T-1}}$ Let $p = 1, q = 2$

Then the equation is $x^2 - 3x + 2 = 0$

Compare with $x^2 + mx + m^2 + a = 0$

So, $m = -3, m^2 + a = 2 \Rightarrow a = -7$

$p^2 + q^2 + pq = 1^2 + 2^2 + 1 \times 2 = 7 = -a$

5. **(c)** $\boxed{\text{Using T-1}}$ Let $\alpha = 1, \beta = 2$

Then the equation is $x^2 - 3x + 2 = 0$

Comparing with $ax^2 + bx + c = 0 \Rightarrow a = 1, b = -3, c = 2$

$S_n = \alpha^n + \beta^n = 1 + 2^n$

Then $a\,S_{n+1} + cS_{n-1} = 1 \times (1 + 2^{n+1}) + 2\,(1 + 2^{n-1})$

$= 1 + 2 \times 2^n + 2 + 2^n$

$= 3 + 3 \times 2^n = 3\,(1 + 2^n)$

$= -bS_n$ [$\because b = -3, S_n = 1 + 2^n$]

6. **(d)** $\boxed{\text{Using T-1}}$ Let $a = 1, b = 2$, then the equation is

$(x-1)(x-2) = 1 \Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \dfrac{3 \pm \sqrt{5}}{2}$

$\Rightarrow x = \dfrac{3+\sqrt{5}}{2} \in (b, \infty)$ and $x = \dfrac{3-\sqrt{5}}{2} \in (-\infty, a)$

7. **(a)** $\boxed{\text{Using T-1}}$ Let $\alpha = 1$, $\beta = 2$ are two consecutive integers which are roots of the equation

Then, $x^2 - 3x + 2 = 0$

Comparing with $x^2 - bx + c = 0 \Rightarrow a = 1, b = 3, c = 2$

$\therefore b^2 - 4c = 9 - 4 \times 2 = 1$

8. **(b)** $\boxed{\text{Using T-1}}$ Let k = –3

Then, the equation is $x^2 - 6x + 4 = 0 \Rightarrow x = 3 \pm \sqrt{5}$

So, roots are real and unequal.

9. **(a)** $\boxed{\text{Using T-2}}$ $\because P + Q + R = a(b-c) + b(c-a) + c(a-b) = 0$

$\Rightarrow$ $P = R$

$\Rightarrow$ $a(b-c) = c(a-b)$

$\Rightarrow$ $ab - ac = ac - bc$

$\Rightarrow$ $ab + bc = 2ac$

$\Rightarrow$ $b(a+c) = 2ac$

$\Rightarrow$ $b = \dfrac{2ac}{a+c}$

$\Rightarrow$ a, b, c are in H.P.

10. **(d)** $\boxed{\text{Using T-3}}$ $\because$ $2\log_{10}x - \log_x 0.01 = 2\log_{10}x - \log_x(10^{-2})$

$= 2\log_{10}x + 2\log_x 10$

Applying AM $\geq$ GM

$$\frac{2\log_{10}x + 2\log_x 10}{2} \geq (2\log_{10}x \cdot 2\log_x 10)^{1/2}$$

$$\Rightarrow 2\log_{10}x + 2\log_x 10 \geq 2\left(4\log_{10}x \cdot \frac{1}{\log_{10}x}\right)^{1/2} \quad \left[\because \log_x 10 = \frac{1}{\log_{10}x}\right]$$

$\Rightarrow 2\log_{10}x + \log_x 0.01 \geq 2 \times 2$

$\Rightarrow 2\log_{10}x + \log_x 0.01 \geq 4$

$\therefore$ Least value is 4.

14 Matrices and Determinants

MATRICES

Review of Key Notes and Formulae

1. **Definition:** A matrix is a rectangular arrangement of numbers (real or complex)

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$$

★ Order of matrix $A = m \times n$

★**Note:** *Upper Triangular Matrix:* $A = [a_{ij}]_{n \times n}$ is called a upper triangular matrix, if $a_{ij} = 0,\ \forall\ i > j$

Lower Triangular Matrix: $A = [a_{ij}]_{n \times n}$ is called a lower triangular matrix, if $a_{ij} = 0,\ \forall\ i < j$

2. **Alegebra of Matrices:** If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$
   (i) Addition of matrices: $A + B = [a_{ij} + b_{ij}]_{m \times n}$
   (ii) Subtraction of matrices: $A - B = [a_{ij} - b_{ij}]_{m \times n}$
   (iii) Multiplication of a matrix by a scalar: $\Rightarrow kA = [ka_{ij}]_{m \times n}$
   (iv) Multiplication of matrices: Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$. Then, AB is multiplication, is given by $C_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

3. **Positive Integral Powers of a Square Matrix:** If A is a square matrix, then
   (i) $A^{n+1} = A^n . A;\ n \in N$
   (ii) $A^m . A^n = A^{m+n}$
   (iii) $(A^m)^n = A^{mn};\ m, n \in N$

4. **Transpose of a Matrix:** $A' = A^T = [a_{ji}]_{n \times m}$

   Properties:

   (i) $(A^T)^T = A$ (ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(kA)^T = kA^T$ (iv) $(AB)^T = B^T A^T$

(v) $(A^n)^T = (A^T)^n$ (vi) $(ABC)^T = C^T B^T A^T$

5. Symmetric and Skew Symmetric Matrix:

(i) A square matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric, if $A^T = A$ *i.e.* $a_{ij} = a_{ji}$, $\forall$ i and j

(ii) A square matrix A is said to be skew-symmetrics, if $A^T = -A$ *i.e.* $a_{ij} = -a_{ji}$, $\forall$ i and j.

Properties:

(i) Elements of principal diagonals of a skew-symmetric matrix are all zero.

(ii) If A is a square matrix, then

(a) $A + A^T$ is symmetric

(b) $A - A^T$ is a skew-symmetric

(iii) If A is symmetric (or skew symmetric), then kA(k is a scalar) is also symmetric (or skew-symmetric) matrix

(iv) Every square matrix can be expressed as the sum of symmetric and skew-symmetric matrix.

i.e. $$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

6. Elementary Operations:

(i) Interchanging any two rows or columns

$$R_i \leftrightarrow R_j \text{ or } C_i \leftrightarrow C_j.$$

(ii) Multiplication of the element of any row or column by a non-zero scalar quantity.

$$R_i \to kR_i \text{ or } C_i \to kC_j$$

(iii) Addition of constant multiple of the elements of any row to the corresponding element of any row,

$$R_i \to R_i + kR_j \quad \text{or} \quad C_i \to C_i + kC_j$$

7. Trace of a Matrix: Sum of the diagonal elements of a square matrix A is called the trace of A, denoted by $tr(A)$.

8. Some Special Types of Matrices:

(i) Orthogonal matrix $\to AA^T = I_n = A^T A$

(ii) Idempotent matrix $\to A^2 = A$

(iii) Involuntary matrix $\to A^2 = I_n$

(iv) Nilpotent matrix $\to A^m = O$, where 'm' is the least positive integer such that $A^m = O$, then 'm' is called the index of the nilpotent matrix 'A'.

(v) Periodic matrix $\to$ If $A^{k+1} = A$, where 'k' is a positive integer, then A is a periodic matrix and 'k' is known as period of matrix 'A'.

DETERMINANTS

Review of Key Notes and Formulae

1. **Definition:** An expression expressed in equal number of rows and column and put between two vertical lines is called as determinant.

2. **Expansion of Determinant:**

(i) Two order: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$

(ii) Three order: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3)$

$+ c_1(a_2b_3 - b_2a_3)$

3. **Minor**

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow$ Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

4. **Cofactor:** $C_{ij} = (-1)^{i+j} M_{ij}$

Therefore, cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11}$

★**Note:** (i) $\Delta = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13}$; etc.
(ii) $a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} = 0$; etc.

5. **Properties of Determinants:**

P-1: The value of the determinant remains unaltered, if two rows and columns are interchanged

P-2: If any two rows or columns of a determinant are interchanged, the numerical value is unaltered but there will be change in sign.

P-3: If any two rows or columns are identical then the value of determinant will be zero.

P-4: If all the elements of a row or column are multiplied by the same number, then the determinant is multiplied by that number.

P-5: The value of determinant is unaltered, by adding the elements of any row or column to the some multiple of the corresponding elements of any other row or column.

6. **System of Equation:** Three variables equation

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y\ c_3z = d_3$

Find the following determinants:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now follow the steps below.

Check value of Δ

- $\Delta \neq 0$ → Consistent and unique solution
  $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$
- $\Delta = 0$
  - At least one of $\Delta_1, \Delta_2, \Delta_3$ is not zero → Inconsistent solution
  - $\Delta_1 = \Delta_2 = \Delta_3 = 0$ → Consistent and infinite solution

★**Note:** *Trivial solution*: If $\Delta \neq 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then system of equation has trivial solution $(x = y = z = 0)$

TIPS & TRICKS!

TIPS AND TRICKS: (T-1)

To find A^n; where A is 2×2 matrix and 'n' is a multiple of '2' of the type

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$$

Use characteristic Eqn: $A^2 - \text{trace}(A) + |A|\, I = 0$

★**Note:** Trace (A) = Sum of the elements of diagonals.

Illustration 1

If $A = \begin{bmatrix} -7 & 5 \\ -10 & 7 \end{bmatrix}$; then A^{2000} is equal to:

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\because$ trace $(A) = 0$, $|A| = 1$

$\Rightarrow A^2 - tr(A) + |A|\, I = A^2 - 0 + I = 0$

Hence $A^{2000} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Illustration 2

If $A = \begin{bmatrix} 5 & 3 \\ 7 & -5 \end{bmatrix}$; then A^{500} is equal to:

Short-cut solution :

Using T-1 $\because$ tr(A) = 0 and $|A| = -46$

$\Rightarrow A^2 - tr(A) + |A|I = A^2 - 0 - 46I = 0 \Rightarrow A^{500} = (46)^{250}\ I$

TIPS AND TRICKS: (T-2)

Inverse of 2 × 2 matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \Rightarrow A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} a & -b \\ -c & a \end{bmatrix}$

★ **Note:** Interchange the diagonal elements and change the sign of non-diagonal elements.

Illustration 3

Find the inverse of $A = \begin{bmatrix} 3 & 5 \\ 2 & -4 \end{bmatrix}$

Short-cut solution :

Using T-2 $A^{-1} = \dfrac{1}{-22}\begin{bmatrix} -4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2/11 & 5/22 \\ 1/11 & -3/22 \end{bmatrix}$

Illustration 4

If inverse of $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ is equal to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then find the value of $a + b + c + d$

Short-cut solution :

Using T-2 $\because A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(\because |A| = 1)$

$\Rightarrow a + b + c + d = 11$

TIPS AND TRICKS: (T-3)

Determinant of triangular matrix.

If $A=\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ OR $A=\begin{bmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{bmatrix}$

Upper Triangular Matrix   Lower Triangular Matrix

Then, $|A|=$ Product of diagonal elements $=(a.d.f)$

Illustration 5

If $A=\begin{bmatrix} -1 & 22 & 333 \\ 0 & 2 & 44 \\ 0 & 0 & 7 \end{bmatrix}$, then find $|A|$

Short-cut solution :

$\boxed{\text{Using T-3}}$ $|A| = -1 \times 2 \times 7 = -14$

Illustration 6

If $A=\begin{bmatrix} 0 & q & r \\ p & (a-b) & (b-c) \\ 0 & 0 & (c-b) \end{bmatrix}$, then find $|A|$.

Short-cut solution :

(Determinant)

$\boxed{\text{Using T-3}}$ Applying $R_1 \leftrightarrow R_2 \Rightarrow$ $A=(-1)\begin{vmatrix} p & (a-b) & (b-c) \\ 0 & q & r \\ 0 & 0 & (c-b) \end{vmatrix}$

Hence it becomes upper triangular matrix $\Rightarrow |A| = pq(b-c)$

TIPS AND TRICKS: (T-4)

Method of substitution.

Illustration 7

If $\Delta_r=\begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ \alpha & \beta & r \\ 2^{n-1} & 3^{n-1} & 5^{n-1} \end{vmatrix}$, then the value of $\sum_{r=1}^{n}\Delta_r$ is equal to

(a) 0 (b) $\alpha\beta r$

(c) $\alpha+\beta+r$ (d) $\alpha \,.\, 2^n+\beta \,.\, 3^n+r \,.\, 4^n$

Short-cut solution :

$\boxed{\text{Using T-4}}$ Let $r = 1$, $n = 1$

$$\Rightarrow \Delta_1 = \begin{vmatrix} 1 & 2 & 4 \\ \alpha & \beta & r \\ 1 & 2 & 4 \end{vmatrix} = 0 \qquad (\because \text{ Two rows are identical}) \qquad \textbf{Ans. (a)}$$

Illustration 8

If $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & b+c & -2c \end{vmatrix} = \alpha(a+b)(b+c)(c+a) \neq 0$, then '$\alpha$' is equal to

[AIEEE 2012]

(a) 1 (b) $a+b+c$ (c) abc (d) 4

Short-cut solution :

$\boxed{\text{Using T-4}}$ Let $a = 0$, $b = 1$, $c = 2$

$$\therefore \begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = \alpha(1)(3)(2) \Rightarrow 10 + 14 = 6\alpha \Rightarrow \alpha = 4 \qquad \textbf{Ans. (d)}$$

Illustration 9

The value of $\Delta = \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$ is equal to

(a) $(a+b+c)^3$ (b) 0

(c) $2(a+b)(b+c)(c+a)$ (d) $(a+b)^3$

Short-cut solution :

$\boxed{\text{Using T-4}}$ Let $a = b = c = 1$

$$\Rightarrow \Delta = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3(8) + 1(-4) - 4 = 16$$

Now, checking options (a), (b), (c), (d) for $a = b = c = 1$

Hence, option (c) $= 2(2)(2)(2) = 16$ **Ans. (c)**

Illustration 10

If a, b, c are in G.P., then the value of determinant

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & o \end{vmatrix} \text{ is equal to}$$

(a) 0 (b) 1 (c) –2 (d) –1

Short-cut solution :

Using T-4 $\because a, b, c$ are in G.P., Let $a = 1$, $b = 2$ and $c = 4$ and $x = 0$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 0 \end{vmatrix} = 1(-16) - 2(-8) + 2(0) = 0$$

Ans. (a)

Illustration 11

For positive numbers x, y, z the numerical value of the determinant

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} \text{ is}$$

Short-cut solution :

Using T-4 Let $x = y = z = 2$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Illustration 12

The value of $\Delta = \begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$ is

(a) $2x!(x+1)!$ (b) $2x!\,(x+1)!\,(x+2)!$

(c) $2x!\,(x+3)!$ (d) $2(x+1)!\,(x+2)!\,(x+3)!$

Short-cut solution :

Using T-4 Let $x = 0$

$$\Rightarrow \Delta = \begin{vmatrix} 0! & 1! & 2! \\ 1! & 2! & 3! \\ 2! & 3! & 4! \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 6 \\ 2 & 6 & 24 \end{vmatrix} = 4$$

Now, check option (a), (b), (c), (d) for $x = 0$

$\Rightarrow 2(0!)(1!)(2!) = 4$ **Ans. (b)**

TIPS AND TRICKS: (T-5)

Short trick to find inverse of 3×3 matrix.

Finding inverse of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$; follow the steps below (procedure).

Step 1:

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

Copy 1^{st} column and II^{nd} column from A.

Step 2:

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \end{array}$$

Copy first row and II^{nd} row from step 1.

Step 3:

$$\begin{array}{c|cccc} a & b & c & a & b \\ \hline d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \end{array}$$

Neglect first row

Neglect first column

✶ From up to down arrow take positive sign.

✶ From down to up arrow take negative sign.

Therefore, $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} ei - hf & fg - id & dh - eg \\ hc - bi & ai - cg & bg - ah \\ bf - ec & cd - af & ae - bd \end{bmatrix}^T$

★**Note:** We are moving up to down but in the A^{-1} matrix we arrange along left to right.

Illustration 13

Find the inverse of $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Short-cut solution :

$\boxed{\text{Using T-5}}$

$$\begin{array}{c|cccc} 1 & 2 & -2 & 1 & 2 \\ \hline -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array}$$

Neglecting

Neglecting

$$\because |A| = 1 \Rightarrow A^{-1} = 1 \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Illustration 14

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Short-cut solution :

$\boxed{\text{Using T-5}}$

$$\begin{array}{c|cccc} 0 & 1 & 2 & 0 & 1 \\ \hline 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{array}$$

Neglecting

Neglecting

$$\because |A| = -2 \Rightarrow A^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

SHORTCUTS: (SC-1)

(i) $A(\text{adj } A) = (\text{adj } A) . A = |A|\, I_n$

(ii) $\text{adj}(\text{adj } A) = |A|^{n-2} . A$

(iii) $|\text{adj}(\text{adj } A)| = |\,A\,|^{(n-1)^2}$

Illustration 15

If $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 7 & 8 \\ -1 & 2 & 0 \end{bmatrix}_{3\times 3}$, then $|\text{adj}(\text{adj } A)|$ is equal to

(a) ± 12

(b) 12^2

(c) 12^4

(d) 12^6

Short-cut solution :

Using SC-1 (iii) $\because |A| = 2(-16) - 1(8) + 4(7) = -12$

Therefore, $|\text{adj}(\text{adj } A)| = |A|^{(3-1)^2} = (12)^4$ **Ans. (c)**

Illustration 16

If $A = \begin{bmatrix} i & 0 & 0 \\ 0 & -(1+i) & 0 \\ 0 & 0 & i \end{bmatrix}$; where $i = \sqrt{-1}$, then $|\text{adj}(\text{adj } A)|$ is equal to

(a) 4

(b) – 4

(c) ± 4

(d) ± 2

Short-cut solution :

Using SC-1 $\because |A| = -i \times (1 + i)i = 1 + i$ ($\because$ Triangular Matrix)

$\Rightarrow |\text{adj}(\text{adj } A)| = |A|^{(3-1)^2} = (1 + i)^4 = -4$ **Ans. (b)**

TECHNIQUE

Orthogonal matrix:

Suppose A be a square matrix with real entries of order n and A^T be transpose of matrix A, then $AA^T = I$

where I is identity matrix.

If A is orthogonal matrix, then $|A| = 1$ or 1 and $A^{-1} = A^T$.

Illustration 17

Find A^{-1} of the matrix A, where $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

Short-cut solution :

Using Tech.

$$A^T = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = T$$

$$\Rightarrow \quad A^{-1} = A^T$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

Concept Booster Exercise

1. The inverse of $A = \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$ is equal to

(a) $\begin{bmatrix} -2 & 5/2 \\ 3 & -7/2 \end{bmatrix}$ (b) $\begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & -5 \\ -6 & 4 \end{bmatrix}$ (d) None of these

2. If $A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ then A^{780} is equal to

(a) I (b) $2I$ (c) $3I$ (d) $4I$

3. The value of the determinant of $A = \begin{bmatrix} -2 & 4 & 5 \\ 0 & -6 & 12 \\ 0 & 0 & 368 \end{bmatrix}$ is equal to

(a) 4426 (b) 4406 (c) 4436 (d) 4416

4. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then A^{-50}, when $\theta = \dfrac{\pi}{12}$ is **[JEE M 2019]**

(a) $2I$ (b) $3I$ (c) I (d) $4I$

5. The value of determinant $\begin{bmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{bmatrix}$ is _____

(a) 1 (b) 0

(c) $a^2 + b^2 + c^2$ (d) None of these

6. If a, b, c are sides of scalene triangle, then value of $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ is

[JEE M 2013]

(a) negative (b) non-negative

(c) positive (d) non-positive

7. If $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = k\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$, $\lambda \neq 0$, then 'k' is equal to:

[JEE M 2014]

(a) $4\lambda\, abc$ (b) $-4\lambda abc$ (c) $4\lambda^2$ (d) $-4\lambda^2$

8. If $\Delta_r = \begin{vmatrix} r & 2r-1 & 3r-2 \\ \frac{n}{2} & n-1 & a \\ \frac{1}{2}n(n-1) & (n-1)^2 & \frac{1}{2}(n-1)(3n+4) \end{vmatrix}$, then value of $\sum_{r-1}^{n-1} \Delta_r$ is:

[JEE M 2014]

(a) depends only on 'n' (b) depends only on 'a'
(c) independent of both 'a' and 'n' (d) depends on both 'a' and 'n'

9. If $a^2 + b^2 + c^2 = -2$ and $f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$ then $f(x)$ is a polynomial of degree **[AIEEE 2005]**

(a) 1 (b) 0 (c) 3 (d) 2

10. If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = px - 12$, then

(a) $p = -24$ (b) $p = 24$ (c) $p = 12$ (d) $p = 0$

11. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and **[JEE M 2014]**

$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then

'k' is equal to

(a) $\alpha\beta$ (b) $\frac{1}{\alpha\beta}$ (c) 1 (d) -1

12. If $|adj(adj\ A)| < 0$ and order of matrix is less than '4', for real elements, then order of matrix is

(a) 3 (b) 2 (c) can't say (d) None

13. If $|adj(adj\ A)| = 16$ and $|A| = 2$, then order of matrix is

(a) 2 (b) 3 (c) 4 (d) 6

14. Let a, b, c be such that $b(a + c) \neq 0$. If **[AIEEE 2009]**

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0,$$

then value of 'n' is

(a) Any even integer (b) Any odd integer

(c) Any integer (d) Zero

15. If $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$, then inverse of A is

(a) A (b) $\frac{1}{3}A$

(c) A^T (d) $3A^T$

NUMERICAL VALUE PROBLEMS

16. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$ then $f(100)$ is ____

17. If p^{th}, q^{th}, r^{th} term of a GP are l, m, n then the value of $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ is equal to _____ **[AIEEE 2002]**

18. If $a_1, a_2, a_3, \ldots a_n$ are in G.P., then the value of the determinant

$$\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$ is: **[AIEEE 2004]**

19. If $A = \begin{bmatrix} 9 & 10 \\ -8 & -9 \end{bmatrix}$ and $A^{2560} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $a + b + c + d$ _____

20. If $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, then $|adj(adj\ A)|$ is equal to _____

Solutions

1. **(b)** $\boxed{\text{Using T-2}}$ $\because |A| = -2 \Rightarrow A^{-1} = \begin{bmatrix} -7/2 & 5/2 \\ 3 & -2 \end{bmatrix}$

2. **(a)** $\boxed{\text{Using T-1}}$ $\because A^2 - \text{trace }(a) + |A|\, I = 0$ and $|A| = 1$

$\Rightarrow A^2 - (-2+2)\, A + I = 0 \Rightarrow A^{780} = (-I)^{390} = I$

3. **(d)** $\boxed{\text{Using T-3}}$ $\because A$ is upper triangular matrix.

$\Rightarrow |A| = (-2)(-6)(368) = 4416$

4. **(c)** $\boxed{\text{Using T-1}}$ $\because A = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} (R_1 \leftrightarrow R_2)$

$\therefore A^2 - tr(a) + |A|\, I = 0 \Rightarrow A^2 + (-\sin^2\theta - \cos^2\theta)I = 0$

Hence, $A^2 = I \Rightarrow A^{-50} = (I)^{-25} = I$

5. **(b)** $\boxed{\text{Using T-4}}$ Let $a = b = c = 1$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

6. **(a)** $\boxed{\text{Using T-4}}$ $\because \Delta$ is scalene, Let $a = 1, b = 2, c = 3$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} \Rightarrow \Delta = 1(5) - 2(1) + 3(-7) = -18 \text{ (Negative)}$$

7. **(c)** $\boxed{\text{Using T-4}}$ Let $a = 1, b = 2, c = 3$ and $\lambda = 1$

$$\text{Therefore, } \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 0 & 1 & 4 \end{vmatrix} = k \begin{vmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow k = 4$

Now, check options for $a = 1, b = 2, c = 3$ and $\lambda = 1$

$\Rightarrow 4\lambda^2 = 4$

8. **(c)** $\boxed{\text{Using T-4}}$ Let $n = 2$ and $r = 1$

$$\Rightarrow \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & a \\ 1 & 1 & 5 \end{vmatrix} = 0 \Rightarrow \text{Independent of } a \text{ and } n$$

9. **(d)** $\boxed{\text{Using T-4}}$ $\because a^2 + b^2 + c^2 = 1$, Let $a^2 = 0$, $b^2 = -1$, $c^2 = -1$

$$\Rightarrow f(x) = \begin{vmatrix} 1 & 0 & 0 \\ x & 1-x & 0 \\ x & 0 & 1-x \end{vmatrix} = 1 - 2x + x^2$$

$\Rightarrow$ Degree = 2

10. **(b)** $\boxed{\text{Using T-4}}$ Let $x = 1$

$$\Rightarrow \begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = p - 12 \Rightarrow p = 24$$

11. **(c)** $\boxed{\text{Using T-4}}$ Let $\alpha = -1$ and $\beta = 2 \Rightarrow f(n) = (-1)^n + 2^n$

Therefore, $f(1) = 1, f(2) = 5, f(3) = 7, f(4) = 17$

$$\Rightarrow \begin{vmatrix} 3 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{vmatrix} = k(1+1)^2(1-2)^2(-3)^2 \Rightarrow k = 1$$

Now, check options (a), (b), (c), (d) for $\alpha = -1$ and $\beta = 2$

12. **(b)** $\boxed{\text{Using SC-1 (iii)}}$ $\because |\text{adj}(\text{adj } A)| < 0$

$\Rightarrow |A|^{(n-1)^2} < 0 \Rightarrow |A| < 0$ and $(n-1)^2$ is odd

Hence, if $n = 2$ then $(n-1)^2$ is odd.

13. **(b)** $\boxed{\text{Using SC-1 (iii)}}$ $\because |\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$\Rightarrow 16 = (2)^{(n-1)^2} = 2^4 \Rightarrow n = 3$

14. **(b)** $\boxed{\text{Using T-4}}$ $\because b(a+c) \neq 0$, Let $a = b = c = 1$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} + (-1)^n \begin{vmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 8 + (-1)^n\, 8 = 0$ is possible only if 'n' is odd integer.

15. **(c)** $\boxed{\text{Using Tech.}}$ $\because AA^T = I \Rightarrow A^{-1} = A^T$

16. **(0)** $\boxed{\text{Using T-4}}$ $\because f(0) = f(1) = f(-1) = \ldots = 0$

Hence, $f(x)$ is independent of x

$\Rightarrow f(0) = f(100) = 0$

17. **(0)** $\boxed{\text{Using T-4}}$ Let $p = 1, q = 2, r = 3$ and $l = 2, m = 2^2, n = 2^3$

$$\Rightarrow \begin{vmatrix} \log 2 & 1 & 1 \\ \log 2^2 & 2 & 1 \\ \log 2^3 & 3 & 1 \end{vmatrix} = \log 2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = 0$$

18. **(0)** $\boxed{\text{Using T-4}}$ Let the G.P. be $2, 2^2, 2^3, 2^4, \ldots$ and $n = 1$

$$\Rightarrow \begin{vmatrix} \log 2 & \log 2^2 & \log 2^3 \\ \log 2^4 & \log 2^5 & \log 2^6 \\ \log 2^7 & \log 2^8 & \log 2^9 \end{vmatrix} = (\log 2)^3 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

19. **(2)** $\boxed{\text{Using T-1}}$ $\because A^2 - tr(a) + |A|I = 0$ and $|A| = -1$

$$\Rightarrow A^2 - 0 - I = 0 \Rightarrow A^{2560} = (I)^{1280} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $a + b + c + d = 2$

20. **(1)** $\boxed{\text{Using SC-1 (iii)}}$ $\because |adj(adj\ A)| = |A|^{(2-1)^2} = \cos^2 x + \sin^2 x$

$\Rightarrow |adj(adj\ A)| = 1$

15 Permutations and Combinations

Review of Key Notes and Formulae

1. Fundamental Principle of Counting:

(i) *Multiplication principle*: If an event can occur in 'm' different ways following which another event can occur in 'n' different ways then number of ways of two consecutive event occurring in a definite way occurs in '$m \times n$' ways.

(ii) *Addition principle*: If an event occur in 'm' ways and another event, which is independent of the first, can occur in 'n' ways. Then either of the two events can occur in "$m + n$" ways.

2. Permutation: Refers to an arrangements of objects in a definite order, objects may be different or alike taking same or all at a time.

Number of permutations of 'n' distinct objects taken 'r' at a time $0 \le r \le n$ is

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

3. Combination: Refers to selection of objects, order in significant objects may be alike or different taken same or all at a time.
Number of combination of 'n' distinct objects taken 'r' at a time is

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!r!}}$$

4. Formation of Groups :

(i) $(m + n)$ distinct objects to be divided into two groups $\left\langle \begin{matrix} m \\ n \end{matrix} \right.$ $m \neq n$

$\Rightarrow {}^{m+n}C_m$

(ii) $2m$ distinct objects to be divided into two groups $\left\langle \begin{matrix} m \\ m \end{matrix} \right.$ $\Rightarrow \dfrac{{}^{2m}C_m}{2!}$

5. **Circular Permutation:** In a circular permutation first we fix the position of one of the objects and then arrange other objects in all possible ways.

Case 1: When clockwise and anti-clockwise orders are treated as different.

(i) Taking all at a time $= \frac{^nP_n}{n} = (n-1)!$

(ii) Taking 'r' at a time $= \frac{^nP_r}{r}$

Case 2: When clockwise and anti-clockwise orders are treated same.

(i) Taking all at a time $= \frac{^nP_n}{2n} = \frac{(n-1)!}{2}$

(ii) Taking 'r' at a time $= \frac{^nP_r}{2r}$

6. **Dearrangements:** If 'n' distinct objects are arranged in a row, then the number of ways in which they can be dearranged so that no one of them occupies the place assigned to it is:

$$D(n) = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+...(-1)^n\frac{1}{n!}\right]$$

7. **Some Important Results**

(i) Given, 'n' distinct points in the plane, no three of which are collinear

(a) Number of line segments formed $= {^nC_2}$

(b) Number of triangles formed $= {^nC_3}$

(ii) The number of diagonals in a n-sided polygon $= {^nC_2} - n$

(iii) Given, 'n' points on the circumference of a circle then

(a) Number of straight lines $= {^nC_2}$

(b) Number of triangles $= {^nC_3}$

(c) Number of Quadrilaterals $= {^nC_4}$

(iv) Exponent of prime 'p' is $n!$ is

$$E_P(n!) = \left[\frac{n}{P}\right]+\left[\frac{n}{P^2}\right]+\left[\frac{n}{P^3}\right]+...+\left[\frac{n}{P^r}\right]$$

(where, [ ] is greatest Integer function)

TIPS AND TRICKS: (T-1)

Short trick to find rank of the word when arranging letters as in a dictionary.

STEPS

Step 1: Rank the alphabets in a alphabetical order. (Give numbering to each letter)

Step 2: Below each letter write the number of digits which is right side on a particular letter and less than the signed number. (Let $P_1, P_2, P_3, ..., P_n$)

Step 3: Start writing 0!, 1!, 2!, 3!, ..., $(n-1)!$ from the right side.

Step 4: Then rank of a word = $\{P_n 0! + P_{n-1} 1! + ... + a_n (n-1)!\} + 1$

Illustration 1

Find the rank of the word TOUGH arranged in dictionary.

Short-cut solution :

Using T-1

4	3	5	1	2	←	Step 1
T	O	U	G	H		
3	2	2	0	0	←	Step 2
4!	3!	2!	1!	0!	←	Step 3

$\Rightarrow$ Rank = $\{3.4! + 2.3! + 2.2! + 0.1! + 0.0!\} + 1 = 89$

Illustration 2

Find the rank of the word PRIME as arranged in dictionary.

Short-cut solution :

Using T-1

4	5	2	3	1	←	Step 1
P	R	I	M	E		
3	3	1	1	0	←	Step 2
4!	3!	2!	1!	0!		

$\Rightarrow$ Rank = $\{3.4! + 3.3! + 1.2! + 1.1! + 0.0!\} + 1 = 94$

Illustration 3

Find the rank of the word SECRET as arranged in the dictionary.

Short-cut solution :

Using T-1

4	2	1	3	2	5
S	E	C	R	E	T
$\frac{4}{2!}$	$\frac{1}{2!}$	$\frac{0}{2!}$	$\frac{1}{1!}$	$\frac{0}{1!}$	$\frac{0}{2!}$

As 'E' is repeated twice in both letters.

5! 4! 3! 2! 1! 0!

$$\Rightarrow \text{Rank} = \left\{\frac{4\times 5!}{2!}+\frac{4!}{2!}+0+\frac{2!}{1!}+0+0\right\}+1 = 255$$

TIPS AND TRICKS: (T-2)

Short trick to competition/games/matches type problems then,

$$(T) \text{ Total number of matches} = \frac{1}{2} \times N \times (N-1) \times p$$

Where, $N \rightarrow$ Number of teams, $p \rightarrow$ Number of times

Illustration 4

In a hockey league there are 10 teams, each plays every other team once. If they can play 5 games every week then how many weeks long is this season.

Short-cut solution :

Using T-2 Here, $N = 10$ and $p = 1$

$$\Rightarrow \quad T = \frac{1}{2} \times 10 \times 9 \times 1 = 45$$

Since, they can play 5 games every week.

Hence, number of weeks $= \frac{45}{5} = 9$ weeks.

Illustration 5

In a game, 153 matches are played. Every 2 teams played once with each other. Find the number of teams.

Short-cut solution :

Using T-2 Here, $T = 153$, $p = 1$

$$\Rightarrow \quad 153 = \frac{1}{2}N(N-1) \times 1 \Rightarrow N^2 - N - 306 = 0$$

Hence, number of teams = 18.

TIPS AND TRICKS: (T-3)

Short trick to find total number of factors or proper factors.

If 'P' are number of factors of one type, 'q' are number of factors of second type, 'r' are number of factors of third type and 's' are number of factors of fourth type and so on. Then,

Total number of factors = $(p + 1)(q + 1)(r + 1)(s + 1)$.

Illustration 6

Find the total number of factors of 4200.

Short-cut solution :

Using T-3 $\because 4200 = 2^3 \times 3^1 \times 5^2 \times 7^1$

Here, $p = 3$, $q = 1$, $r = 2$, $s = 1$

$\Rightarrow$ Total number of factors $= (3 + 1)(1 + 1)(2 + 1)(1 + 1) = 48$

Illustration 7

Find the proper factors of 2520.

Short-cut solution :

Using T-3 $\because 2520 = 2^3 \times 3^2 \times 5^1 \times 7^1$

Here, $p = 3$, $q = 2$, $r = 1$, $s = 1$

$\Rightarrow$ Total number of factors $= 4 \times 3 \times 2 \times 2 = 48$

Now, proper divisors $= 48 - 2 = 46$

$$\downarrow$$

$$\left\{\begin{matrix}\text{2520 is a factor of itself and} \\ \text{'1' is a factor of every number}\end{matrix}\right\}$$

TIPS AND TRICKS: (T-4)

Short trick to find sum of all the numbers that can be formed using the digits 1 to 9 taking all at a time.

Sum of all numbers $= s \times (n - 1)!\,\dfrac{(10^n - 1)}{10 - 1}$

where, $s \rightarrow$ Sum of all given digits.

$n \rightarrow$ Number of digits

Illustration 8

Find the sum of all numbers that can be formed using the digits 2, 3, 4, 5 taken all at a time.

Short-cut solution :

$\boxed{\text{Using T-4}}$ Here, $s = 14$, $n = 4$

$$\Rightarrow \text{Sum of all numbers} = 14 \times (4-1)!\,\frac{(10^4-1)}{10-1} = 93324$$

Illustration 9

Find the sum of all 4-digit numbers that can be formed using the digits 0, 5, 7, 9 taken all at a time.

Short-cut solution :

$\boxed{\text{Using T-4}}$ If '0' comes first than the numbers will be three digit numbers.

$$\text{So, } \left\{\begin{matrix} \text{Required} & = & \text{"Sum of numbers"} & - & \text{"Sum of numbers"} \\ \text{Sum} & & \text{including zero} & & \text{excluding zero} \\ & & (0,5,7,9) & & (5,7,9) \end{matrix}\right\}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\text{Sum} = \frac{6\times 3!(10^4-9)}{9} - 6 \times 2!\,\frac{(10^3-1)}{9}$$

$$= 39996 - 1332 = 38664$$

TIPS AND TRICKS: (T-5)

Short trick for distribution problems.

Case 1: If 'n' distinct things are distributed among 'm' people

If all things can be given to same person	If all things cannot be given to same person
$\Downarrow$	$\Downarrow$
Number of ways = m^n	Number of ways = $m^n - m$

Case 2: If 'n' identical things are distributed among 'm' people

If all things can be given to same person	If all things cannot be given to same person
$\Downarrow$	$\Downarrow$
Number of ways = ${}^{n+m-1}C_{m-1}$	Number of ways = ${}^{n-1}C_{m-1}$

Illustration 10

Find the number of ways in which 9 distinct toys can be distributed among 5 children.

Short-cut solution :

Using T-5 (Case-1) Here, $n = 9$ and $m = 5$

$\Rightarrow$ Number of ways $= m^n = 5^8 = 390625$

Illustration 11

Find the number of ways in which 11 different prizes can be distributed amongst 7 people so that every person receives almost 10 prizes.

Short-cut solution :

Using T-5 (Case-1) Here, $n = 11$ and $m = 7$

$\Rightarrow$ No. of ways $= m^n - m = 7^{11} - 7$

Illustration 12

Find the number of ways in which 30 apples can be distributed amongst 5 persons.

Short-cut solution :

Using T-5 (Case-2) Here, $n = 30$, $m = 5$

$\Rightarrow$ No. of ways $= {}^{n+m-1}C_{m-1} = {}^{30+5-1}C_{5-1} = {}^{34}C_4$

Illustration 13

Find the number of ways of distribution of 9 identical balls in 4 distinct boxes so that none of the box is empty.

Short-cut solution :

Using T-5 (Case-2) Here, $n = 9$ and $m = 4$

$\Rightarrow$ No. of ways $= {}^{n-1}C_{m-1} = {}^{9-1}C_{4-1} = {}^{8}C_3$

SHORTCUTS: (SC-1)

If family of 'm' parallel lines are intersected by family of 'n' parallel lines then number of parallelogram will be:

$$\text{No. of } ||_{\text{gms}} = \frac{m.n(m-1)(n-1)}{4}$$

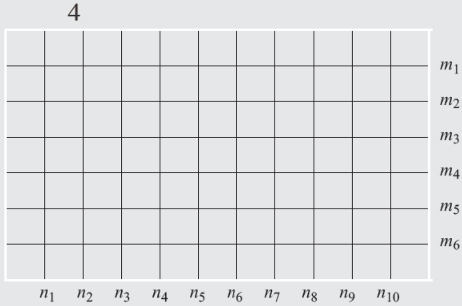


Illustration 14

If family of 4 parallel lines are intersected by family of 3 parallel lines then number of parallelogram is:

Short-cut solution :

Using SC-1 Here, $m = 4, n = 3$

$$\Rightarrow \quad \text{No. of parallelograms} = \frac{mn(m-1)(n-1)}{4} = \frac{4\times3\times3\times2}{4} = 18$$

TECHNIQUE

If n articles (items) taken all at a time when p-items of one type are identical, q-items of second type are identical and r-items of third type are identical, then number of arrangements $= \dfrac{n!}{p!q!r!}$

Illustration 15

Find the number of words that can be formed from the letters of the word HINDUSTAN.

Short-cut solution :

Using Tech. Total number of letters n = 9

N is repeated two times, $p = 2$

$$\therefore \quad \text{Total number of words} = \frac{9!}{2!}$$

Concept Booster Exercise

1. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary.
The number of words that appear before the word COCHIN is **[AIEEE 2007]**
(a) 360 (b) 192 (c) 96 (d) 48

2. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is **[JEE M 2016]**
(a) 46^{th} (b) 59^{th} (c) 52^{nd} (d) 58^{th}

3. If the letters of the word SACHIN are arranged in all possible ways and these words are written out as in dictionary, then the word SACHIN appears at serial number **[AIEEE 2005]**
(a) 601 (b) 600 (c) 603 (d) 602

4. In a football match there are 9 teams. Each team plays each other twice. The total number of matches played are
(a) 56 (b) 72 (c) 64 (d) 102

5. There are '8' teams in a league, each plays with the other exactly once. If each game is between '2' teams, then the total number of games played are
(a) 14 (b) 35 (c) 42 (d) 28

6. The number of proper divisors of $a^p b^q c^r d^s$ where a, b, c, d are primes and $p, q, r, s \in N$, is
(a) $pqrs$ (b) $(p + 1)(q + 1)(r + 1)(s + 1) - 4$
(c) $pqrs - 2$ (d) $(p + 1)(q + 1)(r + 1)(s + 1) - 1$

7. The number of ways in which 10 identical apples can be distributed among 6 children so that each child receives at least one apple is:
(a) 126 (b) 252 (c) 378 (d) None of these

8. The number of ways of distributing 10 identical balls is 4 distinct boxes such that no box is emply is: **[AIEEE 2011]**
(a) ${}^8C_3$ (b) ${}^9C_2$ (c) ${}^8C_2$ (d) ${}^9C_3$

9. The sum of all numbers greater than 10,000 formed using 0, 2, 4, 6, 8 such that no digit being repeated is:
(a) 120×44444 (b) 120×43333 (c) 120 (d) 43333

10. The total number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 = 100$ is
(a) ${}^{102}C_3$ (b) ${}^{101}C_3$ (c) ${}^{103}C_3$ (d) None of these

11. The number of distinct words that starts with E and end with N formed from the letters of the word PERMUTATION is

(a) $\frac{9!}{2!}$ (b) $\frac{11!}{2!}$ (c) 9! (d) 11!

Numerical Value Problems

12. In a hockey league there are 16 teams, each plays every other team twice, then total number of games played are _____.

13. The total numbers of divisors of 75,600 are _____.

14. If chocolates of a particular brand are all identical then the number of ways in which we can choose 6 chocolates out of 8 different brands available in the market is equal to ${}^{P}C_6$ then P is _____.

15. If family of 6 parallel lines are intersected by family of 5 parallel lines then number parallelogram will be _____.

Solutions

1. **(c)** $\boxed{\text{Using T-1}}$

1	5	1	2	3	4
C	O	C	H	I	N
0	4	0	0	0	0
5!	4!	3!	2!	1!	0!

$\Rightarrow$ words before COCHIN $= 4 \times 4! = 96$

2. **(d)** $\boxed{\text{Using T-1}}$

4	3	1	2	2
S	M	A	L	L
$\frac{4}{2!}$	$\frac{3}{2!}$	$\frac{0}{2!}$	$\frac{0}{2!}$	$\frac{0}{1!}$
4!	3!	2!	1!	0!

$$\Rightarrow \text{Position} = \left(\frac{4}{2!} \times 4! + \frac{3.3!}{2} + 0 + 0 + 0\right) + 1 = 58$$

3. **(a)** $\boxed{\text{Using T-1}}$

6	1	2	3	4	5
S	A	C	H	I	N
5	0	0	0	0	0
5!	4!	3!	2!	1!	0!

$\Rightarrow$ Rank $= (5 \times 5! + 0 + 0 + 0 + 0 + 0) + 1 = 601$

4. **(b)** $\boxed{\text{Using T-2}}$ Here, $N = 9$, $p = 2$

$$\Rightarrow T = \frac{1}{2} \times N \times (N-1) \times p = \frac{1}{2} \times 9 \times 8 \times 2 = 72$$

5. **(d)** $\boxed{\text{Using T-2}}$ Here, $N = 8$, $p = 1$

$$\Rightarrow T = \frac{1}{2} \times N \times (N-1) \times p = \frac{1}{2} \times 8 \times 7 \times 1 = 28$$

6. **(d)** $\boxed{\text{Using T-3}}$ $\because$ a, b, c, d are prime numbers

$\Rightarrow$ Proper divisors $= (p+1)(q+1)(r+1)(s+1)$

(since, $abcd$ is multiple of itself)

7. **(a)** $\boxed{\text{Using T-5 (Case-2)}}$ Here, $n = 10$, $m = 6$

$\Rightarrow$ No. of ways $= {}^{n-1}C_{m-1} = {}^{9}C_5 = 126$

8. **(d)** $\boxed{\text{Using T-5 (Case-2)}}$ Here, $n = 10$, $m = 4$

$\Rightarrow$ No. of ways $= {}^{n-1}C_{m-1} = {}^{9}C_3$

9. **(b)** $\boxed{\text{Using T-4}}$ Here, $s = 20$, $n = 5$ and $n = 4$

$$\Rightarrow \text{Sum} = s \times (n-1)!\,\frac{(10^n - 1)}{10-1} - s \times (n-1)!\,\frac{(10^n - 1)}{10-1}$$

$$= 20 \times 4!\,\frac{(10^5 - 1)}{9} - 20 \times 3!\,\frac{(10^4 - 1)}{9}$$

$$= 20 \times 6(4 \times 11111 - 1111) = 120 \times 43333$$

10. **(c)** $\boxed{\text{Using T-5}}$ (Since, no condition is given in the question so, identical things can be distributed to same person also - Case 2)

$\Rightarrow$ No. of ways $= {}^{n+m-1}C_{m-1} = {}^{103}C_3$

11. **(a)** $\boxed{\text{Using Tech.}}$ Total number of letters are 11 out of which two (E and N) are fixed.

So, remaining number of letters $n = 9$

T is repeated twice, $p = 2$

$\therefore$ Number of distinct words that can be formed $= \dfrac{9!}{2!}$

12. **(240)** $\boxed{\text{Using T-2}}$ Here, $N = 16$ and $p = 2$

$$\Rightarrow T = \frac{1}{2} \times N \times (N-1) \times p = \frac{1}{2} \times 16 \times 15 \times 2 = 240$$

13. **(120)** $\boxed{\text{Using T-3}}$ $\because 75600 = 2^4 \times 3^3 \times 5^2 \times 7^1$

Here, $p = 4$, $q = 3$, $r = 2$, $s = 1$

$\Rightarrow$ Total number of factors $= (4 + 1)(3 + 1)(2 + 1)(1 + 1) = 120$

14. **(13)** $\boxed{\text{Using T-5 (Case-2)}}$ Here, $n = 6$ and $m = 8$

$\Rightarrow$ Total number of ways $= {}^{n+m-1}C_{m-1} = {}^{13}C_7 = {}^{13}C_6$

15. **(150)** $\boxed{\text{Using SC-1}}$ Here, $m = 6$ and $n = 5$

$$\Rightarrow \text{No. of } \|_{\text{gms}} = \frac{mn(m-1)(n-1)}{4} = \frac{6 \times 5 \times 5 \times 4}{4} = 150$$

16 Binomial Theorem

Review of Key Notes and Formulae

1. **Binomial Theorem for Positive Integral Index :** If 'n' is any positive integer then $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}.a + {}^nC_2 x^{n-2}.a^2 + ... + {}^nC_n a^n$
where x & a are real and ${}^nC_0, {}^nC_1, ... {}^nC_n$ are called binomial coefficients.
2. **General Term for Positive Integral Index:** General term in the expansion of $(x + a)^n$ is $T_{r+1} = {}^nC_r (x)^{n-r}.a^r$
where, 'a' can be positive, negative or 1.
3. **Properties of Binomial Expansion of $(x + a)^n$:**
   (i) Total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$.
   (ii) The binomial coefficients of term equidistant from the beginning and the end are equal i.e. ${}^nC_r = {}^nC_{n-r}$
4. **Middle Term in a Binomial Expansion:**
   (i) If 'n' is even then middle term $= \left(\frac{n}{2}+1\right)^{\text{th}}$ term
   (ii) If 'n' is odd then middle terms are $= \left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ term.
5. **Greatest Term in the Expansion of $(x + a)^n$:** Calculate $\dfrac{n+1}{\left|\frac{x}{a}\right|+1} = p(\text{say})$
   (i) If 'p' is an integer then T_p and T_{p+1} are the numerically greatest term
   (ii) If 'p' is not an integer with 'm' as integral part of $\dfrac{n+1}{\frac{x}{a}+1}$, then T_{m+1} is the greatest term.
6. **Some Important Results on Binomial Coefficients:**
If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$
   (i) $C_0 + C_1 + C_2 + ... + C_n = 2^n$
   (ii) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(iii) $C_0 - C_1 + C_2 - C_3 + \dots (-1)^n C_n = 0$

(iv) $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = 2^{n-1}(n+2)$

(v) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(vi) $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n$

7. **Binomial Theorem for Any Index.** If 'n' is any rational number then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \dots +, |x| < 1$$

General term: $T_{r+1} = \frac{n(n-1)(n-2)\dots[n-(r-1)]}{r!} = x^r$

TIPS & TRICKS **TIPS AND TRICKS: (T-1)**

Short trick to find the sum of all coefficients in the expansion of

(i) $(ax+by)^n \Rightarrow$ sum of all coefficients $= (a+b)^n$

(ii) $(ax+by+cz)^n \Rightarrow$ sum of all coefficients $= (a+b+c)^n$

Illustration 1

What is the sum of the coefficients of all the terms in the expansion of $(45x-49)^4$?

(a) -256 (b) -100

(c) 100 (d) 256

Short-cut solution :

$\boxed{\text{Using T-1(i)}}$ Here, $a = 45$, $b = -49$ and $n = 4$

$\Rightarrow$ Sum of all coefficients $= (45-49)^4 = 256$ **Ans. (d)**

Illustration 2

In the expansion of $(1+x)^{50}$, sum of coefficients of odd powers of x is:

(a) 2^{26} (b) 2^{49}

(c) 2^{50} (d) 2^{51}

Short-cut solution :

$\boxed{\text{Using T-1(i)}}$ Here, $a = 1$, $b = 1$ and $n = 50$

So, sum of all coefficients $= (1+1)^{50} = 2^{50}$

(As we know that odd power coefficients will be half of total coefficients)

Hence, sum of coefficients of odd powers $= \frac{2^{50}}{2} = 2^{49}$ **Ans. (b)**

Illustration 3

The sum of the coefficients of all the terms in the expansion of $(1 - 3x + x^2)^{10}$ is:

(a) 2 (b) –1

(c) 1 (d) –2

Short-cut solution :

$\boxed{\text{Using T-1(ii)}}$ Here, $a = 1$, $b = -3$, $c = 1$ and $n = 10$

Therefore, sum of all coefficients = $(1 - 3 + 1)^{10} = 1$ **Ans. (c)**

Illustration 4

Sum of all coefficients of all the terms in the expansion of $(2x + 3y + 4z)^{20}$ is:

(a) 9^{10} (b) 20^9

(c) 10^9 (d) 9^{20}

Short-cut solution :

$\boxed{\text{Using T-1(ii)}}$ Here, $a = 2$, $b = 3$, $c = 4$ and $n = 20$

Hence, sum = $(2 + 3 + 4)^{20} = 9^{20}$ **Ans. (d)**

TIPS AND TRICKS: (T-2)

Short trick to find number of integral terms in the expansion of the type $(a^{1/p} + b^{1/q})^n$; where a and b are prime number.

$$\text{Integral term} = \begin{cases} \left[\dfrac{n}{\text{LCM }(p, q)}\right] + 1, & \text{where '}n\text{' is completely divisible by '}p\text{' or '}q\text{'.} \\[2ex] \left[\dfrac{n}{\text{LCM}(p, q)}\right], & \text{when 'n' is not completely divisible by 'p' or 'q'.} \end{cases}$$

where [*] is greatest integer function.

Illustration 5

Find the number of integral terms in the expansion $(3^{1/2} + 5^{1/8})^{256}$

Short-cut solution :

$\boxed{\text{Using T-2}}$ Here, $a = 3$, $b = 5$, $p = 2$, $q = 8$ and $n = 256$

As it is clear that 256 is completely divisible by '2' or '8'.

So, integral terms = $\left[\dfrac{256}{\text{LCM }(2, 8)}\right] + 1 = 32 + 1 = 33$

Illustration 6

Find the number of irrational terms in the expansion of $(\sqrt[8]{5}+\sqrt[6]{2})^{100}$

(a) 96 (b) 97

(c) 98 (d) 99

Short-cut solution :

$\boxed{\text{Using T-2}}$ Irrational terms = Total – $\underbrace{\text{Rational terms}}_{\text{Integral terms}}$

Here, $a = 5$, $b = 2$, $p = 8$, $q = 6$ and $n = 100$

Since '100' is not completely divisible by '6' or '8'.

$$\Rightarrow \text{Integral terms} = \left[\frac{100}{\text{LCM}(8,6)}\right] = \left[\frac{100}{24}\right] = 4$$

Hence, irrational terms = (101 – 4) = 97 **Ans. (b)**

TIPS AND TRICKS: (T-3)

Short trick to find term independent of 'x' of the term $\left(ax^p + \dfrac{R}{bx^q}\right)^n$

where, 'R' can be anything (constant, etc.)

Hence, first of all we will find 'r'.

$r = \dfrac{np}{p+q} \quad \Rightarrow \quad T_{r+1}$ is the term independent of 'x'.

Illustration 7

Find the term independent of x of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^6$

(a) 5^{th} term (b) 4^{th} term

(c) 3^{rd} term (d) 6^{th} term

Short-cut solution :

$\boxed{\text{Using T-3}}$ Here, $n = 6$, $p = 2$ and $q = 1$

$$\Rightarrow \quad r = \frac{np}{p+q} = \frac{6\times 2}{2+1} = 4$$

$\Rightarrow \quad T_{4+1}$ is the term independent of 'x' **Ans. (a)**

Illustration 8

The term independent of $\left(\frac{\sqrt{x}}{\sqrt{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$ is

(a) T_2 (b) T_4

(c) T_3 (d) T_5

Short-cut solution :

$\boxed{\text{Using T-3}}$ Rewrite as $\left(\frac{x^{1/2}}{\sqrt{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$

Here, $n = 10, p = \frac{1}{2}, q = 2$

$$\Rightarrow r = \frac{np}{p+q} = \frac{10\times\frac{1}{2}}{\frac{1}{2}+2} = 2$$

Hence, T_3 is the term independent of 'x'. **Ans. (c)**

TIPS AND TRICKS: (T-4)

Short trick to find number of terms in the expansion of the form $(a_1 + a_2 + a_3 + ... + a_k)^n$ then,

Number of terms $= {}^{n+(k-1)}C_{(k-1)}$

Illustration 9

The number of terms in the expansion of $(1 + 2x - 7y + z)^6$ is:

(a) 83 (b) 84

(c) 85 (d) 80

Short-cut solution :

$\boxed{\text{Using T-4}}$ Here, $n = 6, k = 4$

$\Rightarrow$ Number of terms $= {}^{n+(k-1)}C_{(k-1)} = {}^{6+4-1}C_{(4-1)} = {}^9C_3 = 84$ **Ans. (b)**

Illustration 10

The number of terms in the expansion of $(7x - 4 + y - w + g)^{20}$ is:

(a) 10626 (b) 10620

(c) 620 (d) 265

Short-cut solution :

Using T-4 Here; $n = 20$, $k = 5$

$\Rightarrow$ Number of terms $= {}^{20+5-1}C_{(5-1)} = {}^{24}C_4 = 10626$ **Ans. (a)**

SHORTCUTS: (SC-1)

If the coefficients of T_r, T_{r+1}, T_{r+2} terms in the expansion of $(1 + x)^n$ are in A.P., then the value of 'r' is

$$r = \frac{n \pm \sqrt{n+2}}{2} \ \forall\ n \in \mathbb{N}$$

Illustration 11

If the coefficients of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion of $(1 + y)^m$ are in A.P., then 'm' and 'r' satisfy the equation: **[AIEEE 2005]**

(a) $m^2 - m(4r - 1) + 4r^2 - 2 = 0$
(b) $m^2 - m(4r + 1) = 0$
(c) $m^2 - m(4r + 1) + 4r^2 - 2 = 0$
(d) None of these

Short-cut solution :

Using SC-1 Here, $m = n$

$$\Rightarrow \quad r = \frac{m \pm \sqrt{m+2}}{2} \quad \Rightarrow \quad (2r - m)^2 = (m + 2)$$

Hence, $m^2 - m(4r + 1) + 4r^2 - 2 = 0$ **Ans. (c)**

Illustration 12

If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1 + x)^{23}$ are in A.P.; then $r =$

(a) 7
(b) 14
(c) 8
(d) 16

Short-cut solution :

Using SC-1 Here, $n = 23$

$$\Rightarrow \quad r = \frac{23 \pm \sqrt{23+2}}{2} = 14 \text{ or } 9$$ **Ans. (b)**

SHORTCUTS: (SC-2)

Use of substitution method.

Illustration 13

If $C_0, C_1, C_2, ..., C_n$ denote the coefficients in the binomial expansion $(1 + x)^n$, then $C_1 + 2.C_2 + 3.C_3 + ... + nC_n$ is equal to

(a) $n.2^n$ (b) $n.2^{n-1}$

(c) 2^{n-1} (d) $(n - 1).2^{n-1}$

Short-cut solution :

Using SC-2 Putting $n = 2$

$\therefore$ Expression becomes $= {}^2C_1 + 2.{}^2C_2 = 4$

Now, checking options (a), (b), (c), (d) for $n = 2$

$\Rightarrow n.2^{n-1} = 2.2^{2-1} = 4$ **Ans. (b)**

Illustration 14

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then

$\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + ... + \frac{nC_n}{C_{n-1}}$ is equal to **[AIEEE 2002]**

(a) $\frac{n}{2}$ (b) $n(n + 1)$

(c) $\frac{n(n+1)}{12}$ (d) $\frac{n(n+1)}{2}$

Short-cut solution :

Using SC-2 Putting $n = 2$

$\Rightarrow$ Expression becomes $= \frac{{}^2C_1}{{}^2C_0} + \frac{2.{}^2C_2}{2C_1} = 3$

Now, checking options (a), (b), (c), (d) for $n = 2$

$\Rightarrow \frac{n(n+1)}{2} = \frac{2(2+1)}{2} = 3$ **Ans. (d)**

SHORTCUTS: (SC-3)

Total number of terms of :

(i) $(x + a)^n - (x - a)^n = \frac{n}{2}$

(ii) $(x + a)^n + (x - a)^n = \frac{n}{2} + 1$

Illustration 15

The total number of terms in the expansion of $(x + 3y)^{50} + (x - 3y)^{50}$ is

(a) 25 (b) 26

(c) 27 (d) 24

Short-cut solution :

$\boxed{\text{Using SC-3(ii)}}$ Here, $n = 50$

$\Rightarrow$ Total number of terms $= \frac{n}{2}+1 = \frac{50}{2}+1 = 26$ **Ans. (b)**

Illustration 16

The total number of terms in the expansion of $(x + 2y)^{100} - (x - 2y)^{100}$ is

(a) 51 (b) 49

(c) 48 (d) 50

Short-cut solution :

$\boxed{\text{Using SC-3(i)}}$ Here, $n = 100$

$\Rightarrow$ Total number of terms $= \frac{100}{2} = 50$ **Ans. (d)**

TECHNIQUE

Greatest Coefficient in the Expansion of $(x + a)^n$

The coefficients are ${}^nC_0, {}^nC_1, {}^nC_2, \ldots {}^nC_n$ where the coefficient of the general term is ${}^nC_r$. We have to find the value of r for which ${}^nC_r$ has the greatest value.

We know that if n is even ${}^nC_r$ is greatest when $r = \frac{n}{2}$ and if n is odd ${}^nC_r$ is greatest for $r = \frac{n-1}{2}$ or $r = \frac{n+1}{2}$.

Hence, if n is even the greatest coefficient is ${}^nC_{\frac{n}{2}}$ and if n is odd, the greatest coefficient is ${}^nC_{(n-1)/2}$ or ${}^nC_{(n+1)/2}$ both being equal.

Illustration 17

The greatest coefficient in the expansion of $\left(x + \frac{y}{x^3}\right)^{20}$ is

(a) ${}^{20}C_8$ (b) ${}^{20}C_{10}$

(c) ${}^{20}C_{12}$ (d) ${}^{20}C_{13}$

Short-cut solution :

Using Tech. $\because$ n is even

$\Rightarrow$ the greatest coefficient $= {}^{n}C_{\frac{n}{2}} = {}^{20}C_{10}$ **Ans. (b)**

Illustration 18

The greatest coefficient in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$ is

(a) 126 (b) 127

(c) 128 (d) 130

Short-cut solution :

Using Tech. $\because$ n is odd

$\therefore$ Greatest coefficient $= {}^{n}C_{\frac{n-1}{2}} = {}^{9}C_4$ **Ans. (a)**

Concept Booster Exercise

1. The sum of coefficients of all the odd degree terms in the expansion of $(1+x)^{48}$ is:
(a) 2^{48} (b) 2^{47} (c) 2^{46} (d) 2^{49}

2. The sum of coefficients of all terms in the expansion of $(2+3x)^{99}$ is
(a) 5^{98} (b) 5^{97} (c) 5^{100} (d) 5^{99}

3. The number of integral terms in the expansion of $(5^{1/2}+7^{1/8})^{1024}$ is
(a) 128 (b) 129 (c) 130 (d) 127

4. The number of integral terms in the expansion of $(9^{1/4}+8^{1/6})^{500}$ is
(a) 251 (b) 259 (c) 250 (d) 249

5. The number of terms which are free from radical signs of $(y^{1/5}+x^{1/10})^{55}$ where x and y are prime numbers.
(a) 5 (b) 4 (c) 7 (d) 6

6. The term independent of x of $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$ is:
(a) 7^{th} (b) 4^{th} (c) 6^{th} (d) 5^{th}

7. The term independent of 'x' in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is: **[JEE M 2013]**
(a) 4 (b) 120 (c) 210 (d) 310

8. The number of terms in the expansion of $(7x-5+p)^{10}$ is
(a) 65 (b) 62 (c) 66 (d) 68

9. The number of terms in the expansion of $(100x+y-2)^{101}$ is
(a) 5253 (b) 5251 (c) 5250 (d) 2523

10. If the coefficients of T_r, T_{r+1}, T_{r+2} terms of $(1+x)^{34}$ are in AP, then 'r' is
(a) 20 (b) 13 (c) 12 (d) 40

11. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then for n odd, $C_1^2+C_3^2+C_5^2+\ldots+C_n^2$ is equal to
(a) 2^{2n-2} (b) 2^n (c) $\frac{(2n)!}{2(n!)^2}$ (d) $\frac{(2n)!}{(n!)^2}$

12. If $C_0, C_1, C_2, ..., C_n$ denote the coefficients in the binomial expansion $(1 + x)^n$ then, $\frac{C_0}{1}+\frac{C_2}{3}+\frac{C_4}{5}+...$ is equal to

(a) $\frac{2^{n+1}}{n+1}$ (b) $\frac{2^{n+1}-1}{n+1}$ (c) $\frac{2^n}{n+1}$ (d) None of these

13. The total number of terms in the expansion of $(x + 5z)^{200} - (x - 5z)^{200}$ is

(a) 100 (b) 50 (c) 200 (d) 150

14. The total number of terms in the expansion of $(x + 2000)^{10} + (x - 2000)^{10}$ is

(a) 3 (b) 6 (c) 5 (d) 4

15. $1-{}^nC_1\left\{\frac{1+x}{1+nx}\right\}+{}^nC_2\left\{\frac{1+2x}{(1+nx)^2}\right\}-{}^nC_3\left\{\frac{1+3x}{(1+nx)^3}\right\}+...$ is equal to

(a) 0 (b) n^n (c) x^n (d) n

Numerical Value Problems

16. The sum of the coefficients of the polynomial $(1 + x - 3x^2)^{2163}$ is 'P' then | P | is equal to ________

17. The term independent of x in the expression $\left(8x^2-\frac{3}{x^3}\right)^{20}$ is k^{th} term then 'k' is equal to ________

18. Total number of terms in the expansion of $(3 - 2x^2 + 8x^3 + 2)^6$ is ________

19. Let 'n' be a positive integer. If the coefficients of 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(1 + x)^n$ are in AP, then the value of 'n' is ________

20. The coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^6$ in powers of x, is ___________. **[JEE M 2020]**

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ Here, $a = 1$, $b = 1$ and $n = 48$

So, sum of all coefficients = $(1 + 1)^{48} = 2^{48}$

Hence, sum of odd coefficients will be half = $\frac{2^{48}}{2} = 2^{47}$

2. **(d)** $\boxed{\text{Using T-1}}$ Here, $a = 2$, $b = 3$ and $n = 99$

Hence, sum of all coefficients = $(2 + 3)^{99} = 5^{99}$

3. **(b)** $\boxed{\text{Using T-2}}$ Here, $a = 5$, $b = 7$, $p = 2$, $q = 8$ and $n = 1024$

Since, 1024 is divisible by '2' or '8'

So, integral terms = $\left[\frac{1024}{\text{LCM}(2,8)}\right] + 1 = \left[\frac{1024}{8}\right] + 1 = 129$

4. **(a)** $\boxed{\text{Using T-2}}$ Since, 9 and 8 are not prime

So, we will rewrite as $(3^{2/4} + 2^{3/6})^{500} = (3^{1/2} + 2^{1/2})^{500}$

Here, $a = 3$, $b = 2$, $p = 2$, $q = 2$ and $n = 500$

Since, 500 is divisible by '2'.

$\Rightarrow$ Integral terms = $\left[\frac{500}{\text{LCM}(2,2)}\right] + 1 = 250 + 1 = 251$

5. **(d)** Since, free radical means integrals terms.

$\boxed{\text{Using T-2}}$ Here, $a = y$, $b = x$, $p = 5$, $q = 10$ and $n = 55$

Since, 55 is divisible by 5 or 10

$\Rightarrow$ Integral terms = $\left[\frac{55}{\text{LCM}(5,10)}\right] + 1 = [5.5] + 1 = 6$

6. **(c)** $\boxed{\text{Using T-3}}$ Here, $a = \frac{1}{2}$, $p = \frac{1}{3}$, $q = \frac{1}{5}$ and $n = 8$

So, $r = \frac{np}{p+q} = \frac{8 \times \frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = 5$

$\Rightarrow T_{5+1} \Rightarrow 6^{\text{th}}$ term is independent of x.

7. **(c)** $\boxed{\text{Using T-3}}$ First of all we will simplify the expression

Expression = $\left[\frac{(x^{1/3}+1)(x^{2/3}+1-x^{1/3})}{(x^{2/3}-x^{1/3}+1)} - \frac{(x^{1/2}+1)(x^{1/2}-1)}{x^{1/2}(x^{1/2}-1)}\right]^{10}$

$= (x^{1/3} - x^{-1/2})^{10}$

Here, $a = 1, p = \frac{1}{3}, q = \frac{1}{2}$ and $n = 10$

$$\Rightarrow r = \frac{np}{p+q} = \frac{10 \times \frac{1}{3}}{\frac{1}{3}+\frac{1}{2}} = 4$$

$\Rightarrow T_{4+1} = 5^{th}$ term which is independent of x.

So, $T_5 = {}^{10}C_4 = 210$

8. **(c)** $\boxed{\text{Using T-4}}$ Here, $n = 10, k = 3$

$\Rightarrow$ Number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

9. **(a)** $\boxed{\text{Using T-4}}$ Here, $n = 101, k = 3$

$\Rightarrow$ Number of terms $= {}^{101+3-1}C_{3-1} = {}^{103}C_2 = 5253$

10. **(a)** $\boxed{\text{Using SC-1}}$ Here, $n = 34$

$$\Rightarrow r = \frac{n \pm \sqrt{n+2}}{2} = \frac{34 \pm \sqrt{34+2}}{2} = 20 \text{ or } 14$$

11. **(c)** $\boxed{\text{Using SC-2}}$ Putting $n = 3$ (Since, n is odd)

Expression becomes $= {}^3C_1^2 + {}^3C_3^2 = 9 + 1 = 10$

Now, checking options (a), (b), (c), (d) for $n = 3$

$$\Rightarrow \text{Option (c)} = \frac{(2n)!}{2(n!)^2} = \frac{(2\times 3)!}{2(3!)^2} = 10$$

12. **(c)** $\boxed{\text{Using SC-2}}$ Putting $n = 2$

Now, expression becomes $= \frac{{}^2C_0}{1} + \frac{{}^2C_2}{3} = 1 + \frac{1}{3} = \frac{4}{3}$

Now, checking option (a) (b) (c) (d) for $n = 2$

$$\Rightarrow \text{Option (c)} = \frac{2^n}{n+1} = \frac{2^2}{2+1} = \frac{4}{3}$$

13. **(a)** $\boxed{\text{Using SC-3}}$ Here, $n = 200$

$\Rightarrow$ Total number of terms $= \frac{200}{2} = 100.$

14. **(b)** $\boxed{\text{Using SC-3}}$ Here, $n = 10$

$\Rightarrow$ Total number of terms $= \frac{n}{2} + 1 = \frac{10}{2} + 1 = 6$

15. **(a)** $\boxed{\text{Using SC-2}}$ Putting $n = 1$

$$\Rightarrow \text{Expression} = 1 - {}^1C_1\left(\frac{1+x}{1+x}\right) = 0$$

16. **(1)** $\boxed{\text{Using T-1(ii)}}$ Here, $a = 1$, $b = 1$, $c = -3$

$\Rightarrow$ Sum $= (1 + 1 - 3)^{2163} = -1 = P$

$\Rightarrow |P| = 1$

17. **(9)** $\boxed{\text{Using T-3}}$ Here, $p = 2$, $q = 3$, $n = 20$

$$\Rightarrow \frac{np}{p+q} = \frac{20 \times 2}{2+3} = 8 = r$$

$\Rightarrow T_{8+1} = 9^{th}$ term is independent of x.

18. **(84)** $\boxed{\text{Using T-4}}$ Here, $n = 6$, $k = 4$

$\Rightarrow$ Total number of terms $= {}^{6+4-1}C_{4-1} = {}^{9}C_{3}$

$\Rightarrow$ Hence ${}^{9}C_{3} = 84$

19. **(7)** $\boxed{\text{Using SC-1}}$ Here, $r = 2$

$$\Rightarrow r = \frac{n \pm \sqrt{n+2}}{2} \Rightarrow 2 = \frac{n \pm \sqrt{n+2}}{2}$$

$\Rightarrow (4 - n)^2 = n + 2 \Rightarrow n = 7$ or 2 Rejected.

20. **(120)** Coefficient of x^4 in $\left(\frac{1-x^4}{1-x}\right)^6$ = coefficient of x^4 in $(1-6x^4)(1-x)^{-6}$

= coefficient of x^4 in $(1-6x^4)\left[1+{}^{6}C_1 x+{}^{7}C_2 x^2+....\right]$

$={}^{9}C_4-6\cdot 1=126-6=120.$

17 Sequences and Series

Review of Key Notes and Formulae

1. **Sequence :** An arrangement of numbers according to definite rule or a set of rules is called a sequence.
   A sequence takes the form $a_1, a_2, a_3, ...a_n, ...$
   A sequence is finite if it contains finite number of terms otherwise, it is infinite.
2. **Series :** A series is obtained by adding or subtracting the terms of a sequence.
   If $a_1, a_2, ..., a_n$ is a sequence, then the sum $a_1 + a_2 + ...+ a_n$ is a series.
   The series is finite or infinite according as the given sequence is finite or infinite.
3. **Arithmetic Progression :** A sequence $a_1, a_2, a_3, ..., a_n$ is called arithmetic sequence or progression if $a_{n+1} = a_n + d$, $n \in N$, where a_1 is called the first term and the constant 'd' is called the common difference of the A.P.

   (i) n^{th} term (general term)
   $a_n = a + (n – 1) d$
   where, a = first term,
   d = common difference, n = number of terms

   Note that :
   (a) Common difference can be positive or negative.
   (b) m^{th} term from end of A.P. is $(n – m + 1)^{th}$ term from the beginning of A.P.

   (ii) Sum of n terms
   $$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \text{ or } S_n = \frac{n}{2}\left[a+l\right]$$
   l = the last term

   (iii) Supposition of terms
   When n = 3, we take a – d, a, a + d
   When n = 4, we take a – 3d, a – d, a + d, a + 3d
   When n = 5, we take a – 2d, a – d, a, a + d, a + 2d

   (iv) Arithmetic Mean (A.M.)
   The arithmetic mean of two numbers 'a' and 'b' is given by the formula,
   $$A = \frac{a+b}{2}$$

4. Properties of A.P. :

(i) If a constant is added to (or subtracted from) each term of an A.P., the resulting sequence is also an A.P.

(ii) If each term of an A.P. is multiplied (or divided) by a non-zero constant, the resulting sequence is also an A.P.

5. Insertion of Numbers in A.P. :

Let $A_1, A_2, A_3, \ldots\ldots, A_n$ be n numbers between a & b such that a, $A_1, A_2, \ldots\ldots, A_n$, b is an AP. Here b is $(n+2)^{th}$ term where $d = \left(\frac{b-a}{n+1}\right)$

$$A_1 = a + \frac{b-a}{n+1},\ A_2 = a + \frac{2(b-a)}{n+1},\ A_3 = a + \frac{3(b-a)}{n+1}, \ldots, A_n = a + \frac{n(b-a)}{n+1}.$$

6. Geometric Progression

(i) A sequence $a_1, a_2, a_3, \ldots\ldots, a_n$ is called a geometric progression if each term is non-zero and $\frac{a_{k+1}}{a_k} = r, k \geq 1$

where $a = a_1$ is called the first term and r is called the common ratio of the G.P.

(ii) **Geometric Progression is denoted as :** a, ar, ar^2, ar^3, ...,
where 'a' is the first term and 'r' is the common ratio.
If the number of terms in a G.P is definite it is called finite geometric series, otherwise it is called infinite geometric series.

(a) n^{th} term (general term)
$a_n = ar^{n-1}$
where, a = first term, r = common ratio and n = number of terms

(b) Sum to n terms of a G.P

$$S_n = \frac{a(1-r^n)}{1-r}, r < 1 \text{ or } S_n = \frac{a(r^n - 1)}{r-1}, r > 1$$

(c) Supposition of terms

When n = 3, we take $\frac{a}{r}, ar, ar^3$

When n = 4, we take $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

When n = 5, we take $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

(d) Geometric Mean (G.M.)
The geometric mean between two positive numbers 'a' and 'b' is given by the formula, $G = \sqrt{ab}$

7. Properties of G.P. :

(i) If each term of a G.P. is multiplied or divided by the same non-zero quantity, then resulting sequence is also a G.P.

(ii) If each term of a G.P. is raised to the same power, then resulting sequence is also a G.P.

8. Insertion of Numbers in G.P. :

Let $G_1, G_2, ..., G_n$ be 'n' numbers between positive numbers a & b such that a, $G_1, G_2, G_3, ..., G_n$, b is a G.P.

Here b is the $(n+2)^{th}$ term. Where, $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$$G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, ..., G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

9. Relation between A.M. & G.M

Let A and G be A.M. & G.M. of two given positive real numbers 'a' and 'b' respectively, then $A \geq G$.

10. Some Special Series

(i) $1+2+3+....+n = \frac{n(n+1)}{2}$

(ii) $1^2+2^2+3^2+....+n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $1^3+2^3+3^3+....+n^3 = \left[\frac{n(n+1)}{2}\right]$

(iv) $1+3+5+....+(2n-1) = n^2$

(v) $2+4+6+....+2n = n(n+1)$

(vi) $1^2+3^2+5^2+....+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

(vii) $2^2+4^2+6^2+....+(2n)^2 = \frac{2n(n+1)(2n+1)}{3}$

(viii) $1^3+3^3+5^3+....+(2n-1)^3 = n^2(2n^2-1)$

(ix) $2^3+4^3+6^3+....+(2n)^3 = 2[n(n+1)]^2$

11. Important Points to Remember

(i) If r^{th} term of an A.P.,

$T_r = Ar^3 + Br^2 + Cr + D$, then sum of first n term of this AP,

$$S_n = \sum_{r=1}^{n} T_r = A\sum_{r=1}^{n} r^3 + B\sum_{r=1}^{n} r^2 + C\sum_{r=1}^{n} r + D\sum_{r=1}^{n} 1$$

(ii) If no. of terms n in any series is odd then only one middle term is exist which is term $\left(\frac{n+1}{2}\right)^{th}$

(iii) If no. of terms n in any series is even then $(n/2)^{th}$ term and $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ term are two middle terms, where n is even.

(iv) For every sequence, it is not always possible to write a specific formula.

(v) Sum of n arithmetic means $A_1, A_2, A_3, \ldots, A_n$ inserted between a and b is equal to n times the single A.M. between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where $A = \frac{a+b}{2}$

(vi) Between any two numbers, $\frac{\text{sum of } m \text{ AM's}}{\text{sum of } n \text{ AM's}} = \frac{m}{n}$

(vii) G.M. of n positive numbers $a_1, a_2, a_3, \ldots\ldots, a_n$ is $(a_1 \cdot a_2 \cdot a_3 \ldots\ldots\ldots\ldots a_n)^{1/n}$.

(viii) If a and b are two numbers of opposite signs, then G.M. between them does not exist.

(ix) **Formula for the sum of infinite terms of a G.P.**

If $|r| < 1$, the sum of infinite terms (S) of the G.P. $a + ar + ar^2 + \ldots\ldots\ldots$ to infinity is $S = \frac{a}{1-r}$

12. Harmonic progression (H.P.)

The sequence where each $\langle x_1, x_2, x_3, \ldots\ldots, x_n, \ldots\rangle$ $x_i \neq 0$ is said to be a harmonic progression (H.P.), if the sequence formed by the reciprocals of its terms is an A.P. That is, the sequence $\left\langle \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \ldots\ldots\ldots, \frac{1}{x_n}, \ldots\ldots\ldots \right\rangle$ is an A.P.

The standard form of H.P. is $\left\langle \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots\ldots\ldots \right\rangle$

General term n^{th} term of a H.P. is given by

$$T_n = \frac{1}{a+(n-1)d}$$

13. Relations between A.M., G.M. and H.M.

If a and b are positive numbers, then $A = \frac{a+b}{2}$; $G = \sqrt{ab}$; $H = \frac{2ab}{a+b}$. We have

(a) A.M., G.M., H.M. are in G.P., i.e. $G^2 = AH$

(b) A.M. ≥ G.M. ≥ H.M., i.e., $A \geq G \geq H$. Equality holds if and only if $a = b$

14. Arithmetico-geometric sequence (a.g.s.)

If each term of a sequence is the product of the corresponding terms of an A.P. and a G.P., then it is called arithmetico-geometric sequence (A.G.S.) e.g. $a, (a + d)r, (a + 2d)\, r^2, \ldots\ldots$

The general term (n^{th} term) of an A.G.S. is

$$T_n = [a + (n-1)d]\, r^{n-1}$$

$$S_n = \frac{a}{1-r} + \frac{r.d\left(1-r^{n-1}\right)}{(1-r)^2} \text{ and } S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

15. Important Points to Remember.

(i) a, b, c are in A.P. and H.P. ⇒ a, b, c are in G.P.

(ii) If a, b, c are in A.P. then $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in A.P.

(iii) If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(iv) If a, b, c are in G.P. then a^2, b^2, c^2 are in G.P.

(v) If a, b, c, d are in G.P. then $a+b, b+c, c+d$ are in G.P.

(vi) If a, b, c are in H.P. then $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

TIPS & TRICKS!

TIPS AND TRICKS: (T-1)

Sum of all interior angles of polygon has n sides is $(n-2) \times 180°$

Illustration 1

The internal angles of a convex polygon are in A.P. The smallest angle is 100° and the common difference is 8°. The number of sides of the polygon is ________.

(a) 8 (b) 6 (c) 4 (d) 7

Short-cut solution :

$\boxed{\text{Using T-1}}$ Here a = 100°, d = 8°

$\therefore$ Sum of interior angle of polygon.

$$(n-2) \times 180 = \frac{n}{2}[2a + (n-1)d]$$

$$(n-2) \times 180 = \frac{n}{2}[200 + (n-1)\,8]$$

$$(n-2) \times 180 = 96n + 4n^2$$

$\Rightarrow$ $n^2 - 21n + 90 = 0$

$\Rightarrow$ n = 6 or n = 15 not possible. **Ans. (b)**

TIPS & TRICKS!

TIPS AND TRICKS: (T-2)

If the ratio of sum of n terms of two A.P is $f(n) : g(n)$ then ratio of their nth term will be $f(2n-1) : g(2n-1)$.

Illustration 2

The ratio of the sum of n terms of two A.P.'s is (3n – 13) : (5n + 21). The ratio of 24th terms of the two progressions is

(a) 1 : 2 (b) 2 : 3 (c) 3 : 5 (d) 7 : 11

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\because \quad \dfrac{S_n}{S'_n} = \dfrac{3n-13}{5n+21}$ (Given)

$\therefore \quad \dfrac{a_n}{a'_n} = \dfrac{3(2n-1)-13}{5(2n-1)+21} \Rightarrow \dfrac{a_{24}}{a'_{24}} = \dfrac{3(48-1)-13}{5(48-1)+21} = \dfrac{128}{256} = \dfrac{1}{2}$ **Ans. (a)**

TIPS & TRICKS

TIPS AND TRICKS: (T-3)

If the ratio of n terms of two A.P. is $f(n) : g(n)$ then ratio of sum of n terms will be $f\left(\dfrac{n+1}{2}\right) : g\left(\dfrac{n+1}{2}\right)$

Illustration 3

If the ratio of first n terms of two A.P's is $\dfrac{14n-6}{8n+23}$, then find the ratio of sum of first n terms.

Short-cut solution :

$\boxed{\text{Using T-3}}$ $\because \quad \dfrac{a_n}{a'_n} = \dfrac{14n-6}{8n+23}$ (Given)

$$\therefore \quad \frac{S_n}{S'_n} = \frac{14\left(\dfrac{n+1}{2}\right)-6}{8\left(\dfrac{n+1}{2}\right)+23} = \frac{7(n+1)-6}{4(n+1)+23}$$

$$= \frac{7n+1}{4n+27}.$$

TIPS & TRICKS

TIPS AND TRICKS: (T-4)

If arithmetic mean and geometric mean of two numbers a and b $(a > b)$ are in ratio $m : n$, then

$$\frac{a}{b} = \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}}$$

Illustration 4

If α and β be the roots of equation $x^2 - 6x + 4 = 0$, then find the ratio of α and β.

Short-cut solution :

$\boxed{\text{Using T-4}}$ Here $\alpha + \beta = 6$

$\Rightarrow$ A.M. (m) $= \dfrac{\alpha+\beta}{2} = 3$

and $\alpha.\beta = 4$

$\Rightarrow$ G.M. (n) $= \sqrt{\alpha\beta} = 2$

$$\therefore \quad \frac{\alpha}{\beta} = \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}} = \frac{3+\sqrt{9-4}}{3-\sqrt{9-4}} = \frac{3+\sqrt{5}}{3-\sqrt{5}}$$

SHORTCUTS: (SC-1)

If for an A.P., p^{th} term is q and q^{th} term is p then m^{th} term is $p + q - m$.

Illustration 5

If p^{th} term of an A.P. is q and its q^{th} term is p, then what is the common defference?

(a) –1 (b) 0 (c) 2 (d) 1

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $\because$ $T_m = p + q - m$

$\therefore$ $T_1 = p + q - 1$ and $T_2 = p + q - 2$

So, $d = T_2 - T_1 = (p + q - 2) - (p + q - 1) = -1$. **Ans. (a)**

Illustration 6

If 19^{th} term of an A.P. is 39 and its 39^{th} term is 19, then value of 53^{th} term is _____

(a) 5 (b) 58 (c) 39 (d) 19

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Here p = 19 and q = 39

$\because$ $T_m = p + q - m$

$\therefore$ $T_{53} = 19 + 39 - 53 = 58 - 53 = 5$ **Ans. (a)**

SHORTCUTS: (SC-2)

If p^{th} term of an A.P. is $\dfrac{1}{q}$ and qth term is $\dfrac{1}{p}$, then

(i) $a = \dfrac{1}{pq}$ and $d = \dfrac{1}{pq}$ (ii) $a_m = \dfrac{m}{pq}$

(iii) $S_m = \dfrac{m(m+1)}{2pq}$

Illustration 7

If the 10^{th} term of an A.P. is $\frac{1}{20}$ and its 20th term is $\frac{1}{10}$, then the sum of its first 200 terms is: **[JEE M 2020]**

(a) 50 (b) $50\frac{1}{4}$ (c) 100 (d) $100\frac{1}{2}$

Short-cut solution :

Using SC-2(iii) Here p = 10 and q = 20

$$\because \quad S_m = \frac{m(m+1)}{2pq}$$

$$\therefore \quad S_{200} = \frac{200\times 201}{2\times 10\times 20} = \frac{201}{2} = 100\frac{1}{2}.$$

Ans. (d)

SHORTCUTS: (SC-3)

Sum of infinite series of type

$$\frac{1}{a_1b_1}+\frac{1}{a_2b_2}+\frac{1}{a_3b_3}+....\infty = \frac{1}{a_1(b_1-a_1)}$$

where a_1, a_2, a_3.... are first A.P and b_1, b_2 b_3.... are second A.P and $b_1 - a_1 = b_2 - a_2 = ...$

Illustration 8

Sum of the following series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+..........$ is-

(a) ∞ (b) 1

(c) 0 (d) None of these

Short-cut solution :

Using SC-3 $S = \frac{1}{a_1(n_1-a_2)} = \frac{1}{(1)(2-1)} = 1$

Ans. (b)

SHORTCUTS: (SC-4)

For $x \in$ R, Let $[x]$ denote the greatest integer $\le x$, then the sum of the series

$$[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+....\left[x+\frac{n-1}{n}\right]=[nx]$$

Illustration 9

For $x \in R$, let $[x]$ denote the greatest integer $\le x$, then the sum of the series is

$$\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+\cdots+\left[-\frac{1}{3}-\frac{99}{100}\right]$$ **[JEE M 2019]**

(a) –153 (b) –133 (c) –131 (d) –135

Short-cut solution :

Using SC-4 $\because \quad [x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]....\left[x+\frac{n-1}{n}\right]=[nx]$

and $[x] + [-x] = -1$ $(x \notin z)$

$$\therefore \quad \left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-100\right]+...+\left[-\frac{1}{3}-\frac{99}{100}\right]$$

$$=-100-\left\{\left[\frac{1}{3}\right]+\left[\frac{1}{3}+\frac{1}{100}\right]+...\left[\frac{1}{3}+\frac{99}{100}\right]\right\}$$

$$=-100-\left[\frac{100}{3}\right]=-133$$ **Ans. (c)**

SHORTCUTS: (SC-5)

To convert recurring decimals in fraction form Remove decimals and recurring then subtract that number which has no recurring decimals, then divided by many nine's as many digit has recurring decimal and many zero's as many digits has no recurring decimal.

Illustration 10

The fractional value of $0.2\overline{35}$ is-

(a) $\frac{235}{1000}$ (b) $\frac{235}{990}$ (c) $\frac{233}{990}$ (d) $\frac{233}{900}$

Short-cut solution :

Using SC-5 $0.2\overline{35}=\frac{235-2}{990}=\frac{233}{990}$

(Non recurring no = 2. After decimal two recurring digit use 99 and one non recurring digit use 0.) **Ans. (c)**

SHORTCUTS: (SC-6)

Sum of series in the form K + KK + KKK + ...up to n terms is

$\frac{K}{81}\left[10^{n+1}-9n-10\right]$ where K is non zero digits.

Illustration 11

The sum of the following series 7 + 77 + 777 + ... to n term is

(a) $\frac{7}{81}(10^{n+1}-9^n-10)$ (b) $\frac{7}{9}(10^n-9n)$

(c) $\frac{7}{81}(10^n-9n-1)$ (d) $\frac{7}{81}(10^{n+1}-9n)$

Short-cut solution :

Using SC-6 $S_n=\frac{7}{81}[10^{n+1}-9^n-10]$ **Ans. (a)**

SHORTCUTS: (SC-7)

Sum of Series in the form 0.K + 0.KK + 0.KKK +....up to n terms is

$\frac{K}{81}\left[9n+10^{-n}-1\right]$ where K is non zero digits.

Illustration 12

The sum of the following series 0.5 + 0.55 + 0.555 + ... to n terms is

(a) $\frac{5}{81}(9n-10^{-n}-1)$ (b) $\frac{5}{81}(10^{-n}-9n)$

(c) $\frac{5}{81}(10^{-n}-1)$ (d) $\frac{5}{81}(10^{-n}+9n-1)$

Short-cut solution :

Using SC-7 $S_n=\frac{5}{81}[9n+10^{-n}-1]$ **Ans. (d)**

SHORTCUTS: (SC-8)

Product of n GM's inserted between 'a' and 'b' is equal to n^{th} power of the single GM between 'a' and 'b' i.e.

$\prod_{r=1}^{n}G_r=(G)^n$ where $G=\sqrt{ab}$

Illustration 13

The product of 8 geometric means between 8 and $\frac{1}{64}$ is P. Then value of P is ___

(a) 2^{12} (b) 2^{-12} (c) 2^{8} (d) 2^{-8}

Short-cut solution :

Using SC-8 Geometrical mean between 8 and $\frac{1}{64}$

$$G = \left(8 \times \frac{1}{64}\right)^{1/2} = \left(\frac{1}{8}\right)^{1/2}$$

$\therefore$ Product of 8 geometrical mean

$$P = (G)^8 = \left(\frac{1}{8}\right)^{1/2} = \left(\frac{1}{8}\right)^4 = 2^{-12}$$

Ans. (b)

SHORTCUTS: (SC-9)

Method of Substitution check the option by substitute the value of terms a_1, a_2, a_3.... and number of terms n.

Illustration 14

If S_n denotes the sum of n term of an A.P., then $S_{n+3} - 3S_{n+2} + 3\,S_{n+1} - S_n$ is equal to

(a) 0 (b) 1 (c) 1/2 (d) 2

Short-cut solution :

Using SC-9 Let the A.P. is 1, 2, 3, 4, 5, 6 .....

$\therefore S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$

(Put $n = 1$) = $S_4 - 3S_3 + 3S_2 - S_1 = 10 - 3(6) + 3(3) + 1 = 0.$ **Ans. (a)**

Illustration 15

If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+d}$ are in

(a) A.P. (b) G.P. (c) H.P. (d) None of these

Short-cut solution :

Using SC-9 Let $a^2 = 1$, $b^2 = 25$ and $c^2 = 49$

Now $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} = \frac{1}{12}, \frac{1}{8}, \frac{1}{6}$

= 2, 3, 4 (Multiply by 24)

which are in A.P. **Ans. (a)**

Note : We can give direct answer from point to remember

TECHNIQUE

Sum of series by method of difference

Sometimes, the nth term of a series cannot be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its nth term by the method of difference and then find the sum of the series by using the formulas for Σn, Σn^2, and Σn^3.

Illustration 16

Let a, b, c $\in$ R. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to : **[JEE M 2017]**

(a) 255 (b) 330 (c) 165 (d) 190

Short-cut solution :

$\boxed{\text{Using Tech.}}$ $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 3 \Rightarrow f(1) = 3$

Now $f(x + y) = f(x) + f(y) + xy$ ...(i)

Put $x = y = 1$ in eqn (i)

$f(2) = f(1) + f(1) + 1 = 2f(1) + 1$

$f(2) = 7$

$\Rightarrow f(3) = 12$

$S_n = 3 + 7 + 12 + \text{.........} + t_n$

$S_n = \quad 3 + 7 + 12 + \text{.........} + t_n - 1 + t_n$

$- \quad - \quad - \quad - \quad - \quad -$

$0 = 3 + 4 + 5 \text{ ....to n term} - t_n$

$t_n = 3 + 4 + 5 + \text{....}$ upto n terms

$$t_n = \frac{(n^2 + 5n)}{2}$$

$$S_n = \sum t_n = \sum \frac{(n^2 + 5n)}{2}$$

$$S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}\right]$$

$$= \frac{n(n+1)(n+8)}{6}$$

$$S_{10} = \frac{10 \times 11 \times 18}{6} = 330$$

Ans. (b)

Concept Booster Exercise

1. The interior angles of a polygon are in arithmetic progression. The smallest angle is 120°, and the common difference is 5°. The number of sides of the polygon is
(a) 9 (b) 6 (c) 7 (d) 8

2. The ratio of the sum of n terms of two A.P's is $(7n + 1) : (4n + 27)$. The ratio of 13th terms is
(a) $\frac{99}{79}$ (b) $\frac{176}{127}$ (c) $\frac{170}{127}$ (d) $\frac{127}{176}$

3. If the ratio of first n terms of two A.P's is $(6n - 16) : (10n + 16)$, then ratio of sum of 15th terms is ____.
(a) 1:2 (b) 7:11 (c) $\frac{35}{101}$ (d) $\frac{101}{35}$

4. If sum of two number is 10 and product is 16 then their ratio is ____.
(a) 4 : 1 (b) 1 : 4 (c) 5 : 4 (d) 4 : 5

5. If the j^{th} term and k^{th} term of an A.P are k and j respectively, the $(k + j)^{th}$ term is.
(a) 0 (b) 1 (c) $k + j + 1$ (d) $k + j - 1$

6. Let T_r be the r^{th} term of an A.P., for $r = 1, 2, 3, ...$ If for some positive integers m, n we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} equals
(a) $\frac{1}{mn}$ (b) $\frac{1}{m}+\frac{1}{n}$ (c) 1 (d) 0

7. Sum of the following series
$\frac{1}{1-4}+\frac{1}{4-7}+\frac{1}{7-10}+....$is
(a) 1/2 (b) 1 (c) 1/3 (d) None

8. If $[x]$ stand for the greater integer functions, then the value of
$\left[\frac{1}{2}+\frac{1}{1000}\right]+\left[\frac{1}{2}+\frac{2}{1000}\right]+....+\left[\frac{1}{2}+\frac{999}{1000}\right]$ is equal to
(a) 497 (b) 498 (c) 500 (d) 502

9. The fractional value of $2.3\overline{57}$ is
(a) $\frac{1167}{450}$ (b) $\frac{1167}{495}$ (c) $\frac{1177}{450}$ (d) $\frac{1177}{495}$

10. The sum of the following series $5 + 55 + 555 + ...$ to n terms is
(a) $\frac{5}{81}\left[10^{n+1}-9n\right]$ (b) $\frac{5}{81}\left[10^{n+1}-9n-10\right]$
(c) $\frac{5}{81}\left[10^{n+1}+9n-10\right]$ (d) $\frac{5}{81}\left[10^{n+1}-9n+10\right]$

11. The Sum of the following series 0.6 + 0.66 + 0.666 +.... to n terms is.

(a) $\frac{6}{81}\left(10^{-n}-9n-1\right)$ (b) $\frac{6}{81}\left(10^{-n}+9n\right)$

(c) $\frac{2}{27}\left(10^{-n}+9n-1\right)$ (d) $\frac{1}{81}\left(10^{-n}+9n-1\right)$

12. If first and 7^{th} term of a G.P is 2 and 1458 respectively then product of terms between first and 7^{th} term is–

(a) $2^6 3^{18}$ (b) $2^5 3^{15}$ (c) 2.3^3 (d) 2.3^5

13. The Sum of the series $1+\frac{4}{3}+\frac{10}{9}+\frac{28}{27}$ +.... up to n terms is: **[AIEEE-2012]**

(a) $n+\frac{1}{2}-\frac{1}{2-3^{n-1}}$ (b) $n-\frac{1}{3}+\frac{1}{3.2^{n-1}}$

(c) $\frac{5}{3}n-\frac{7}{6}+\frac{1}{2.3^{n-1}}$ (d) $\frac{7}{6}n+\frac{1}{6}-\frac{1}{3.2^{n-1}}$

14. If the Sum of n terms of an A.P. is Cn^2, then the Sum of square of these n terms are: **[AIEEE-2009]**

(a) $\frac{n\left(4n^2-1\right)C^2}{6}$ (b) $\frac{4\left(4n^2+1\right)C^2}{3}$

(c) $\frac{n\left(4n^2-1\right)C^2}{3}$ (d) $\frac{n\left(4n^2+1\right)C^2}{6}$

15. The sum $\sum_{k=1}^{20} k\frac{1}{2^k}$ is equal to : **[JEE M 2019]**

(a) $2-\frac{3}{2^{17}}$ (b) $1-\frac{11}{2^{20}}$ (c) $2-\frac{11}{2^{19}}$ (d) $2-\frac{21}{2^{20}}$

Numerical Value Problems

16. Fifth term of a GP is 2, then the product of its 9 terms is

17. If the arithmetic mean of two numbers a,b; $a > b > 0$ is five times their geometric mean and $\frac{a+b}{a-b}$ is equal to $\frac{5\sqrt{6}}{K}$, then value of k is _____. **[JEE M 2017]**

18. If the sum and product of the first three terms in an A.P. are 33 and 1155, respectively, then absolute value of its 11^{th} term is

19. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set XVY is _____. **[Adv. 2018]**

20. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? **[Adv. 2018]**

Solutions

1. **(a)** Using T-1 Let there be n sides in the polygon.

$\therefore$ Sum of all n interior angles of polygon = $(n - 2) \times 180°$

Since the angles are in A.P. with the smallest angle 120° and common difference 5°.

$\therefore$ Sum of all interior angles of polygon

$$= \frac{n}{2}[2 \times 120 + (n-1) \times 5]$$

$$\therefore \quad \frac{n}{2}[2 \times 120 + (n-1) \times 5] = (n-2) \times 180$$

$$\Rightarrow \quad \frac{n}{2}[5n + 235] = (n-2) \times 180$$

$$\Rightarrow \quad 5n^2 + 235n = 360n - 70$$

$\therefore$ $n^2 - 25n + 144 = 0$ $\therefore$ $n = 16, 9$

Also if n = 16, then 16th angle = 120 + 15 × 5 = 195° > 180°, which is not possible. $\therefore$ $n = 9$.

2. **(b)** Using T-2

$$\because \frac{S_n}{S'_n} = \frac{7n+1}{4n+27} \qquad \text{(given)}$$

$$\therefore \frac{a_n}{a'_n} = \frac{7(2n-1)+1}{4(2n-1)+27}$$

$$\Rightarrow \frac{a_{13}}{a'_{13}} = \frac{7(26-1)+1}{4(26-1)+27} = \frac{176}{127}$$

3. **(c)** Using T-3

$$\because \frac{a_n}{a'_n} = \frac{6n-16}{10n+16)} \quad \text{(given)}$$

$$\therefore \frac{S_n}{S'_n} = \frac{6\left(\frac{n+1}{2}\right)-16}{10\left(\frac{n+1}{2}\right)+16} = \frac{3(n+1)-16}{5(n+1)-16}$$

$$\Rightarrow \frac{S_{15}}{S'_{15}} = \frac{3\,(17)-16}{5\,(17)+16} = \frac{35}{101}$$

4. **(a)** Using T-4

Here $a + b = 10$

$$\Rightarrow \text{A.M. } (m) = \frac{a+b}{2} = \frac{10}{2} = 5 \text{ and } ab = 16$$

$\Rightarrow$ G.M $(n) = \sqrt{16} = 4$

$$\therefore \frac{a}{b} = \frac{m+\sqrt{m^2-n^2}}{m-\sqrt{m^2-n^2}} = \frac{5+\sqrt{25-6}}{5-\sqrt{25-16}}$$

$$= \frac{5+3}{5-3} = \frac{8}{2} = \frac{4}{1} = 4:1$$

5. **(a)** Using SC-1

Here $P = j$ and $q = k$

$\therefore T_m = P + q - m$

$\therefore T_{(k+j)} = (j+k) - (j+\hat{k}) = 0$

6. **(c)** Using SC-2(ii)

Here $P = m$ and $q = n$

$$\because T_m = \frac{m}{Pq}$$

$$\therefore T_{mn} = \frac{mn}{m.n} = 1$$

7. **(c)** Using SC-3

$$S = \frac{1}{a_1(b_1 - a_2)} = \frac{1}{1(4-1)} = \frac{1}{3}.$$

8. **(c)** Using SC-4

$$S = [n\,x] = [1000 \times \frac{1}{2}] = [500] = 500$$

9. **(b)** Using SC-5

$$2.3\overline{57} = \frac{2357-23}{990} = \frac{2334}{990} = \frac{1167}{495}$$

10. **(b)** Using SC-6

$$S_n = \frac{5}{81}\left[10^{n+1} - 9n - 10\right]$$

11. **(c)** Using SC-7

$$S_n = \frac{6}{81}\left(10^{-n} + 9n - 1\right) = \frac{2}{27}\left(10^{-n} + 9n - 1\right)$$

12. **(c)** Using SC-8

$\because$ $G = (2 \times 1458)^{1/2} = (2^2 3^6)^{1/2} = 2 \times 3^3$.

$\therefore$ Product of terms between first and 7th term is $= (2 \times 3^3)^5 = 2^5 3^{15}$.

13. **(a)** Using SC-9

Let $n = 2$ then Sum $= 1 + \frac{4}{3} = \frac{7}{3}$.

Now put $n=2$ in options, then option (a) $n+\frac{1}{2}-\frac{1}{2.3^{n-1}}=2+\frac{1}{2}-\frac{1}{6}=\frac{7}{3}$ is current answer.

14. **(c)** Using SC-9

$T_n = S_n - S_{n-1} = cn^2 - c(n-1)^2$
$= c(2n-1)$

$\Rightarrow \quad (Tn)^2 = c^2(2n-1)^2 = c^2[4n^2 - 4n + 1]$

$S = c^2[4\Sigma n^2 - 4\Sigma n + \Sigma 1]$

$$= c^2\left[4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n\right]$$

Put $n = 2$

$S = c^2$

Now put $n = 1$ in the options, then

$(c) = n\ \dfrac{(4n^2-1)c^2}{3} = c^2$ is correct option.

15. **(c)** Using Tech. Let, $S = \sum_{k=1}^{20} k.\frac{1}{2^k}$

$$S = \frac{1}{2} + 2.\frac{1}{2^2} + 3.\frac{1}{2^3} + ... + 20.\frac{1}{2^{20}} \quad ...(i)$$

$$\frac{1}{2}S = \frac{1}{2^2} + 2.\frac{1}{2^3} + ... + 19\frac{1}{2^{20}} + 20\frac{1}{2^{21}} \quad ...(ii)$$

On subtracting equations (ii) by (i),

$$\frac{S}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{20}}\right) - 20\frac{1}{2^{21}}$$

$$= \frac{\frac{1}{2}\left(1 - \frac{1}{2^{20}}\right)}{1 - \frac{1}{2}} - 20.\frac{1}{2^{21}} = 1 - \frac{1}{2^{20}} - 10.\frac{1}{2^{20}}$$

$$\frac{S}{2} = 1 - 11.\frac{1}{2^{20}} \Rightarrow S = 2 - 11.\frac{1}{2^{19}} = 2 - \frac{11}{2^{19}}$$

16. **(512)** Using Key Notes $ar^4 = 2$

$a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 \times ar^6 \times ar^7 \times ar^8$

$= a^9 r^{36} = (ar^4)^9 = 2^9 = 512$

17. **(12)** Using T-4 $\dfrac{A.M}{G.M} = \dfrac{5}{1}$

$$\therefore \frac{a}{b} = \frac{5+\sqrt{24}}{5-\sqrt{24}} \Rightarrow \frac{a+b}{a-b} = \frac{5\sqrt{6}}{12} \Rightarrow k = 12.$$

18. **(25)** $\boxed{\text{Using Key Notes}}$ Let three terms of A.P. are $a-d, a, a+d$

Sum of terms is, $a-d+a+a+d=33 \Rightarrow a=11$
Product of terms is, $(a-d)\,a\,(a+d)=11(121-d^2)=1155$
$\Rightarrow 121-d^2=105 \Rightarrow d=\pm 4$
if $d=4$
$T_{11}=T_1+10d=7+10(4)=47$
if $d=-4$
$T_{11}=T_1+10d=15+10(-4)=-25$
Hence, absolute value of $T_{11}=|T_{11}|=|-25|=25$

19. **(3748)** $\boxed{\text{Using Key Notes}}$ The given sequences upto 2018 terms are

1, 6, 11, 16, ..., 10086 and 9, 16, 23, ..., 14128
The common terms are
16, 15, 86, ... upto n terms, where
$16+(n-1)\,35 \le 10086$
$\Rightarrow \quad 35n-19 \le 10086$
$$\Rightarrow \quad n \le \frac{10105}{35}=288.7$$
$\therefore \quad n=288$
$\therefore \quad n(X \cup Y)=n(X)+n(Y)-n(X \cap Y)$
$\quad = 2018+2018-288=3748$

20. **(12)** $\boxed{\text{Using Key Notes}}$ Let the sides be a – d, a, a + d where d is positive. Using

Pythagoras theorem,
$(a+d)^2=(a-d)^2+a^2 \Rightarrow a=4d$
$\therefore$ Sides are 3d, 4d, 5d
Area = 24 $\Rightarrow \frac{1}{2}\times 3d \times 4d=24$
$\Rightarrow \quad d=4$
$\therefore$ Smallest side = 3d = 12

18 Probability

Review of Key Notes and Formulae

1. Basic Terms

(i) *Random experiment :* It is an experiment which when repeated under identical conditions does not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes.

Possible result of a random experiment is called its outcome.

An experiment is called random experiment if it satisfies the following two conditions:

(a) It has more than one possible outcome.
(b) It is not possible to predict the outcome in advance.

(ii) *Sample space :* The set of all possible outcomes of an experiment is called the sample space. It is denoted by S. An element of S is called a sample point.

2. Event and its Occurence

Any subset of a sample space is called an event. Any event (say E) of a sample space (say S) is said to have occured if the outcome (say ω) of the experiment is such that $\omega \in E$. If the outcome ω is such that $\omega \notin E$, we say that the event E has not occured.

3. Types of Events

(i) *Impossible event :* The null set ϕ is called the impossible event or null event.

(ii) *Sure or certain event :* The entire sample space is called the certain event.

(iii) *Simple event :* An event which consists of only one sample point of a sample space is a simple event.

(iv) *Compound event :* An event which consists of more than one sample point is called a compound event.

(v) *Mutually exclusive events :* If two events cannot occur simultaneously then they are called mutually exclusive. If A and B are mutually exclusive, then $A \cap B = \phi$

(iv) *Exhaustive events :* If the union of given events is the sample space S, then those events are exhaustive events.

In general, if $E_1, E_2, \ldots, E_n$ are n events of a sample space S and if $E_1 \cup E_2 \cup E_3 \ldots \cup E_n = S$ then $E_1, E_2, \ldots, E_n$ are called exhaustive events.

4. Algebra of Events

(i) *Complementary event :* For every event A, there corresponds another event A′ called complementary event to A. It is also called event 'not A'. It is the set of All sample points other than the sample points of A, i.e., $A' = S - A$

(ii) *The event (A or B) :* Event (A or B) contains all those elements which are either in A or in B or in both. It is also denoted by $(A \cup B)$. Therefore,

Event (A or B) $= A \cup B$

$= \{\omega : \omega \in A \text{ or } \omega \notin B\}$

(iii) *The event (A and B) :* Event (A and B) is the set of all those elements which are common to both A and B. i.e., which belongs to both A and B. It is also denoted by $A \cap B$.

$\therefore A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

(iv) *The event (A but not B) :* It is also denoted by $(A - B)$ and $(A \cap B')$. It is the set of all those elements which are in A but not in B.

5. Probability of Equally Likely Outcomes

Let S be a sample space and E be an event such that $n(S) = n$ and $n(E) = m$. If each outcome is equally likely, then we have

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total possible outcomes}}$$

6. Axiomatic Approach to Probability

The probability P is a real valued function whose domain is the power set of S and range is the interval [0, 1] such that,

(i) For any event E, $P(E) \geq 0$

(ii) $P(S) = 1$

(iii) $P(E \cup F) = P(E) + P(F)$, if E & F are mutually exclusive events

Let S be a sample space containing outcomes, $\omega_1, \omega_2, \ldots, \omega_n$

$S = \{\omega_1, \omega_2, \ldots, \omega_n\}$

Then

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \Sigma P(\omega_i)$, $\omega_i \in A$

7. **Probability of the Event 'A or B'**

For any two non-disjoint sets A and B, the probability of the event (A or B) is given by

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are disjoint sets i.e., they are mutually exclusive events, then $A \cap B = \phi$

Thus for mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B)$$

Note that $P(\phi) = 0$, $P(S) = 1$

8. **Probability of Event 'not A'**

Consider an event A associated with the sample space S. Then the probability of the event 'not A' is given by $P(A') = 1 - P(A)$

9. **Odds in Against and Odds in Favour of an Event**

Let there be $(m + n)$ equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases.

Then by definition of probability of occurrences of event A, $P(A) = \frac{m}{m+n}$.

The probability of non-occurrence of event A, $P(A') = \frac{n}{m+n}$

$\therefore\ P(A) : P(A') = m : n$

Odd in favour of occurrences of the event A are defined by $m : n$ *i.e.* $P(A) : P(A')$; and the odds against the occurrence of the event A are defined by $n : m$ *i.e.* $P(A') : P(A)$.

10. **Conditional Probability**

If E and F are two events associated with the sample space of a random experiment, the conditional probability of the event E given that F has occured is given as:

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{n(E \cap F)}{n(F)}, P(F) \neq 0$$

Properties of Conditional Probability :

(i) Let E & F be events of sample space S of an experiment, then we have $P(S/F) = P(F/F) = 1$.

(ii) If A and B are any two events of a sample space S & F is an event of S such that $P(F) \neq 0$, then

$P((A \cup B)/F) = P(A/F) + P(B/F) - P((A \cap B)/F)$

In particular if A and B are disjoint events, then

$P((A \cup B)/F) = P(A/F) + P(B/F)$

(iii) $P(E'/F) = 1 - P(E/F)$

11. Multiplication Theorem on Probability

For two events E & F associated with a sample space S, we have $P(E \cap F) = P(E)\, P(F/E)$

$= P(F)\, P(E/F)$

provided $P(E) \neq 0$ & $P(F) \neq 0$

The above result is known as Multiplication Rule of Probability.

12. Independent Events

Two or more events are said to be independent if occurrence or non-occurrence of any of them does not affect the probability of occurrence or non-occurrence of other events. For example, when two cards are drawn from a pack of 52 playing cards with replacement (the first card drawn is put back in the pack & then second card in drawn).

(i) If E & F are independent, then

$$P(E \cap F) = P(E)\, P(F)$$
$$P(E/F) = P(E),\ P(F) \neq 0$$
$$P(F/E) = P(F),\ P(E) \neq 0$$

(ii) Three events A, B & C are said to be mutually independent, if

$$P(A \cap B) = P(A)\, P(B)$$
$$P(A \cap C) = P(A)\, P(C)$$
$$P(B \cap C) = P(B)\, P(C)$$

& $P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$

If at least one of the above is not true for three given events, we say that the events are not independent.

13. Baye's Theorem

(i) *Partition of a sample space :* A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of the sample space S if

(a) $E_i \cap E_j = \phi$, i, j, i, j = 1, 2, 3, ... n

(b) $E_1 \cup E_2 \cup ... \cup E_n = S$

(c) $P(E_i) > 0$ for all i = 1, 2, ... n

(ii) *Theorem of total probability :* Let $\{E_1, E_2, ..., E_n\}$ be a partition of the sample space S, and suppose that each of the events $E_1, E_2, ..., E_n$ has nonzero probability of occurence. Let A be any event associated with S, then

$$P(A) = \sum_{j=1}^{n} P(E_j)\, P(A/E_j)$$

(iii) *Baye's Theorem :* If $E_1, E_2, ..., E_n$ are non-empty events which constitute a partition of sample space S & A is any event of non-zero probability.

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{j=1}^{n} P(E_j)P(A/E_j)} \qquad \text{for any } i = 1, 2, 3, ..., n$$

14. Random Variable & its Probability Distributions

A random variable is a real valued function whose domain is the sample space of a random experiment.

The probability distribution of a random variable X is the system of numbers.

X :	x_1	x_2	...	x_n
P(X):	p_1	p_2	...	p_n

where, $p_i > 0, \sum_{i=1}^{n} p_i = 1, i = 1, 2, ..., n$

The real numbers $x_1, x_2, ..., x_n$ are the possible values of the random variable X and $P_i (i = 1, 2, ..., n)$ is the probability of the random variable X taking the value x_i i.e., $P(X = x_i) = p_i$

15. Mean of a Random Variable

The mean (μ) of a random variable X is also called the expectation of X, denoted by E(X)

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i$$

Here $x_1, x_2, ..., x_n$ are possible values of random variable X, occuring with probabilities $p_1, p_2, ..., p_n$ respectively.

16. Variance of a Random Variable

Let X be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p(x_1), p(x_2), ..., p(x_n)$ respectively. Also let $\mu = E(X)$ be the mean of X, then the variance of X is given as:

$$\text{Var (X) or } \sigma_x^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i) = E(X-\mu)^2 = E(X^2) - [E(X)]^2$$

The non-negative number, $\sigma_x = \sqrt{\text{Var}(X)}$ is called the Standard Deviation of random variable X.

17. Bernoulli Trials & Binomial Distribution

(i) *Bernoulli trials:* Trials of a random experiments are called Bernoulli trials, if they satisfy the following conditions:

(a) There should be a finite number of trials.

(b) The trials should be independent.
(c) Each trial has exactly two outcomes: success or failure.
(d) The probability of success remains same in each trial.

(ii) *Binomial distribution:* The probability distribution of number of success in an experiment consisting of n Bernoulli trials may p + q = 1. Hence, this distribution (also called Binomial distribution B(n, p)) of number of success X can be written as:

X	0	1	2	– – –	x	n
P(x)	${}^nC_0q^n$	${}^nC_1q^{n-1}p^1$	${}^nC_2q^{n-2}p^2$		${}^nC_xq^{n-x}p^x$	${}^nC_np^n$

The probability of x successes P(X = x) is also denoted by P(x) is given as:
$P(x) = {}^nC_xq^{n-x}p^x$, x = 0, 1, ..., n (q = 1 – p).
This P(x) is called the probability function of the binomial distribution.

18. Extension of Multiplication Theorem for Independent Events

If $A_1, A_2...,A_n$ are independent events associated with a random experiment, then $P(A_1 \cap A_2 \cap A_3 \cap ... \cap A_n) = P(A_1)P(A_2)... P(A_n)$.

19. Probability of Occurence of at Least One of the *n* Independent Events

If $p_1, p_2, p_3, ..., p_n$ be the probabilities of happening of *n* independent events $A_1, A_2, A_3, ..., A_n$ respectively, then

(a) Probability of happening none of them

$$= P(\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 .... \cap \overline{A}_n)$$
$$= P(\overline{A}_1).P(\overline{A}_2).P(\overline{A}_3)....P(\overline{A}_n)$$
$$= (1 - p_1)(1 - p_2)(1 - p_3)....(1 - p_n)$$

(b) Probability of happening at least one of them

$$= P(A_1 \cup A_2 \cup A_3 .... \cup A_n)$$
$$= 1 - P(\overline{A_1 \cup A_2 \cup ...... \cup A_n})$$
$$= 1 - P(\overline{A}_1)P(\overline{A}_2)P(\overline{A}_3)....P(\overline{A}_n)$$
$$= 1 - (1 - p_1)(1 - p_2)(1 - p_3)...(1 - p_n)$$

20. Probability Regarding *n* Letters and Their Envelopes

If *n* letters corresponding to *n* envelopes are placed in the envelopes at random, then

(a) Probability that all letters are in right envelopes $= \dfrac{1}{n!}$

(b) Probability that all letters are not in right envelopes

$$= 1 - \frac{1}{n!}$$

(c) Probability that no letters is in right envelopes

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ....+ (-1)^n \frac{1}{n!}$$

(d) Probability that exactly r letters are in right envelopes

$$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots.. + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

21. Mode of Binomial Distribution

The mode of a binomial distribution is r if for X = r, the probability function p(X) is the maximum.

i.e., $p(r) > p(r-1)$ and $p(r) > p(r+1)$

22. Method of Finding Mode

Let X, n, p be respectively binomial variate and the two parameters

(i) *Case I: If (n + 1)p is a positive integer :* Let $(n+1)p = k$. The binomial distribution will be bi-modal, the modal values of X being k and k – 1.

(ii) *Case II: If (n + 1)p is not an integer :* Let $(n+1)p = k + f$, $0 < f < 1$ and $k \in N$. The binomial distribution will have unique mode (unimodal) the modal value of x being k.

TIPS! & TRICKS

TIPS AND TRICKS: (T-1)

For two events A and B

(i) P(at least one out of them) = $P(A) + P(B) - P(A \cap B)$

(ii) P(exactly one out of them) = $P(A) + P(B) - 2P(A \cap B)$

Illustration 1

Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(i) the problem is solved

(ii) exactly one of them solves the problem.

Short-cut solution :

$\boxed{\text{Using T-1 (i)}}$ (i) Here $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.

Since A and B Solve the problem independently.

$\therefore P(A \cap B) = P(A) \cdot P(B)$

$\therefore$ P(the problem is solved) = P(at least one solved a problem)

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A) \cdot P(B)$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} \times \frac{1}{3} = \frac{4}{6} = \frac{2}{3}$$

(ii) Using T-1 (ii)

P(exactly one solves the problem)

$= P(A) + P(B) - 2P(A \cap B)$

$= P(A) + P(B) - 2P(A) \cdot P(B)$

$= \frac{1}{2} + \frac{1}{3} - 2 \times \frac{1}{2} \times \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

Illustration 2

In a class of 60 students, 40 opted for NCC, 30 opted for NSS and 20 opted for both NCC and NSS. If one of these students is selected at random, then the probability that the student selected has opted neither for NCC nor for NSS is : **[JEE M 2019]**

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{5}{6}$

Short-cut solution :

Using T-1 P = Set of students who opted for NCC

Q = Set of Students who opted for NSS

$n(P) = 40$, $n(Q) = 30$, $n(P \cap Q) = 20$

$n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

$= 40 + 30 - 20 = 50$

$\therefore$ Hence, required probability $= 1 - \frac{50}{60} = \frac{1}{6}$ **Ans. (a)**

TIPS & TRICKS!

TIPS AND TRICKS: (T-2)

For three events A, B and C

(i) P(at least one out of them) $= P(A \cup B \cup C)$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two out of them) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P (exactly two out of them) $= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P (exactly one out of them) $= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Illustration 3

The probabilities of three events A, B and C are $P(A) = 0.6$,

$P(B) = 0.4$, $P(C) = 0.5$, $P(A \cap B) = 0.3$

$P(A \cap C) = 0.3$, $P(B \cap C) = 0.2$ and $P(A \cap B \cap C) = 0.2$ then

(i) Find the probability of at least two out of A, B and C

(ii) Find the probability of exactly two-out of them.

(iii) Find the probability one out of them.

Short-cut solution :

$\boxed{\text{Using T-2 (ii)}}$ (i) P(at least two out of them)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

$= 0.3 + 0.2 + 0.3 - 2 \times 0.2 = 0.4$

$\boxed{\text{Using T-2 (iii)}}$ (ii) P(exactly two out of them)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

$= 0.3 + 0.2 + 0.3 - 3 \times 0.2 = 0.2$

$\boxed{\text{Using T-2 (iv)}}$ (iii) P(exactly one out of them)

$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2(C \cap A) + 3P(A \cap B \cap C)$

$= 0.6 + 0.4 + 0.5 - 2(0.3) - 2(0.2) - 2(0.3) + 3(0.2) = 0.5$

SHORTCUTS: (SC-1)

Expected value (Mean) E(X).

Let X and Y be random variables on the sample space S and K are real number. Then

(i) $E(K) = K$

(ii) $E(KX) = KE(X)$

(iii) $E(X \pm Y) = E(X) \pm E(Y)$

(iv) $E(KX + b) = KE(X) + b$

(v) If X and Y are independent then $E(XY) = E(X)\,E(Y)$.

Variance V(X) :

$\because V(X) = E(X^2) - [E(X)]^2$

(vi) $V(KX) = K^2V(X)$

(vii) $V(K) = 0$

Illustration 4

If $E(X) = 2$ and $E(Y) = 4$ then $E(Y - X)$ is equal to

(a) 2 (b) 6 (c) 0 (d) None

Short-cut solution :

$\boxed{\text{Using SC-1(iii)}}$ $E(Y - X) = E(Y) - E(X) = 4 - 2 = 2.$ **Ans. (a)**

Illustration 5

If $Y = 3X + 4$ and $E(X) = 5$ then value of E(Y) is

(a) 5 (b) 15 (c) 19 (d) 4

Short-cut solution :

$\boxed{\text{Using SC-1(iv)}}$ $\because Y = 3X + 4 \therefore E(Y) = E(3X + 4) = 3E(X) + 4$

$= 3 \times 5 + 4 = 19.$ **Ans. (c)**

SHORTCUTS: (SC-2)

Mean, variance and standard deviation of binomial distribution are np, npq, $\sqrt{npq}$ respectively.

Note: If np = integer, the binomial distribution will be unimodal and the mean = mode.

Illustration 6

The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probability of 2 successes is **[AIEEE 2004]**

(a) $\frac{28}{256}$ (b) $\frac{219}{256}$ (c) $\frac{128}{256}$ (d) $\frac{37}{256}$

Short-cut solution :

Using SC-2 Given that mean = $np = 4$ and variance = $npq = 2$

$\Rightarrow p = q = \frac{1}{2}$ and $n = 8$

$\therefore \quad P(2 \text{ success}) = {}^8C_2 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2$

$= \frac{28}{2^8} = \frac{28}{256}$ **Ans. (a)**

TECHNIQUE

Boole's Inequality.

For any two events A and B

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(A \cup B) \le P(A) + P(B)$ $\{\because P(A \cap B) \ge 0\}$

For any three events A, B and C

$P(A \cup B \cup C) \le P(A) + P(B) + P(C)$

In general for any n events $A_1, A_2, \ldots\ldots\ldots, A_n$;

$P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) \le P(A_1) + P(A_2) + \ldots\ldots\ldots + P(A_n)$

Illustration 7

If A and B are arbitrary events, then

(a) $P(A \cap B) \ge P(A) + P(B)$ (b) $P(A \cup B) \le P(A) + P(B)$

(c) $P(A \cap B) = P(A) + P(B)$ (d) None of these

Short-cut solution :

Using Tech. We know,

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le P(A) + P(B)$ $[\because P(A \cap B) \ge 0]$

Ans. (b)

Concept Booster Exercise

1. Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is: **[JEE M 2020]**
   (a) 0.02 (b) 0.20 (c) 0.01 (d) 0.10
2. If A,B and C are three events, then which of the following is/are not correct?
   (a) P(exactly two of A, B and C occur) $\le P(A \cap B) + P(B \cap C) + P(C \cap A)$
   (b) $P(A \cup B \cup C) \le P(A) + P(B) + P(C)$
   (c) P(exactly one of A, B and C occur) $\le P(A) + P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B)$
   (d) P(at least two out of them) $\ge P(A \cap B) + P(B \cap C) + P(C \cap A)$
3. If $E[X] = 1$ and $V(X) = 5$, then value of $E[(2 + X)^2]$ is
   (a) 14 (b) 9 (c) 27 (d) 49
4. If $E(X) = 5$ and $V(X) = 2$ then value of $V(4 + 3X)$ is
   (a) 6 (b) 18 (c) 22 (d) 10
5. The mean and variance of a random variable X having binomial distribution are 4 and 2 respectively, then $P(X = 1)$ is **[AIEEE 2003]**
   (a) $\frac{1}{4}$ (b) $\frac{1}{32}$ (c) $\frac{1}{16}$ (d) $\frac{1}{8}$
6. If X has a binomial distribution, B(n, p) with parameters n and p such that $P(X = 2) = P(X = 3)$, then E(X), the mean of variable X, is **[JEE M 2014]**
   (a) $2 - p$ (b) $3 - p$ (c) $\frac{p}{2}$ (d) $\frac{p}{3}$
7. If the mean and the variance of a binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than or equal to one is : **[JEE M 2015]**
   (a) $\frac{9}{16}$ (b) $\frac{3}{4}$ (c) $\frac{1}{16}$ (d) $\frac{15}{16}$

Numerical Value Problems

8. Four fair dice are thrown independently 27 times. Then the expected number of times, at least two dice show up a three or a five, is ______. **[JEE M 2020]**
9. In a bombing attack, there is 50% chance that a bomb will hit the target. At least two independent hits are required to destroy the target completely. Then the minimum number of bombs, that must be dropped to ensure that there is at least 99% chance of completely destroying the target, is _________. **[JEE M 2020]**

10. The probability of a man hitting a target is $\frac{1}{10}$. The least number of shots required, so that the probability of his hitting the target at least once is greater than $\frac{1}{4}$, is ____________. **[JEE M 2020]**

11. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha\ 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). **[JEE Adv. 2013]**

Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}$

12. Let S be the sample space of all 3×3 matrices with entries from the set $\{0,1\}$. Let the events E_1 and E_2 be given by $E_1 = \{A \in S : det\ A = 0\}$ and $E_2 = \{A \in S$: sum of entries of A is $7\}$.
If a matrix is chosen at random from S, then the conditional probability $P(E_1/E_2)$ equals _____. **[JEE Adv. 2019]**

Solutions

1. **(d)** $\boxed{\text{Using T-1 (ii)}}$ $P(\text{exactly one}) = \frac{2}{5}$

$\Rightarrow P(A) + P(B) - 2P(A \cap B) = \frac{2}{5}$ ...(i)

$P(A \text{ or } B) = P(A \cup B) = \frac{1}{2}$

$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{1}{2}$ ...(ii)

From (i) and (ii)

$\therefore\ P(A \cap B) = \frac{1}{2} - \frac{2}{5} = \frac{5-4}{10} = \frac{1}{10} = 0.10$

2. **(d)** $\boxed{\text{Using T-2 (iii)}}$ $\because$ P(exactly two of A, B and C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

$\leq P(A \cap B) + P(B \cap C) + P(C \cap A)$

$\therefore$ Option (a) is correct.

$\boxed{\text{Using Tech.}}$ $\because P(A \cap B \cap C) = P(A \cup B) + P(C) - P\{(A \cup B) \cap C\}$

$\leq P(A \cup B) + P(C) \leq P(A) + P(B) + P(C)$

$\therefore$ Option (b) is correct.

$\because$ $\boxed{\text{Using T-2 (iv)}}$ P(exactly one of A, B and C occur)

$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(C \cap A)$

$+ 3P(A \cap B \cap C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) - \{P(A \cap B)$

$+ P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)\}$

$\leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)$

$\therefore$ Option (c) is correct.

$\boxed{\text{Using T-2 (ii)}}$ Also P(at least two out of them)

$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C) \leq P(A \cap B)$

$+ P(B \cap C) + P(C \cap A)$

$\therefore$ Option (d) is not correct.

3. **(a)** $\boxed{\text{Using SC-1}}$

$E[(2 + X)^2] = E[4 + 4X + X^2] = 4 + 4\,E(X) + E(X^2)$

$= 4 + 4 + V(X) + (E[X])^2$

$= 4 + 4 + 5 + 1^2 = 14.$

4. **(b)** $\boxed{\text{Using SC-1}}$

$\because V[4 + 3X] = V[4] + 3^2 V[X]$

$= 0 + 3^2 \times 2 = 18$

5. **(b)** $\boxed{\text{Using SC-2}}$ Given that $np = 4$ and $npq = 2$

$\Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 8$

$P(X = 1) = {}^8C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^7 = 8.\frac{1}{2^8} = \frac{1}{2^5} = \frac{1}{32}$

6. **(b)** $\boxed{\text{Using SC-2}}$ Since X has a binomial distribution, B (n, p)

$\therefore\ P(X = 2) = {}^nC_2\,(p)^2\,(1 - p)^{n-2}$

and $P(X = 3) = {}^nC_3\,(p)^3\,(1 - p)^{n-3}$

Given $P(X = 2) = P(X = 3)$

$\Rightarrow\ {}^nC_2\,p^2\,(1 - p)^{n-2} = {}^nC_3\,(p)^3\,(1 - p)^{n-3}$

$\Rightarrow \frac{n!}{2!(n-2)!}.\frac{p^2(1-p)^n}{(1-p)^2} = \frac{n!}{3!(n-3)!}.\frac{p^3(1-p)^n}{(1-p)^3}$

$\Rightarrow \frac{1}{n-2} = \frac{1}{3}.\frac{p}{1-p} \Rightarrow 3\,(1 - p) = p\,(n - 2)$

$\Rightarrow 3 - 3p = np - 2p$

$\Rightarrow np = 3 - p$

$\Rightarrow E(X) = \text{mean} = 3 - p \qquad (\because \text{mean of B}\,(n, p) = np)$

7. **(d)** $\boxed{\text{Using SC-2}}$ Let mean = np = 2 ...(i)

and variance = npq = 1 ...(ii)

On solving eqns. (i) and (ii), we get

$q = \frac{1}{2}$ and $p = \frac{1}{2}$

From eqn (i), we have

n = 4

$P(X \geq 1) = {}^4C_1p^1q^3 + {}^4C_2p^2q^2 + {}^4C_3p^3q + {}^4C_4p^4$

$= 1 - {}^4C_0p^0q^4 = 1-\left(\frac{1}{2}\right)^4 = 1-\frac{1}{16}=\frac{15}{16}$

8. **(11)** $\boxed{\text{Using SC-2}}$

Probability of getting at least two 3's or 5's in one trial

$$= {}^4C_2\left(\frac{2}{6}\right)^2\left(\frac{4}{6}\right)^2 + {}^4C_3\left(\frac{2}{6}\right)^3\left(\frac{4}{6}\right) + {}^4C_4\left(\frac{2}{6}\right)^4$$

$$= \frac{33}{3^4} = \frac{11}{27}$$

$$E(X) = np = 27\left(\frac{11}{27}\right) = 11.$$

9. **(11.00)** $\left[\begin{array}{l}\textbf{Tricks}\\ P(X \geq 2) = 1 - P(X=0) - P(X=1)\end{array}\right]$

Let 'n' bombs are required, then

$$P(X \geq 2) = 1 - {}^nC_1 \cdot \left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{n-1} - {}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n \geq \frac{99}{100}$$

$$\Rightarrow \frac{1}{100} \geq \frac{n+1}{2^n} \Rightarrow 2^n \geq 100(n+1) \Rightarrow n \geq 11$$

10. **(3.00)** $p = \frac{1}{10}, q = \frac{9}{10}$

$\left[\begin{array}{l}\textbf{Tricks}\\ P(X \geq 1) = 1 - P(X=0)\end{array}\right]$

$$P\text{ (not hitting target in } n \text{ trials)} = \left(\frac{9}{10}\right)^n$$

$$P\text{ (at least one hit)} = 1-\left(\frac{9}{10}\right)^n$$

$$\because 1-\left(\frac{9}{10}\right)^n > \frac{1}{4} \Rightarrow (0.9)^n < 0.75$$

$$\therefore n_{\text{minimum}} = 3.$$

11. **(6)** Let $P(E_1) = x; P(E_2) = y, P(E_3) = z$

$$\begin{bmatrix} \textbf{Tricks} \\ \text{For independent} \\ P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \\ = [1 - P(A)] \cdot [1 - P(B)] \cdot [1 - P(C)] \end{bmatrix}$$

$P(\text{only } E_1) = P(E_1 \cap \bar{E}_2 \cap \bar{E}_3) = x(1 - y)(1 - z) = \alpha$

$P(\text{only } E_2) = P(\bar{E}_1 \cap E_2 \cap \bar{E}_3) = (1 - x)\, y\, (1 - z) = \beta$

$P(\text{only } E_3) = P(\bar{E}_1 \cap \bar{E}_2 \cap E_3) = (1 - x)(1 - y)\, z = \gamma$

$P(\text{none}) = P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) = (1 - x)(1 - y)(1 - z) = p.$

Now given $(\alpha - 2\beta)\, p = \alpha\beta \Rightarrow x = 2y$

and $(\beta - 3r)p = 2\beta r \Rightarrow y = 3z \therefore x = 6z$

Hence $\dfrac{P(E_1)}{P(E_3)} = \dfrac{x}{z} = 6$

12. **(0.50)** $\begin{bmatrix} \textbf{Tricks} \\ \text{Use permutation and combination} \end{bmatrix}$

Total number of 3×3 matrices with 0 or $1 = 2^9 = 512$

E_2 contains those matrices in which sum of entries is 7.

$\therefore$ It will be contains 7 one's and 2 zeroe's.

$\therefore$ $n(E_2) = {}^9C_2 = 36$

$E_1 \cap E_2$ contains those matrices in which 7 ones, 2 zeroes and its det is zero.

Det(A) = 0. This can be occurs when two rows/columns are identical.

e.g.

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \text{ or } \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore$ $n(E_1 \cap E_2) = {}^3C_1 \times {}^3C_1 \times 2 = 18$

$\therefore$ $P(E_1 / E_2) = \dfrac{P(E_1 \cap E_2)}{P(E_2)} = \dfrac{18/512}{36/512} = \dfrac{1}{2} = 0.50$

UNIT-III : TRIGONOMETRY

19 Trigonometric Ratios

Review of Key Notes and Formulae

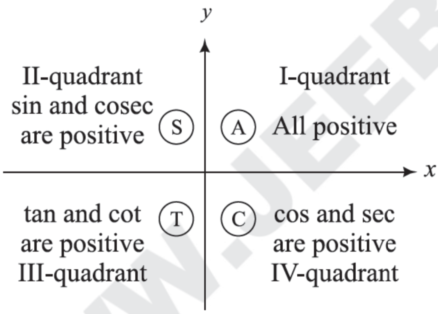
1. Fundamental Trigonometric Identities

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

2. Sign of Trigonometric Functions



Cruse Aid to Memorize
"All Students to California"

3. Trigonometric Ratios of Compound Angles

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(iv) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

4. Transformation Formulae

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = (\sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

(v) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(vi) $\sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$

(vii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(viii) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

5. Trigonometric Ratios of Multiple Angles

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

(iv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(v) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(vi) $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

6. Trigonometric Ratios of Sub-Multiple Angles

(i) $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$

(ii) $\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$

(iii) $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

7. Maximum and minimum value of $E = a\sin\theta + b\cos\theta$

Maximum value $= \sqrt{a^2+b^2}$, Minimum value $= -\sqrt{a^2+b^2}$

8. Conditional Trigonometric Identities

If $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$

(iii) $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(v) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(vi) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(vii) $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

(viii) $\tan\dfrac{A}{2}\tan\dfrac{B}{2} + \tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} = 1$

Some Useful Series

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left[\alpha + \left(\dfrac{n-1}{2}\right)\beta\right]\left[\sin\left(\dfrac{n\beta}{2}\right)\right]}{\sin\left(\dfrac{\beta}{2}\right)};\ \beta \neq 2n\pi$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left[\alpha + \left(\dfrac{n-1}{2}\right)\beta\right]\left[\sin\left(\dfrac{n\beta}{2}\right)\right]}{\sin\left(\dfrac{\beta}{2}\right)};\ \beta \neq 2n\pi$$

10. Some Important Results to Remember

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

(iv) $\tan A = \cot A - 2\cot(2A)$

TIPS & TRICKS

TIPS AND TRICKS: (T-1)

If $\theta_1 + \theta_2 = 90^\circ \Rightarrow \begin{cases} \tan\theta_1 . \tan\theta_2 = 1 \\ \cot\theta_1 . \cot\theta_2 = 1 \end{cases}$

Illustration 1

The value of $\tan 3^\circ \tan 20^\circ \tan 40^\circ \tan 45^\circ \tan 50^\circ \tan 70^\circ \tan 87^\circ$ is equal to

Short-cut solution :

Using T-1 $\tan 3° \tan 87° \tan 20° \tan 70° \tan 40° \tan 50° \tan 45°$

$\because \theta_1 + \theta_2 = 90° \Rightarrow 1 \times 1 \times 1 \times 1 = 1$

Illustration 2

If $\tan 2\theta \,.\, \tan 7\theta = 1$, then $\tan 3\theta$ is equal to:

Short-cut solution :

Using T-1 $\because\; 2\theta + 7\theta = 90° \Rightarrow \theta = 10°$

Now, $\tan 3(10°) = \tan 30° = \dfrac{1}{\sqrt{3}}$

Illustration 3

If $\tan(x+y) \,.\, \tan(x-y) = 1$ then $\tan\left(\dfrac{2x}{3}\right)$ is equal to:

Short-cut solution :

Using T-1 $x + y + x - y = \dfrac{\pi}{2} \Rightarrow x = \dfrac{\pi}{4}$

Hence, $\tan\left(\dfrac{2}{3}.\dfrac{\pi}{4}\right) = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

TIPS AND TRICKS: (T-2)

$$\frac{\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \ldots + \sin\theta_n}{\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \ldots + \cos\theta_n} = \tan\left(\frac{\theta_1 + \theta_2 + \ldots + \theta_n}{n}\right)$$

Illustration 4

$\dfrac{\sin 2\theta + \sin 5\theta + \sin 4\theta + \sin 7\theta}{\cos 2\theta + \cos 5\theta + \cos 4\theta + \cos 7\theta}$ is equal to:

Short-cut solution :

Using T-2 $\tan\left(\dfrac{2\theta + 5\theta + 4\theta + 7\theta}{4}\right) = \tan\left(\dfrac{9\theta}{2}\right)$

Illustration 5

$\dfrac{\cos 2\theta + \cos 5\theta + \cos 8\theta}{\sin 2\theta + \sin 5\theta + \sin 8\theta}$ is equal to:

Short-cut solution :

$\boxed{\text{Using T-2}}$ $\cot\left(\frac{2\theta+5\theta+8\theta}{3}\right) = \cot(5\theta)$

TIPS AND TRICKS: (T-3)

If $\sec\theta + \tan\theta = x$, then, $\sec\theta - \tan\theta = \frac{1}{x}$ and

If $\text{cosec}\,\theta + \cot\theta = y$, then, $\text{cosec}\,\theta - \cot\theta = \frac{1}{y}$

Illustration 6

If $\sec\theta + \tan\theta = e^x$, then $\cos\theta$ is equal to:

Short-cut solution :

$\boxed{\text{Using T-3}}$ $\sec\theta - \tan\theta = e^{-x}$ ...(1)

$\sec\theta + \tan\theta = e^x$ ...(2)

On adding eqn (1) and (2)

$$\Rightarrow \quad \sec\theta = \frac{e^x + e^{-x}}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{2}{e^x + e^{-x}}$$

Illustration 7

If $\text{cosec}\,\theta + \cot\theta = \frac{11}{2}$, then $\tan\theta$ is equal to:

Short-cut solution :

$\boxed{\text{Using T-3}}$ $\because$ $\text{cosec}\,\theta - \cot\theta = \frac{2}{11}$ ...(1)

$\text{cosec}\,\theta + \cot\theta = \frac{11}{2}$ ...(2)

On subtracting eqns (1) and (2)

$$\Rightarrow \quad \cot\theta = \frac{117}{44}$$

$$\Rightarrow \quad \tan\theta = \frac{44}{117}$$

TIPS AND TRICKS: (T-4)

To find range of trigonometric functions of the type:

		Minimum	Maximum
(i) $a\sin^2 x + b\operatorname{cosec}^2 x$	$\Rightarrow$	$2\sqrt{ab}$	∞
(ii) $a\cos^2 x + b\sec^2 x$	$\Rightarrow$	$2\sqrt{ab}$	∞
(iii) $a\tan^2 x + b\cot^2 x$	$\Rightarrow$	$2\sqrt{ab}$	∞

★**Note:** This can be proved by using concept of AM $\geq$ GM.

Illustration 8

The range of $y = 4\sin^2 x + 9\operatorname{cosec}^2 x$ is equal to:

Short-cut solution :

Using T-4 Minimum $= 2\sqrt{ab} = 2\sqrt{4\times 9} = 12$

Maximum $\to \infty$

$\Rightarrow$ Range : $y \in [12, \infty)$

Illustration 9

The range of $y = 9\tan^2 x + 16\cot^2 x$ is equal to:

Short-cut solution :

Using T-4 Minimum $= 2\sqrt{ab} = 2\sqrt{9\times 16} = 24$

Maximum $\to \infty$

Hence, range is : $y \in [24, \infty)$

TIPS AND TRICKS: (T-5)

Method of Substitution:

If an expression is independent of an unknown parameter, then we can put any suitable value for that parameter to minimize the calculation.

Illustration 10

If $p = \dfrac{2\sin\theta}{1+\cos\theta+\sin\theta}$, then $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta}$ is equal to:

(a) $1-p$ (b) p (c) $1+p$ (d) $1/p$

Short-cut solution :

Using T-5 Put $\theta = 0 \Rightarrow p = 0$

Now, expression $\dfrac{1+\sin(0)-\cos(0)}{1+\sin(0)} = 0$ **Ans. (b)**

Illustration 11

If $\tan \alpha = \dfrac{n}{n+1}$ and $\tan \beta = \dfrac{1}{2n+1}$, then $\alpha + \beta$ is equal to:

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

Short-cut solution :

Using T-5 Put $n = 0 \Rightarrow \tan \alpha = 0 \Rightarrow \alpha = 0°$

and $\tan \beta = 1 \Rightarrow \beta = \dfrac{\pi}{4}$

Hence, $\alpha + \beta = \pi/4$ **Ans. (a)**

Illustration 12

If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to

(a) $2n$ (b) 2 (c) $2n - 2$ (d) $2n - 1$

Short-cut solution :

Using T-5 Since, $x = \dfrac{\pi}{2}$ satisfies in $\sin x + \operatorname{cosec} x = 2$

then, $\left(\sin\dfrac{\pi}{2}\right)^n + \left(\operatorname{cosec}\dfrac{\pi}{2}\right)^n = 2$ **Ans. (b)**

Illustration 13

If $x = \sin \theta + \cos \theta$ and $y = \sin \theta \cos \theta$, then, $x^4 - 4x^2y - 2x^2 + 4y^2 + 4y + 1$ is equal to

(a) 1 (b) 0 (c) –2 (d) 4

Short-cut solution :

Using T-5 Putting $\theta = \dfrac{\pi}{2} \Rightarrow x = 1$ and $y = 0$

Now, expression $(1)^4 - 0 - 2 + 0 + 0 + 1 = 0$ **Ans. (b)**

Illustration 14

The value of the expression

$\cos A \sin(B - C) + \cos B \sin(C - A) + \cos C \sin(A - B)$ is equal to:

(a) 0 (b) 1 (c) –1 (d) 2

Short-cut solution :

Using T-5 Putting $A = B = C = 0$ in the expression $\Rightarrow 0 + 0 + 0 = 0$

Ans. (a)

Illustration 15

In triangle ABC, $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}$ is equal to:

Short-cut solution :

Using T-5 Since, $A+B+C=\pi$

Putting $A=B=C=\pi/3$

$$\Rightarrow \sum\tan\frac{A}{2}\tan\frac{B}{2} = 3\left(\tan\frac{\pi}{6}.\tan\frac{\pi}{6}\right) = 1$$

Illustration 16

If $(\cos\theta+\sin\theta)^2+m\cos\theta\sin\theta=1, \forall\,\theta\in\{R-n\pi\}$ then value of 'm' is equal to:

(a) -1 (b) 1 (c) -2 (d) 2

Short-cut solution :

Using T-5 Putting $\theta=\pi/4 \Rightarrow m=-2$ **Ans. (c)**

Illustration 17

The value of $\tan A+2\tan 2A+4\tan 4A+8\cot 8A$ is equal to:

(a) $\tan 2A$ (b) $\tan A$ (c) $\cot 2A$ (d) $\cot A$

Short-cut solution :

Using T-5 Putting $A=\frac{\pi}{12}$

Expression $\tan\frac{\pi}{12}+2\tan\frac{\pi}{6}+4\tan\frac{\pi}{3}+8\cot\left(\frac{2\pi}{3}\right)=2+\sqrt{3}$

On checking options $\Rightarrow \cot A=\frac{\pi}{12}=2+\sqrt{3}$ **Ans. (d)**

Illustration 18

The value of $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ $(0\le\theta\le 15°)$ is equal to:

(a) $\cos\theta$ (b) $\sin\theta$ (c) $2\sin\theta$ (d) $2\cos\theta$

Short-cut solution :

Using T-5 Putting $\theta=0° \Rightarrow$ L.H.S. $=2$

Hence, checking options $\Rightarrow 2\cos(0)=2$ **Ans. (d)**

Illustration 19

If $\tan\theta = \dfrac{y}{x}$, then $\sqrt{\dfrac{x+y}{x-y}} + \sqrt{\dfrac{x-y}{x+y}}$ is equal to

(a) $\dfrac{2\cos\theta}{\sqrt{\cos 2\theta}}$ (b) $\dfrac{2\sin\theta}{\sqrt{\sin 2\theta}}$ (c) $\dfrac{2\sin\theta}{\sqrt{\cos 2\theta}}$ (d) $\dfrac{2\cos\theta}{\sqrt{\sin 2\theta}}$

Short-cut solution :

$\boxed{\text{Using T-5}}$ Putting $x = a, y = 0 \Rightarrow \theta = 0°$

$\Rightarrow$ Expression $\sqrt{\dfrac{a+0}{a-0}} + \sqrt{\dfrac{a-0}{a+0}} = 2$

Now, checking options $\Rightarrow \dfrac{2\cos(0)}{\sqrt{\cos(0)}} = 2$ **Ans. (a)**

SHORTCUTS: (SC-1)

Use of some important series:

(i) $\sin\theta \,.\, \sin(60° - \theta) \,.\, \sin(60° + \theta) = \dfrac{1}{4}\sin 3\theta$

(ii) $\cos\theta \,.\, \cos(60° - \theta) \,.\, \cos(60° + \theta) = \dfrac{1}{4}\cos 3\theta$

(iii) $\tan\theta .\ \tan(60° - \theta) \,.\, \tan(60° + \theta) = \tan 3\theta$

Illustration 20

The value of sin 10° sin 50° sin 60° sin 70° is equal to:

Short-cut solution :

$\boxed{\text{Using SC-1(i)}}$ $\underbrace{\sin 10° \sin(60° - 10°)\sin(60° + 10°)}.\sin 60°$

$\Rightarrow$ Expression is $\dfrac{1}{4}\sin(3.10°) \,.\, \sin 60° = \dfrac{\sqrt{3}}{16}$

Illustration 21

The value of cos 20° cos 30° cos 40° cos 80° is equal to:

Short-cut solution :

$\boxed{\text{Using SC-1(ii)}}$ $\cos 20° \cos(60° - 20°) \cos(60° + 20°) \,.\, \cos 30°$

$\Rightarrow$ Expression is $\dfrac{1}{4}\cos(3.20) \,.\, \cos 30° = \dfrac{\sqrt{3}}{16}$

Illustration 22

The value of tan 40° tan 20° tan 100° . tan 45°

Short-cut solution :

$\boxed{\text{Using SC-1(iii)}}$ tan 40° tan(60° – 40°) . tan(60° + 40) . tan 45°

$\Rightarrow$ Expression is tan (3.40) . tan 45 = $-\sqrt{3}$

TECHNIQUE

An increasing product series. Let S be an increasing product series of cos such that

$$S = \cos\alpha \cdot \cos 2\alpha \cdot \cos 2^2\alpha \, ... \, \cos(2^{n-1}\alpha)$$

Then $$S = \begin{cases} \dfrac{\sin 2^n\alpha}{2^n \sin\alpha}, & \alpha \neq n\pi \\ 1, & \alpha = 2n\pi \\ -1, & \alpha = (2n+1)\pi \end{cases}$$

Illustration 23

Find the value of $\cos\dfrac{\pi}{9}\cdot\cos\dfrac{2\pi}{9}\cdot\cos\dfrac{3\pi}{9}\cos\dfrac{4\pi}{9}$.

Short-cut solution :

$\boxed{\text{Using Tech.}}$ $\left(\cos\dfrac{\pi}{9}\cdot\cos\dfrac{2\pi}{9}\cdot\cos\dfrac{4\pi}{9}\right)\cos\dfrac{3\pi}{9}$

$$= \frac{\sin 2^3\dfrac{\pi}{9}}{2^3\sin\dfrac{\pi}{9}}\cdot\cos\frac{\pi}{3} = \frac{\sin\dfrac{8\pi}{9}}{8\sin\dfrac{\pi}{9}}\cdot\frac{1}{2}$$

$$= \frac{1}{16}\frac{\sin\dfrac{\pi}{9}}{\sin\dfrac{\pi}{9}} = \frac{1}{16}.$$

Concept Booster Exercise

1. If $\cot(11\theta) . \cot(7\theta) = 1$ then $\tan(9\theta)$ is equal to:
   (a) -1 (b) 1 (c) $\sqrt{2}$ (d) 2
2. Find value of $\tan 1° \tan 2° \tan 3° ... \tan 89°$
   (a) 1 (b) 2 (c) $\sqrt{2}$ (d) -1
3. The value of $\dfrac{\sin 1° + \sin 2° + \sin 3° + \sin 4° \sin 5°}{\cos 1° + \cos 2° + \cos 3° + \cos 4° + \cos 5°}$ is equal to:
   (a) $\tan 1°$ (b) $\tan 2°$ (c) $\tan 3°$ (d) $\tan 4°$
4. If $\cot x + \operatorname{cosec} x = 5$, then $\sin x$ is equal to
   (a) $\dfrac{12}{7}$ (b) $\dfrac{7}{12}$ (c) $\dfrac{12}{5}$ (d) $\dfrac{5}{13}$
5. If $\sec\theta + \tan\theta = n$, then $\dfrac{n^2-1}{n^2+1}$ is equal to:
   (a) $\sin\theta$ (b) $\cos\theta$ (c) $\operatorname{cosec}\theta$ (d) $\tan\theta$
6. The minimum value of $8\sec^2 2x + 6\cos^2 2x$ is equal to:
   (a) $4\sqrt{3}$ (b) $2\sqrt{3}$ (c) $\sqrt{3}$ (d) $8\sqrt{3}$
7. If $\dfrac{\tan 3x}{\tan x} = k$, then $\dfrac{\sin 3x}{\sin x}$ is equal to:
   (a) $\dfrac{2k}{k-1}$
   (b) $\dfrac{2k}{k-1}; k \in \left[\dfrac{1}{3}, 3\right]$
   (c) $\dfrac{2k}{k-1}; k \notin \left[\dfrac{1}{3}, 3\right]$
   (d) $\dfrac{k-1}{2k}; k \in R$
8. If $\alpha + \beta + \gamma = \pi$, then $\sin^2\alpha + \sin^2\beta - \sin^2\gamma$ is equal to:
   (a) $2\sin\alpha\sin\beta\sin\gamma$
   (b) $2\sin\alpha\cos\beta\cos\gamma$
   (c) $2\sin\alpha\sin\beta\cos\gamma$
   (d) $\sin\alpha\sin\beta\cos\gamma$
9. If $\theta \in \left(0, \dfrac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then **[AIEEE 2006]**
   (a) $t_1 > t_2 > t_3 > t_4$
   (b) $t_2 < t_1 < t_3 < t_4$
   (c) $t_3 > t_1 > t_2 > t_4$
   (d) $t_2 > t_3 > t_1 > t_4$

10. If $A + B + C = 180°$, then $\tan A + \tan B + \tan C$ is equal to:

(a) $\tan A \tan B \cot C$ (b) $\tan A \cot B \tan C$

(c) $\tan A \tan B \tan C$ (d) $\cot A \tan B \tan C$

11. The value of $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$ is

(a) 1 (b) 3 (c) –1 (d) –3

12. If $\theta = \frac{\pi}{(2^n - 1)}$, then $\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta$ is

(a) $\frac{1}{2^n}$ (b) $-\frac{1}{2^n}$ (c) $\frac{1}{2^n - 1}$ (d) $-\frac{1}{2^n - 1}$

Numerical Value Problems

13. The minimum value of $3\sec^2 x + 12\cos^2 x$ is equal to _____

14. The value of $\tan(15°)\tan(45°)\tan(75°)$ is equal to _____

Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ $\because \cot(11\theta) \,.\, \cot(7\theta) = 1 \Rightarrow 11\theta + 7\theta = 90^\circ \quad \Rightarrow \theta = 5^\circ$

Hence, $\tan(9 \times 5) = \tan 45^\circ = 1$

2. **(a)** $\boxed{\text{Using T-1}}$ $\because (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ) \ldots (\tan 45^\circ)$

and $1^\circ + 89^\circ = 2^\circ + 88^\circ = 3^\circ + 87^\circ = 90^\circ \quad \Rightarrow 1$

3. **(c)** $\boxed{\text{Using T-2}}$ Expression $= \tan\left(\dfrac{1^\circ + 2^\circ + 3^\circ + 4^\circ + 5^\circ}{5}\right) = \tan 3^\circ$

4. **(d)** $\boxed{\text{Using T-3}}$ $\operatorname{cosec} x - \cot x = \dfrac{1}{5}$ ...(1)

and $\quad \operatorname{cosec} x + \cot x = 5$ ...(2)

On adding eqn. (1) and eqn. (2) we get

$$2 \operatorname{cosec} x = \frac{26}{5} \quad \Rightarrow \quad \sin x = \frac{5}{13}$$

5. **(a)** $\boxed{\text{Using T-3}}$ $\sec\theta - \tan\theta = \dfrac{1}{n}$ ...(1)

and $\quad \sec\theta + \tan\theta = n$ ...(2)

On adding and subtracting eqn. (1) and (2) we get

$$2 \sec\theta = \frac{n^2+1}{n} \qquad \text{...(3)}$$

$$2 \tan\theta = \frac{n^2-1}{n} \qquad \text{...(4)}$$

From eqns. (3) and (4) we get: $\sin\theta = \dfrac{n^2-1}{n^2+1}$

6. **(d)** $\boxed{\text{Using T-4(ii)}}$ $(8\sec^2 2x + 6\cos^2 2x)_{\text{Min}} = 2\sqrt{8 \times 6} = 8\sqrt{3}$

7. **(c)** $\boxed{\text{Using T-5}}$ Putting $x = \dfrac{\pi}{3} \Rightarrow k = 0$ and $\dfrac{\sin\pi}{\sin\left(\dfrac{\pi}{3}\right)} = 0$

Now, cheking options for $k = 0$

$\Rightarrow k \neq 1$ which satisfies the condition $k \notin \left[\dfrac{1}{3}, 3\right]$

8. **(c)** $\boxed{\text{Using T-5}}$ Putting $\alpha = \beta = \gamma = \dfrac{\pi}{3}$

$$\Rightarrow \sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$$

Now, checking options for $\alpha = \beta = \gamma = \frac{\pi}{3}$

$= 2\sin\left(\frac{\pi}{3}\right) . \sin\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{\pi}{3}\right) = \frac{3}{4}$

9. (b) $\boxed{\text{Using T-5}}$ $\because \theta \in \left(0, \frac{\pi}{4}\right)$, putting $\theta = \frac{\pi}{6}$

$\Rightarrow t_4 = \left(\cot\left(\frac{\pi}{6}\right)\right)^{\cot(\pi/6)} = \left(\sqrt{3}\right)^{\sqrt{3}}$ which is greatest in option (b).

10. (c) $\boxed{\text{Using T-5}}$ Putting $A = B = C = 60°$

Expression is $\tan(60°) + \tan(60°) + \tan(60°) = 3\sqrt{3}$

Now, checking options for $A = B = C = 60°$

$\Rightarrow$ option (c) is $\tan(60°)\tan(60°)\tan(60°) = 3\sqrt{3}$

11. (a) $\boxed{\text{Using Tech.}}$ Let $\frac{2\pi}{15} = A$, $\frac{4\pi}{15} = 2A$, $\frac{8\pi}{15} = 4A$, $\frac{16\pi}{15} = 8A$ or $15A = 2\pi$

So, $16\cos A \cos 2A \cos 4A \cos 8A = 16(\cos A \cos 2A \cos 2^2A \cos 2^3 A)$

$= 16\left(\frac{\sin 2^4 A}{2^4 \sin A}\right)$

$= 16\frac{\sin(15A + A)}{16\sin A}$

$= \frac{\sin(2\pi + A)}{\sin A}$ $(\because 15A = 2\pi)$

$= \frac{\sin A}{\sin A} = 1$

12. (b) $\boxed{\text{Using Tech.}}$ $\theta \Rightarrow (2^n - 1)\pi\ 2^n\theta - \theta = \pi \Rightarrow 2^n\theta = (\pi + \theta)$

$\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \ ..... \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta} = \frac{1}{2^n}\frac{\sin(\pi + \theta)}{\sin\theta}$

$= \frac{1}{2^n}\left(-\frac{\sin\theta}{\sin\theta}\right) = -\frac{1}{2^n}$

13. (12) $\boxed{\text{Using T-4(ii)}}$ Here, $a = 3$, $b = 12$

$\Rightarrow$ Minimum Value $= 2\sqrt{ab} = 2\sqrt{3 \times 12} = 12$

14. (1) $\boxed{\text{Using SC-1(iii)}}$ $\because \tan\theta \tan(60° - \theta) . \tan(60° + \theta) = \tan 3\theta$

$\Rightarrow$ Here, $\theta = 15° \Rightarrow \tan(3 \times 15°) = 1$

20 Trigonometric Equations

Review of Key Notes and Formulae

1. **Definition:** An equation involving one or more trigonometric ratios of unknown angle is called a trigonometric equation.
2. **Solution of Trigonometric Equation:**

   The trigonometric equation may have infinite number of solutions.

   (i) *Principal Solution:* The solutions of trigonometric equation which lie in the interval $[0, 2\pi)$ are called principal solutions.

   (ii) *General Solution:* The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called general solution.

   (a) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\,\alpha;\ \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$

   (b) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha;\ \alpha \in [0, \pi], n \in I$

   (c) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + 2;\ \alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], n \in I$

   (d) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

   (e) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

   (f) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

   [★**Note:** 'α' is called the principal angle]
3. **Important Results to Remember**

   (i) $\sin\theta_1 + \sin\theta_2 + \ldots + \sin\theta_n = n \Rightarrow \sin\theta_1 = \sin\theta_2 = \ldots = \sin\theta_n = 1$

   (ii) $\cos\theta_1 + \cos\theta_2 + \ldots + \cos\theta_n = n \Rightarrow \cos\theta_1 = \cos\theta_2 = \ldots = \cos\theta_n = 1$

   (iii) $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta = 1$

   (iv) $\sin\theta + \operatorname{cosec}\theta = -2 \Rightarrow \sin\theta = -1$

   (v) $\cos\theta + \sec\theta = 2 \Rightarrow \cos\theta = 1$

   (vi) $\cos\theta + \sec\theta = -2 \Rightarrow \cos\theta = -1$

SHORTCUTS: (SC-1)

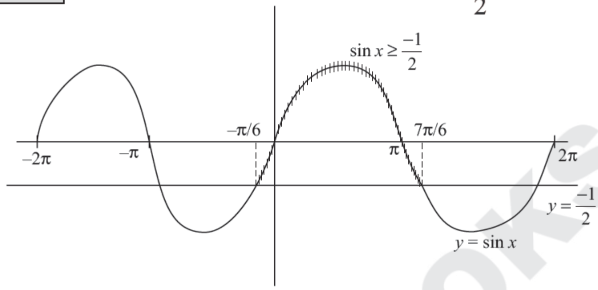
Solving trigonometric inequations using graphs.

Illustration 1

Solve the inequality: $\sin x \geq \frac{-1}{2}$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Drawing graphs of $y = \sin x$ and $y = \frac{-1}{2}$



From above figure, $\sin x \geq \frac{-1}{2}$ when $\frac{-\pi}{6} \leq x \leq \frac{7\pi}{6}$

Hence, the general solution

$\Rightarrow 2n\pi - \frac{\pi}{6} \leq x \leq 2n\pi + \frac{7\pi}{6}$; $n \in I$

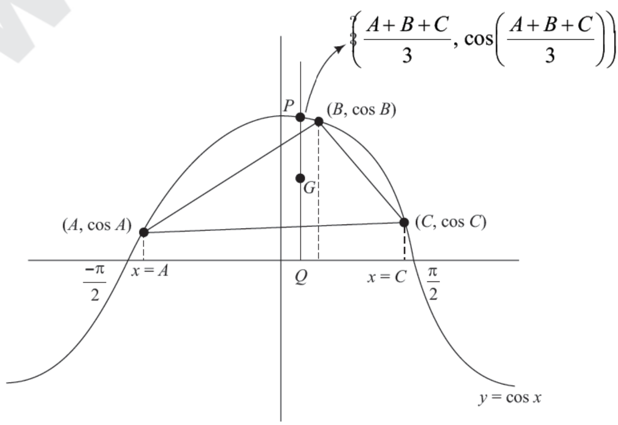
Illustration 2

If $A + B + C = \pi$, then prove that:

$\cos A + \cos B + \cos C < \frac{3}{2}$; when A, B, C are distinct.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Let, $x = A$, $x = B$, $x = C$ be three points on $f(x) = \cos x$. So that $A + B + C = \pi$



where, 'G' be centroid of triangle given by

$$G \equiv \left(\frac{A+B+C}{3}, \frac{\cos A+\cos B+\cos C}{3}\right)$$

Hence, from figure it is clear that P, G, Q are collinear points, where, ordinate of $GQ <$ ordinate of PQ.

$$\frac{\cos A+\cos B+\cos C}{3} < \cos\left(\frac{A+B+C}{3}\right)$$

$\Rightarrow \cos A + \cos B + \cos C < \dfrac{3}{2}$ Hence, proved.

TECHNIQUE

(i) The general solution of an equation of the form

$a \sin x + b \cos x = c$

Put $\dfrac{a}{r} = \sin\theta,\ \dfrac{b}{r} = \cos\theta,\ r = \sqrt{a^2+b^2}$

So, $r \sin\theta.\sin x + r \cos\theta.\cos x = c$

$\cos (x - \theta) = \dfrac{c}{r}$

Then, the general solution of the equation $a \sin x + b \cos x = c$ is $x = 2n\pi \pm \alpha + \theta$, where $n \in Z$, $\cos \alpha = \dfrac{c}{r}$, $\alpha \in [0, \pi]$

(ii) $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Illustration 3

Find solution of $\sin x + \sqrt{3}\cos x = \sqrt{2}$

Short-cut solution :

Using Tech.

$$r = \sqrt{a^2+b^2} = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$$

$$\therefore \quad \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2} \Rightarrow \sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x = \frac{1}{\sqrt{2}}$$

$$\cos\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4} \Rightarrow x = 2n\pi + \frac{\pi}{6} \pm \frac{\pi}{4}$$

Concept Booster Exercise

1. If $A + B + C = \pi$ and A, B, C are angles of triangle; then prove that:

$\sin A + \sin B + \sin C < \frac{3\sqrt{3}}{2}$

2. Solve the inequility: $\tan x < 3$.

3. If $A + B + C = \pi$, then prove that: $\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} > 1$

4. The value of $\tan 20° + \tan 40° + \sqrt{3}\ \tan 20° \tan 40° + \tan 60° + 2\sqrt{3}$ is equal to:

(a) $\frac{1}{\sqrt{3}}$ (b) $4\sqrt{3}$

(c) 1 (d) $\sqrt{2}$

5. The value of $\tan 30° + \tan 15° + \tan 30° \tan 15° + \tan 60°$, is equal to

(a) $1-\sqrt{3}$ (b) $\sqrt{3}$ (c) $1+\sqrt{3}$ (d) $-\sqrt{3}$

6. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solutions of the equation $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to **[JEE Adv. 2016]**

(a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$ (c) 0 (d) $\frac{5\pi}{9}$

7. The number of integral values of k for which the equation $7 \cos x + 5 \sin x = 2k + 1$ has a solution is **[AIEEE 2002S]**

(a) 4 (b) 8 (c) 10 (d) 12

8. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by:

(a) $x = 2n\pi;\ n = 0, \pm1, \pm2$ ....

(b) $x = 2n\pi + \pi/2;\ n = 0, \pm1, \pm2$ ....

(c) $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$, where $n = 0, \pm1, \pm2...$

(d) none of these

Solutions

1. $\boxed{\text{Using SC-1}}$

Drawing graph of $y = \sin x$

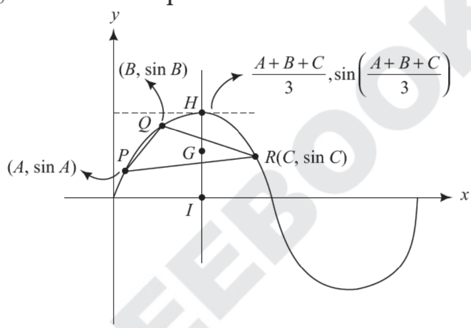
Let $x = A, x = B, x = C$ be three points on $f(x) = \sin x$

So that $A + B + C = \pi$,

where 'G' be the centeroid of triangle.

$$G \equiv \left(\frac{A+B+C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right)$$

where; G, H, I are collinear points



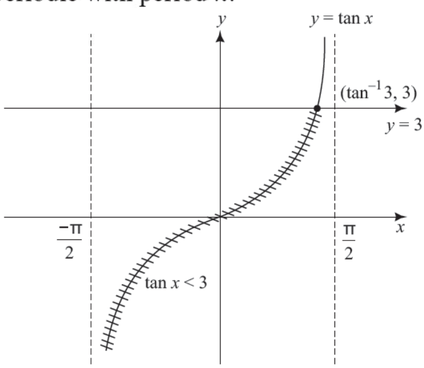
Hence, from the figure it is clear that, ordinate of $H >$ ordinate of G

$$\Rightarrow \quad \sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A + \sin B + \sin C}{3}$$

$$\Rightarrow \quad \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C \qquad \text{Hence, proved.}$$

2. $\boxed{\text{Using SC-1}}$ Drawing graphs $y = \tan x$ and $y = 3$

Since $\tan x$ is periodic with period π.



So, we will check solution on $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

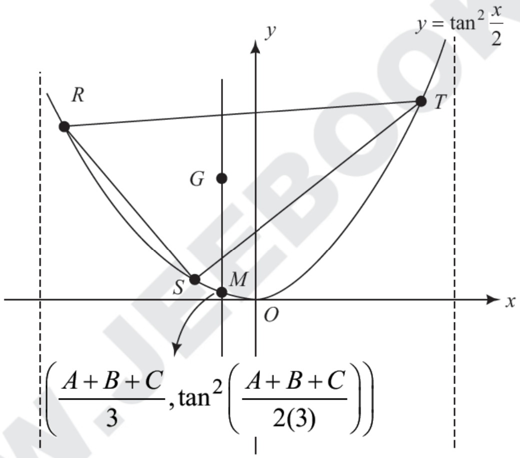
Now, it is clear from figure

$$\frac{-\pi}{2} < x < \tan^{-1} 3$$

$\Rightarrow$ General solution

$$2n\pi - \frac{\pi}{2} < x < 2n\pi + \tan^{-1} 3$$

3. $\boxed{\text{Using SC-1}}$ Drawing graph of $y = \tan^2 \frac{x}{2}$



Let $x = A$, $x = B$, $x = B$ be three points on $f(x) = \tan^2 \frac{x}{2}$

So that $A + B + C = \pi$

Where 'G' is the centeroid of triangle.

$$G \equiv \left(\frac{A+B+C}{3}, \frac{\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2}}{3}\right)$$

Now, from figure it is clear that,

ordinate of $G >$ ordinate of M

$$\Rightarrow \quad \frac{\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}{} > \tan^2\left(\frac{A+B+C}{6}\right)$$

$\Rightarrow \quad \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2} > 1$ Hence, proved.

4. **(b)** $\boxed{\text{Using SC-1}}$ $\tan 60° = (\tan(20° + 40°))$

$$\sqrt{3} = \frac{\tan 20° + \tan 40°}{1-\tan 20° \tan 40°} \qquad \left(\because \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}\right)$$

$\Rightarrow \quad \sqrt{3}-\sqrt{3}\ \tan 20° \tan 40° = \tan 20° + \tan 40°$ ...(1)

On substituting (1) in the expression

$\Rightarrow \quad \sqrt{3}-\sqrt{3}\ \tan 20° \tan 40° + \sqrt{3}\ \tan 20° \tan 40° + \sqrt{3}+2\sqrt{3} = 4\sqrt{3}$

5. **(c)** $\boxed{\text{Using SC-1}}$ $\tan 45° = \tan(15° + \tan 30°)$

$$1 = \frac{\tan 15° + \tan 30°}{1-\tan 15° \tan 30°}$$

$\Rightarrow \quad \tan 15° + \tan 30° = 1 - \tan 15° \tan 30°$ ...(1)

Substituting (1) in the expression

$\Rightarrow \quad 1 - \tan 15° \tan 30° + \tan 15° \tan 30° + \sqrt{3} = 1 + \sqrt{3}$

6. **(c)** $\boxed{\text{Using Tech.}}$ $\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$

$\Rightarrow \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos\left(x-\frac{\pi}{3}\right) = \cos 2x \Rightarrow x-\frac{\pi}{3} = 2n\pi \pm 2x$

$\Rightarrow x = \frac{2n\pi}{3}+\frac{\pi}{9}$ or $x = -2n\pi - \frac{\pi}{3}$

For $x \in S$, $n = 0 \Rightarrow x = \frac{\pi}{9}, -\frac{\pi}{3}$

Now, $n = 1 \Rightarrow x = \frac{7\pi}{9}$; and $n = -1 \Rightarrow x = \frac{-5\pi}{9}$

Hence, sum of all values of $x = \frac{\pi}{9}-\frac{\pi}{3}+\frac{7\pi}{9}-\frac{5\pi}{9} = 0$

7. **(b)** $\boxed{\text{Using Tech.}}$ We know, $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

$\Rightarrow \quad -\sqrt{74} \le 7\cos x + 5\sin x \le \sqrt{74}$

$\Rightarrow \quad -\sqrt{74} \le 2k+1 \le \sqrt{74} \quad \Rightarrow \quad -8.6 \le 2k+1 \le 8.6$

$\Rightarrow \quad -4.8 \le k \le 3.8$

Hence, k can take only 8 integral values.

8. **(c)** $\boxed{\text{Using Tech.}}$ Given : $\sin x + \cos x = 1$

$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} = \sin \frac{\pi}{4}$

$\Rightarrow \quad \sin (x + \pi/4) = \sin \pi/4$

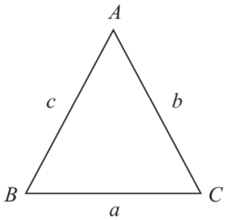
$\Rightarrow \quad x + \pi/4 = n\pi + (-1)^n \pi/4, n \in Z$ (the set of integers)

$\Rightarrow \quad x = n\pi + (-1)^n \pi/4 - \pi/4;$

where $n = 0, \pm 1, \pm 2, ...$

21 Solution of Triangles (Properties of Triangles)

Review of Key Notes and Formulae



Semi-perimeter $= \dfrac{a+b+c}{2}$

Where a, b, c are sides of triangle ABC.

1. **Sine Rule:** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ where, R is circumradius of ΔABC.

2. **Cosine Rule:** $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$,

   $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

3. **Projection Rule:** $a = b\cos C + c\cos B$, $b = c\cos A + a\cos C$,
   $c = a\cos B + b\cos A$

4. **Napier's Analogy:** $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$,

   $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$, $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$.

5. **Trigonometric Ratios of Half Angles of a Triangle:**

   (i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ac}}$,

   $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$, $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ac}}$, $\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

(iii) $\tan\dfrac{A}{2}=\dfrac{\Delta}{s(s-a)}$, $\tan\dfrac{B}{2}=\dfrac{\Delta}{s(s-b)}$, $\tan\dfrac{C}{2}=\dfrac{\Delta}{s(s-c)}$

6. Area of Triangle:

(i) $\Delta=\dfrac{1}{2}bc\sin A$, $\Delta=\dfrac{1}{2}ca\sin B$, $\Delta=\dfrac{1}{2}ab\sin C$

(ii) $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$, it is known as Heron's formula.

(iii) $\Delta=\dfrac{abc}{4R}=rs$, where R and r are radii of the circumcircle and the incircle of ΔABC.

7. Circumcircle of a Triangle:

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}$$

8. Incircle of a Triangle:

(i) $r=\dfrac{\Delta}{s}$

(ii) $r=(s-a)\tan\dfrac{A}{2}$, $r=(s-b)\tan\dfrac{B}{2}$ and $r=(s-c)\tan\dfrac{C}{2}$

(iii) $r=\dfrac{a\sin\frac{B}{2}.\sin\frac{C}{2}}{\cos\frac{A}{2}}$, $r=\dfrac{b\sin\frac{C}{2}.\sin\frac{A}{2}}{\cos\frac{B}{2}}$, $r=\dfrac{c\sin\frac{A}{2}.\sin\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r=4R\,.\sin\dfrac{A}{2}\,.\sin\dfrac{B}{2}\,.\sin\dfrac{C}{2}$

9. Excircles of a Triangle:

(i) $r_1=\dfrac{\Delta}{s-a}$, $r_2=\dfrac{\Delta}{s-b}$, $r_3=\dfrac{\Delta}{s-c}$

(ii) $r_1=s\tan\dfrac{A}{2}$, $r_2=s\tan\dfrac{B}{2}$, $r_3=s\tan\dfrac{C}{2}$

(iii) $r_1=\dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$, $r_2=\dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$, $r_3=\dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

(iv) $r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

$r_2 = 4R \cos\frac{A}{2} \sin\frac{B}{2} \cos\frac{C}{2}$

$r_3 = 4R \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}$

10. Orthocentre of a Triangle:

If AD, BE and CF are perpendiculars from the vertices A, B, C on the opposite sides BC, CA, AB of ΔABC, respectively meet at 'O', which is orthocentre of ΔABC.

(i) $OA = 2R \cos A$, $OB = 2R \cos B$, $OC = 2R \cos C$

(ii) $OD = 2R \cos B \cos C$, $OE = 2R \cos C \cos A$, $OF = 2R \cos A \cos B$

(iii) Circumradius of the pedal triangle $= \frac{R}{2}$

(iv) The area of pedal triangle $= 2\Delta \cos A \cos B \cos C$

11. Length of Altitude, Angle Bisector and Median:

(i) Length of Altitude from vertex $A = \dfrac{a}{\cot B + \cot C}$

(ii) Length of Angle bisector from vertex $A = \dfrac{2bc\cos\frac{A}{2}}{b+c}$

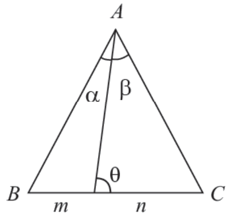
(iii) Length of Median from vertex $A = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

★**Note:** Similarly find lengths for others using symmetry.

12. m – n Rule:

(i) $(m + n) \cot \theta = m \cot \alpha - n \cot \beta$

(ii) $(m + n) \cot \theta = n \cot B - m \cot C$



13. Regular Polygon:

Area of polygon $= \frac{na^2}{4}\cot\left(\frac{\pi}{n}\right) = nr^2\left(\frac{\pi}{n}\right)$

$= \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$

★ **Note:**

(i) Distance between the circumcentre and orthocentre

$= R\sqrt{1-8\cos A\cos B\cos C}$

(ii) Distance between the circumcentre and incentre $= \sqrt{R^2-2Rr}$

(iii) Distance between incentre and orthocentre $= \sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(iv) Distance between circumcentre and centroid $= R^2 - \frac{1}{9}(a^2+b^2+c^2)$

SHORTCUTS: (SC-1)

Use of Substitution Method.

Assume triangle to be an equilateral triangle and proceed with $a = b = c$ and $\angle A = \angle B = \angle C$.

★**Note:** In equilateral triangle $\frac{r}{R} = \frac{1}{2}$ $\left(\text{where, } \begin{array}{l} r \to \text{inradius} \\ R \to \text{circumradius} \end{array}\right)$

Illustration 1

In ΔABC, cosec A (sin B cos C + cos B sin C) is equal to

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Let ABC be equilateral triangle

Then, $A = B = C = 60°$

$\Rightarrow$ cosec A (sinA cosC + cosB sinC) $= \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\right) = 1$

Illustration 2

In a ΔABC, among the following which one is true? **[AIEEE 2005]**

(a) $(b+c)\cos\frac{A}{2} = a\sin\left(\frac{B+C}{2}\right)$

(b) $(b+c)\cos\left(\frac{B+C}{2}\right) = a\sin\frac{A}{2}$

(c) $(b-c)\cos\left(\frac{B-C}{2}\right) = a\cos\frac{A}{2}$

(d) $(b-c)\cos\frac{A}{2} = a\sin\left(\frac{B-C}{2}\right)$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Let ΔABC be equilateral triangle.

Then, $A = B = C = 60°$ and a = b = c

$$(b-c)\cos\frac{A}{2} = (b-b)\cos\frac{60}{2} = 0$$

$$a\sin\left(\frac{B-C}{2}\right) = a\sin\left(\frac{B-B}{2}\right) = 0$$

Ans. (d)

Illustration 3

In ΔABC, $\frac{b-c}{3r_1} + \frac{c-a}{3r_2} + \frac{a-b}{3r_3}$ is equal to

(a) 1 (b) 0 (c) abc (d) $r_1 r_2 r_3$

Short-cut solution :

Using SC-1 Let the triangle be equilateral then, the expression = 0

Ans. (b)

Illustration 4

In ΔABC, $1 - \cot A \cot B$ is equal to

(a) $\frac{2c}{a+b+c}$ (b) $\frac{a}{a+b+c}$ (c) $\frac{2}{a+b+c}$ (d) $\frac{4a}{a+b+c}$

Short-cut solution :

Using SC-1 Let the triangle be equilateral triangle.

$\Rightarrow a = b = c$ and $A = B = 60° \Rightarrow 1 - \cot(60°)\cot(60°) = \frac{2}{3}$

$$\frac{2c}{a+b+c} = \frac{2c}{3c} = \frac{2}{3}$$

Ans. (a)

Illustration 5

$\frac{a\cos A + b\cos B + c\cos C}{2(a+b+c)}$ is equal to

(a) $1/r$ (b) $r/2R$ (c) R/r (d) $1/R$

Short-cut solution :

Using SC-1 Let the triangle be equilateral triangle

$\therefore a = b = c$ and $A = B = C = 60° \Rightarrow \frac{a\cos A + b\cos B + c\cos C}{2(a+b+c)} = \frac{1}{2\times 2} = \frac{1}{4}$

$\because \frac{r}{R} = \frac{1}{2} \Rightarrow \frac{r}{2R}$

Ans. (b)

Concept Booster Exercise

1. In a ΔABC, $\tan A + \tan B + \tan C$ is equal to

(a) $\sqrt{3}$ (b) $2\sqrt{3}$ (c) $3\sqrt{3}$ (d) $4\sqrt{3}$

2. If P_1, P_2, P_3 are the altitudes of a triangle from the vertices A, B, C and 'Δ' the area of the triangle, then $\frac{1}{P_1}+\frac{1}{P_2}-\frac{1}{P_3}=\frac{Kab}{(a+b+c)\Delta}\cos^2\frac{C}{2}$ then 'K' is equal to

(a) 1 (b) 2 (c) –1 (d) –2

3. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is equal to **[AIEEE 2010]**

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

4. $\cot\left(\frac{A+B}{2}\right) . \tan\left(\frac{A-B}{2}\right)$ is equal to _____ for ΔABC.

(a) $\frac{a+b}{a-b}$ (b) $\frac{a-b}{a+b}$ (c) $\frac{a}{a+b}$ (d) $\frac{a}{a-b}$

5. In ΔABC $a^2 \sin 2B + b^2 \sin 2A$ is equal to

(a) 2Δ (b) 3Δ (c) 4Δ (d) Δ

6. In ΔABC, $\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$

(a) $\cos^2 A$ (b) $\cos^2 B$ (c) $\sin^2 A$ (d) $\sin^2\left(\frac{A+B}{3}\right)$

7. In ΔABC, $a(b\cos C - c\cos B)$ is equal to

(a) b^2+c^2 (b) b^2-c^2 (c) $\frac{1}{b}+\frac{1}{c}$ (d) $\frac{1}{a}+\frac{1}{b}$

Numerical Value Problems

8. In ΔABC, $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C$ is equal to $K(a+b+c)$; then 'K' is equal to _____

9. In ΔABC, $1-\tan\frac{A}{2}\tan\frac{B}{2}$ is equal to $\frac{Kc}{a+b+c}$ the value of 'K' is _____.

10. In ΔABC, $\angle C = 60°$, then $\frac{1}{a+c}+\frac{1}{b+c}$ is equal to $\frac{K}{a+b+c}$, then 'K' is equal to _____.

Solutions

1. **(c)** $\boxed{\text{Using SC-1}}$ Let ΔABC be equilateral triangle.

$\therefore A = B = C = 60° \Rightarrow \tan 60° + \tan 60° + \tan 60° = 3\sqrt{3}$

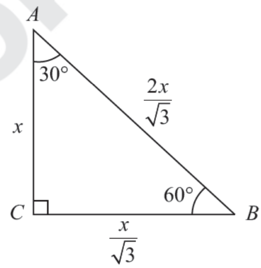
2. **(b)** $\boxed{\text{Using SC-1}}$ Let ΔABC be equilateral triangle.

$$\therefore P_1 = P_2 = P_3,\ a = b = c,\ \Delta = \frac{\sqrt{3}}{4}(\text{side})^2,\ P_1 = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \frac{2}{\sqrt{3}a} = \frac{K.a^2 \times 4}{3a(\sqrt{3})a^2} \times \frac{3}{4} \Rightarrow K = 2$$

3. **(d)** Since, A, B, C are in AP for ΔABC

Let angles be $A = 30°, B = 60°, C = 90°$



$$\Rightarrow \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$$

$$= \frac{1}{2}\sin(180°) + 2\sin(60°)$$

$$= \sqrt{3}$$

4. **(b)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$$\therefore A = B = C \Rightarrow \cot\left(\frac{A+A}{2}\right)\tan\left(\frac{A-A}{2}\right) = 0$$

Now, checking options for $a = b = c \Rightarrow \dfrac{a-a}{a+a} = 0$

5. **(c)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore a = b = c$ and $A = B = C = 60°$

$$\Rightarrow 2a^2\sin(120°) = \frac{\sqrt{3}a^2 \times 4}{4} = 4\Delta$$

6. **(c)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore a = b = c$

$$\Rightarrow \frac{3a(a)(a)(a)}{4a^2.a^2} = \frac{3}{4}$$

Now, checking options for $A = B = C = 60°$

$$\sin^2 A = \sin^2 60 = \frac{3}{4}$$

7. **(b)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore\ a = b = c$ and $A = B = C$

$\Rightarrow b(b \cos C - b \cos B) = 0$

Now, check options for $a = b = c$

$b^2 - c^2 = b^2 - b^2 = 0$

8. **(1)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore\ a = b = c$ and $A = B = C = 60°$

$\because\ (b + c) \cos A + (c + a) \cos B + (a + b) \cos C$

$= 6a \cos 60° = 3a$

and $K(a + b + c) = 3aK$

$\therefore\ 3a = K(3a) \Rightarrow K = 1$

9. **(2)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore\ A = B = C = 60°$ and $a = b = c$

$$1 - \tan\frac{60}{2}\tan\frac{60}{2} = \frac{Kc}{3c}$$

$$1 - \frac{1}{3} = \frac{K}{3}$$

$$\frac{2}{3} = \frac{K}{3}$$

$\Rightarrow K = 2$

10. **(3)** $\boxed{\text{Using SC-1}}$ Let ΔABC be an equilateral triangle.

$\therefore\ a = b = c$ and $A = B = C = 60°$

$$\Rightarrow \frac{1}{2a} + \frac{1}{2a} = \frac{K}{3a} \Rightarrow K = 3.$$

22 Straight Line

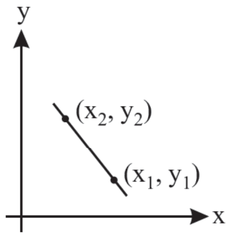
Review of Key Notes and Formulae

1. **Definition:** A straight line is the locus of all those points which are collinear with two given points.

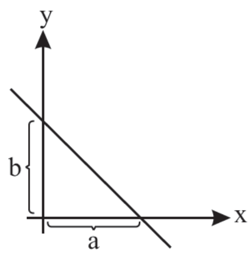
2. **Slope (Gradient) of a Line:** If 'θ' is the angle made by a line with the positive direction of x-axis in anticlockwise sense, then the slope of the line, $m = \tan\theta$.
   (i) Slope of a line parallel to x-axis, $m = 0$
   (ii) Slope of line parallel to y-axis, $m = \infty$
   (iii) Slope of a line equally inclined with axes is 1 or –1 as it makes an angle of 45° or 135° with x-axis.
   (iv) Slope of line joining points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

3. **Different Forms of the Equation of Straight Line:**
   (i) *Slope form:*
   (a) $y = mx + c$; where 'c' is intercept on y-axis.
   (b) $y - y_1 = m(x - x_1)$

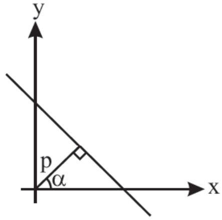
   (ii) *Point form:* $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



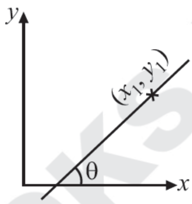
   (iii) *Intercept form:* $\dfrac{x}{a} + \dfrac{y}{b} = 1$



(iv) *Normal form:* $x \cos \alpha + y \sin \alpha = p$



(v) *Parametric form:* $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$



4. **Position of a Point w.r.t. Line:** Let point be (x_1, y_1) and line be $ax + by + c = 0$

★ if $b(ax_1 + by_1 + c) > 0 \Rightarrow$ Point lies above the line

★ if $b(ax_1 + by_1 + c) < 0 \Rightarrow$ Point lies below the line

5. **Relative Position of Two Points w.r.t. Line:** Let points be (x_1, y_1) and (x_2, y_2) and line be $ax + by + c = 0$

★ if $(ax_1 + by_1 + c)(ax_2 + by_2 + c) > 0 \Rightarrow$ Both points lies in same side of line.

★ if $(ax_1 + by_1 + c)(ax_2 + by_2 + c) < 0 \Rightarrow$ Both points lies in opposite side of line.

6. **Angle between Two Straight Lines:** The angle between two straight line having slopes m_1 and m_2 is $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

7. **Perpendicular Distance of a Point From a Line:** The perpendicular distance of the point $A(x_1, y_1)$ from the line $ax + by + c = 0$ is

$$P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

8. **Perpendicular Distance between Two Parallel Lines:** The perpendicular distance between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

$$d = \frac{|c_1 - c_2|}{\sqrt{a_1^2 + b_1^2}}$$

9. **Condition of Concurrency of Three Lines:** $a_i + b_i + c_i = 0;\ i = 1, 2, 3$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

10. Family of Lines: Equation of line passing through intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$

11. Equation of Angle Bisectors: The equation of angle bisector of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$$

12. Homogeneous Equation of Second Degree:

(i) $ax^2 + 2hxy + by^2 = 0$

★If $h^2 > ab \Rightarrow$ Two distinct real lines

★ If $h^2 = ab \Rightarrow$ Coincident real lines

★ If $h^2 < ab \Rightarrow$ Imaginary lines

(ii) Acute angle between pair of lines

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{|a+b|}$$

★If lines are perpendicular $\Rightarrow a + b = 0$

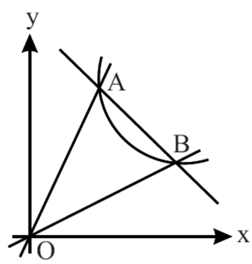
★If lines are coincident $\Rightarrow h^2 = ab$

★If lines are equally inclined to x-axis $\Rightarrow$ Coefficient of $xy = 0$ *i.e.* $h = 0$

(iii) Pair of angle bisectors to pair of lines $ax^2 + by^2 + 2hxy = 0$ is

$$\frac{x^2-y^2}{a-b} = \frac{xy}{h}$$

13. Homogenisation: A two degree curve and given a line intersecting the curve at two points A and B, then equation of pair of lines OA and OB where 'O' is origin, is obtained by homogenisation.



Let equation of curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the equation of line AB: $lx + my + n = 0$

Then, equation of pair of lines OA and OB is

★ $ax^2 + 2hxy + by^2 + (2gx + 2fy)$

$$\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

14. Important Points to Remember:

(i) *Locus:* It is the path or the curve traced by a moving point satisfying the given condition.

(ii) *Area of a general quadrilateral ABCD*

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$$

where, 's' is semi-perimeter and $s = \dfrac{a+b+c+d}{2}$

and $\alpha = \dfrac{A+C}{2}$ or $\alpha = \dfrac{2\pi-(B+D)}{2}$

(iii) *Ptolemy's Theorem:*

If □ABCD is cyclic quadrilateral

$\Rightarrow$ (AC)(BD) = (AB)(CD) + (BC)(AD)

(iv) *Harmonic conjugates:*

Two points are said to be harmonic conjugates if they divide a line segment say (AB) in the same ratio.

$$\Rightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}$$

(v) *Area of the parallelogram formed by the lines:*

$a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$$\text{Area} = \frac{|c_1 - c_2||d_1 - d_2|}{|m_1 - m_2|}$$

TIPS AND TRICKS: (T-1)

The ratio in which the line $ax + by + c = 0$ divides the line segment joining the points (x_1, y_1) and (x_2, y_2)

$$\Rightarrow \text{Ratio} = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right)$$

★ Simply put the coordinates in the equation of line followed by negative sign.

Illustration 1

Determine the ratio in which line $3x + 4y - 9 = 0$ divides the line segment joining the points (1, 3) and (2, 7).

Short-cut solution :

$\boxed{\text{Using T-1}}$ Putting the coordinates in the equation

$$\text{Ratio} = -\left[\frac{3(1)+4(3)-9}{3(2)+4(7)-9}\right] = \frac{-6}{25}$$

$\Rightarrow$ Line divides in the ratio 6 : 25 externally.

Illustration 2

The ratio in which the line $x + y = 4$ divides the line segment joining the points (–1, 1) and (5, 7) is

(a) 1 : 3 (b) 2 : 1 (c) 1 : 2 (d) 2 : 5

Short-cut solution :

$\boxed{\text{Using T-1}}$ Putting the points in the equation

$$\Rightarrow \text{Ratio} = -\left[\frac{-1+1-4}{5+7-4}\right] = \frac{4}{8} = \frac{1}{2}$$

Hence, line divides internally in the ratio 1 : 2. **Ans. (c)**

TIPS AND TRICKS: (T-2)

Short trick to find the intersection point of two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

$$\Rightarrow x = \frac{b_1c_2 - c_1b_2}{a_1b_2 - b_1a_2} \text{ and } y = \frac{-(a_1c_2 - c_1a_2)}{a_1b_2 - b_1a_2}$$

Illustration 3

Find the intersection point of the lines $3x + y + 2 = 0$ and $5x + 3y + 5 = 0$

Short-cut solution :

$\boxed{\text{Using T-2}}$ $x = \dfrac{5-6}{3\times3-5\times1} = \dfrac{-1}{4}$

$$y = \frac{-(15-10)}{3\times3-5\times1} = \frac{-5}{4}$$

TIPS AND TRICKS: (T-3)

If a point (x_1, y_1) divides the line segment between the x-axis and y-axis in the ratio $m : n$, then the equation of line is

$$\frac{nx}{x_1} + \frac{my}{y_1} = m + n$$

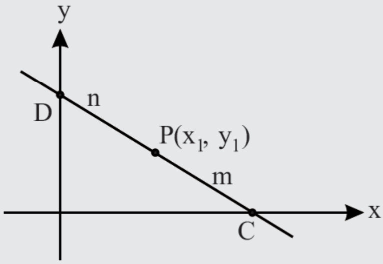


Illustration 4

A straight line through the point A(3, 4) is such that its intercept between the axes is bisected at A, then its equation **[AIEEE 2006]**

(a) $3x - 4y + 7 = 0$ (b) $4x + 3y = 24$

(c) $3x + 4y = 25$ (d) $x + y = 7$

Short-cut solution :

Using T-3 Here, $m : n = 1 : 1$ and $(x_1, y_1) = (3, 4)$

Hence, equation is $\frac{(1)x}{3} + \frac{(1)y}{4} = 1 + 1 \Rightarrow 4x + 3y = 24$ **Ans. (b)**

Illustration 5

If the point (2, 3) divides the line between the x-axis and y-axis in the ratio 3 : 4 the equation of the line is:

(a) $2x + 3y = 7$ (b) $2x + 3y = 4$

(c) $2x - 3y = 7$ (d) $2x - 3y = 4$

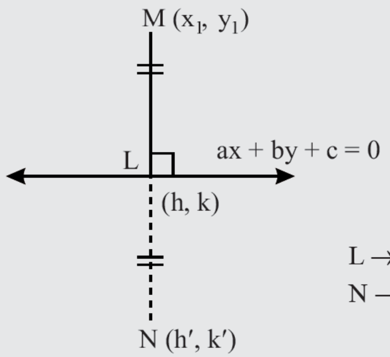
Short-cut solution :

Using T-3 Here, $m : n = 3 : 4$ and $(x_1, y_1) = (2, 3)$

Hence, equation is $\frac{4x}{2} + \frac{3y}{3} = 3 + 4 \Rightarrow 2x + 3y = 7$ **Ans. (a)**

SHORTCUTS: (SC-1)

Short trick to find "Foot of perpendicular and reflection of point with respect to a line".



L → Foot of perpendicular

N → Reflection of M w.r.t. line

(i) To determine 'L' (Foot of perpendicular)

$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$$

(ii) To determine 'N' (Image w.r.t. $ax + by + c = 0$)

$$\frac{h'-x_1}{a}=\frac{k'-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

Illustration 6

The foot of perpendicular from point (2, 3) on the line $x + y = 2$ is

(a) $\left(\frac{1}{2},\frac{-3}{2}\right)$ (b) $\left(\frac{-1}{2},\frac{3}{2}\right)$ (c) $\left(\frac{3}{2},\frac{1}{2}\right)$ (d) $\left(\frac{1}{2},\frac{3}{2}\right)$

Short-cut solution :

$\boxed{\text{Using SC-1}}$ $\frac{h-2}{1}=\frac{k-3}{1}=\frac{-(2+3-2)}{1+1}$

$\Rightarrow h=\frac{1}{2}$ and $k=\frac{3}{2}$ **Ans. (d)**

Illustration 7

The reflection of the point (4, –3) in the line $2x + 3y + 5 = 0$ is

(a) $\left(\frac{-1}{2},\frac{3}{5}\right)$ (b) $\left(\frac{36}{13},\frac{-63}{13}\right)$

(c) $\left(\frac{1}{2},\frac{3}{5}\right)$ (d) $\left(\frac{-36}{13},\frac{63}{13}\right)$

Short-cut solution :

Using SC-1 $\dfrac{h'-4}{2}=\dfrac{k'+3}{3}=\dfrac{-2(8-9+5)}{4+9}$

$\Rightarrow h'=\dfrac{36}{13}$ and $k'=\dfrac{-63}{13}$ **Ans. (b)**

SHORTCUTS: (SC-2)

Shortcut method to find "number of points having both coordinates as an integers that lies in the interior region of a triangle" with vertices

(0, 0), (0, k) and (k, 0); $k \in I^{+}$ is $\dfrac{(k-1)(k-2)}{2}$

Illustration 8

The number of points having both coordinates as an integers that lies in the interior region of the triangle with vertices (0, 0), (0, 41) and (41, 0)

[JEE M 2015]

(a) 901 (b) 861 (c) 820 (d) 780

Short-cut solution :

Using SC-2 Here, $k = 41$

$\Rightarrow$ No. of points $=\dfrac{(41-1)(41-2)}{2}=780$ **Ans. (d)**

Illustration 9

The number of points having both coordinates as an integers that lies in the interior of the triangle with vertices (0, 0), (0, 61) and (61, 0) is

(a) 1772 (b) 1770 (c) 1660 (d) 1550

Short-cut solution :

Using SC-2 Here, $k = 61$

$\Rightarrow$ No. of points $=\dfrac{(61-1)(61-2)}{2}=1770$ **Ans. (b)**

SHORTCUTS: (SC-3)

In ΔPQR if X, Y, Z are the mid points of triangle, then we can find the vertices of the triangle by using shortcut method.

$$\boxed{\begin{aligned} P &= X+Z-Y \\ Q &= X+Y-Z \\ R &= Y+Z-X \end{aligned}}$$

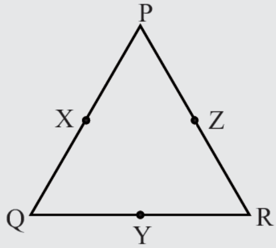


Illustration 10

The mid-points of sides of a triangle are (3, 4), (4, 2) and (5, 4). Then the coordinates of its vertices are

(a) (4, 6), (2, 2), (6, 2)
(b) (2, 5), (5, 9), (2, 6)
(c) (4, 9), (2, 5), (2, 4)
(d) None of these

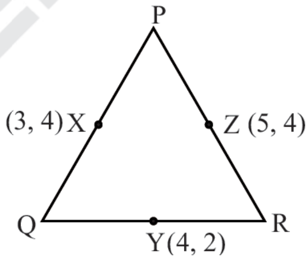
Short-cut solution :

Using SC-3

$P = X+Z-Y = (4,6)$

$Q = X+Y-Z = (2,2)$

$R = Y+Z-X = (6,2)$



Ans. (a)

TECHNIQUE

To find the point of intersection of a pair of straight lines:

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines.

Then, the point of intersection of the pair of straight lines is

$$x = \frac{hf - bg}{ab - h^2}, y = \frac{gh - af}{ab - h^2}$$

Illustration 11

The point of intersection of lines represented by the equation

$6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$ is

(a) (1, 1)
(b) $\left(\frac{5}{23}, -\frac{7}{23}\right)$
(c) $\left(\frac{17}{23}, \frac{32}{23}\right)$
(d) $\left(-\frac{32}{23}, \frac{17}{23}\right)$

Short-cut solution :

$\boxed{\text{Using Tech.}}$ $\because\ a = 6, b = -21, f = 19, g = \dfrac{13}{2}, h = \dfrac{5}{2}$

$$x = \frac{\frac{5}{2}\times 19 - (-21)\times\frac{13}{2}}{6\times(-21) - \left(\frac{5}{2}\right)^2} = -\frac{32}{23}$$

$$y = \frac{\frac{13}{2}\times\frac{5}{2} - 6\times 19}{6\times(-21) - \left(\frac{5}{2}\right)^2} = \frac{17}{23}$$

$\therefore$ Point of intersection $= \left(-\dfrac{32}{23}, \dfrac{17}{23}\right)$ **Ans. (d)**

Concept Booster Exercise

1. The ratio in which the line segment joining the points (5, – 4) and (2, 3) is divided by the x-axis is

   (a) 3 : 4 (b) 4 : 3 (c) 1 : 2 (d) 2 : 1

2. The ratio in which the line segment joining the points (2, 3) and (4, 5) is divided by the line joining the points (6, 8) and (–3, –2) is:

   (a) 5 : 7 (internally) (b) 5 : 7 (externally)
   (c) 7 : 6 (internally) (d) 6 : 7 (externally)

3. The coordinates of point of intersection of the line $5x + 2y - 34 = 0$ and $3x + 4y - 26 = 0$ is :

   (a) (2, 6) (b) (2, 4) (c) (4, 2) (d) (6, 2)

4. If a line intercepted between the coordinate axes is trisected at a point A(4, 3) which is nearer to x-axis, then its equation is : **[JEE M 2014]**

   (a) $4x - 3y = 7$ (b) $3x + 8y = 36$
   (c) $3x + 2y = 18$ (d) $x + 3y = 13$

5. If a point (2, 5) bisects the line segment intercepted between coordinate axes, then its equation is :

   (a) $5x + 2y = 20$ (b) $5x - 2y = 20$
   (c) $2x + 5y = 20$ (d) $2x - 5y = 20$

6. The coordinates of foot of perpendicular drawn from (2, 4) to the line $x + y = 1$ is

   (a) $\left(\frac{1}{3}, \frac{3}{2}\right)$ (b) $\left(\frac{-1}{2}, \frac{3}{2}\right)$ (c) $\left(\frac{4}{3}, \frac{1}{2}\right)$ (d) $\left(\frac{3}{4}, \frac{-1}{2}\right)$

7. The image of a point A(3, 8) in the line $x + 3y - 7 = 0$ is

   (a) (–1, –4) (b) (–3, –8) (c) (4, 1) (d) (3, 8)

8. The coordinates of the foot of perpendicular from the point (2, 3) on the line $x + y - 11 = 0$ are

   (a) (–6, 5) (b) (5, 6) (c) (–5, 6) (d) (6, 5)

9. The reflection of the point (4, –13) in the line $5x + y + 6 = 0$ is

   (a) (–1, –14) (b) (3, 4) (c) (1, 2) (d) (– 4, 13)

10. The number of points having both coordinates as integers that lie in the interior of the triangle with vertices (0, 0), (0, 31) and (31, 0) is:

(a) 465 (b) 496 (c) 435 (d) 935

Numerical Value Problems

11. Number of integral points exactly in the interior of the triangle with vertices (0, 0), (0, 15) and (15, 0) is _____

12. The mid-points of sides of a triangle are (2, 1), (–1, –3) and (4, 5). Then the sum of x-coordinates of the vertices of the triangle is _____

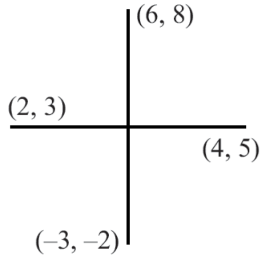
Solutions

1. **(b)** $\boxed{\text{Using T-1}}$ Since, equation of line is $y = 0$

Putting co-ordinates in the equation $y = 0$

$\Rightarrow$ Ratio $= -\left(\dfrac{-4}{3}\right)$

2. **(b)** $\boxed{\text{Using T-1}}$ Finding equation of line



$$y - 8 = \frac{-10}{-9}(x - 6) \Rightarrow 10x - 9y + 12 = 0$$

Now, putting (2, 3) and (4, 5) in the equation of line.

$$\Rightarrow \text{Ratio} = -\left(\frac{10\times2-9\times3+12}{10\times4-9\times5+12}\right) = \frac{-5}{7}$$

3. **(d)** $\boxed{\text{Using T-2}}$ $x = \dfrac{-52+136}{20-6} = 6;\ y = \dfrac{-(-130+102)}{20-6} = 2$

4. **(c)** $\boxed{\text{Using T-3}}$ Since, line is trisected by the point

Here, $m_1 : m_2 = 1 : 2$, $(x_1, y_1) = (4, 3)$

Hence, equation is $\dfrac{2x}{4} + \dfrac{y}{3} = 1 + 2$

$\Rightarrow 3x + 2y = 18$

5. **(a)** $\boxed{\text{Using T-3}}$ Since, line is bisected

Here, $m_1 : m_2 = 1 : 1$, $(x_1, y_1) = (2, 5)$

Hence, equation is $\dfrac{x}{2} + \dfrac{y}{5} = 1 + 1$

$\Rightarrow 5x + 2y = 20$

6. **(b)** $\boxed{\text{Using SC-1(i)}}$ Here, $(x_1, y_1) = (2, 4)$

$$\frac{h-2}{1} = \frac{k-4}{1} = \frac{-(2+4-1)}{1+1}$$

$$\Rightarrow h = \frac{-1}{2} \text{ and } k = \frac{3}{2}$$

7. **(a)** $\boxed{\text{Using SC-1(ii)}}$ Here, $(x_1, y_1) = (3, 8)$

$$\frac{h'-3}{1} = \frac{k'-8}{3} = \frac{-2(3+24-7)}{1+9}$$

$$\Rightarrow h' = -1 \text{ and } k' = -4$$

8. **(b)** $\boxed{\text{Using SC-1(i)}}$ Here, $(x_1, y_1) = (2, 3)$

$$\frac{h-2}{1} = \frac{k-3}{1} = \frac{-(2+3-11)}{1+1}$$

$$\Rightarrow h = 5 \text{ and } k = 6$$

9. **(a)** $\boxed{\text{Using SC-1(ii)}}$ Here, $(x_1, y_1) = (4, -13)$

$$\frac{h'-4}{5} = \frac{k'+13}{1} = \frac{-2(20-13+6)}{25+1}$$

$$\Rightarrow h' = -1 \text{ and } k' = -14$$

10. **(c)** $\boxed{\text{Using SC-2}}$ Here, $k = 31$

$$\Rightarrow \text{Number of points} = \frac{(k-1)(k-2)}{2} = \frac{(31-1)(31-2)}{2} = 435$$

11. **(91)** $\boxed{\text{Using SC-2}}$ Here, $k = 15$

$$\Rightarrow \text{Number of points} = \frac{(k-1)(k-2)}{2} = \frac{(15-1)(15-2)}{2} = 91$$

12. **(5)** $\boxed{\text{Using SC-3}}$ x-coordinates of

$$\left.\begin{array}{l} P = 2+4-(-1) = 7 \\ Q = 2-1-4 = -3 \\ R = 4-1-2 = 1 \end{array}\right\} \Rightarrow \text{Sum of } x\text{-coordinate} = 7 - 3 + 1 = 5$$

23 Circle

Review of Key Notes and Formulae

1. Equation of Circle in Different Forms:

(i) The circle with centre (h, k) and radius 'r' has the equation

$(x - h)^2 + (y - k)^2 = r^2$

(ii) General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ with centre as $(-g, -f)$ & radius $= \sqrt{g^2 + f^2 - c}$

(iii) *Diametric form:* The equation of circles with (x_1, y_1) and (x_2, y_2) as its diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

2. Intercepts Made by a Circle: The intercepts made by the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

x - intercept $\Rightarrow$ $2\sqrt{g^2 - c}$

y - intercept $\Rightarrow$ $2\sqrt{f^2 - c}$

3. Position of Point w.r.t. a Circle:

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

If $S_1 > 0$ $\Rightarrow$ Lies outside the circle

If $S_1 < 0$ $\Rightarrow$ Lies inside the circle

where S_1 is $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

4. Line and Circle: Let $L = 0$ be a line and $S = 0$ be a circle. If r is the radius of the circle and 'p' is the length of the perpendicular from the centre on the line, then

If $p > r$ $\Rightarrow$ line is neither tangent nor secant

If $p = r$ $\Rightarrow$ tangent

If $p < 0$ $\Rightarrow$ secant

If $p = 0$ $\Rightarrow$ line is diameter of circle.

5. Parametric Form of Circle:

(i) The parametric equations of a circle $x^2 + y^2 = r^2$ are $x = r\cos\theta, y = r\sin\theta$

(ii) The parametric equation of a circle $(x - h)^2 + (y - k)^2 = r^2$ are
$x = h + r\cos\theta, y = k + r\sin\theta$.

6. Equation of Tangent:

(i) Point form (T = 0):
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(ii) Slope form: $y = mx \pm r\sqrt{1+m^2}$

(iii) Parametric form:
$(x - h)\cos\theta + (y - k)\sin\theta = r$

'T' represents (replace)

$x \to \dfrac{x+x_1}{2}$	$y^2 \to yy_1$
$y \to \dfrac{y+y_1}{2}$	$xy \to \dfrac{xy_1 + yx_1}{2}$
$x^2 \to xx_1$	

7. Equation of Normal: Normal will always be perpendicular to tangent and passes through centre which P_s diameter of circle. But point of contact of tangent & normal is same.

8. Length of tangent = $\sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

9. Pair of tangent = $T^2 = SS_1$

10. Chord of contact = T = 0

11. Equation of whose mid point is given $\Rightarrow T = S_1$

12. Director Circle: Director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of given circle *i.e.* $x^2 + y^2 = 2r^2$

13. Pole and Polar:

(i) The equation of polar is T = 0

(ii) Pole of polar $Ax + By + C = 0$ w.r.t. circle $x^2 + y^2 = r^2$ is $\left(\dfrac{-Ar^2}{C}, \dfrac{-Br^2}{C}\right)$

14. Family of Circles: The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$

15. Common Tangents to Two Circles: Let $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$ be the centre of two circles and r_1, r_2 be their radius and C_1C_2 be the distance b/w them.

(i) If $C_1C_2 > r_1 + r_2 \Rightarrow$ 4 common tangents

(ii) If $C_1C_2 = r_1 + r_2 \Rightarrow$ 3 common tangents

(iii) If $C_1C_2 < |r_1 - r_2| \Rightarrow$ 0 common tangents

(iv) If $C_1C_2 = |r_1 - r_2| \Rightarrow$ 1 common tangent

(v) If $|r_1 - r_2| < C_1C_2 < r_1 + r_2 \Rightarrow$ 2 common tangents

16. Radical Axis and Radical Centre:

(i) Equation of radical axis is $S_1 - S_2 = 0$

(ii) Radical centre is the point of intersection of the radical axis of three circles taken two at a time.

17. **Orthogonality of Two Circles:** Two circles $S_1 = 0$ and $S_2 = 0$ are said to be orthogonal if the tangents at their point of intersection include a right angle.
Condition: $2g_1g_2 + 2f_1f_2 = C_1 + C_2$

TIPS AND TRICKS: (T-1)

The locus of centre of the circle if intercepts of x and y axis are $2l_1$ and $2l_2$ respectively is $y^2 - y^2 = l_1^2 - l_2^2$

Illustration 1

The locus of centre of the circle if intercepts of x and y axis are 6 and 4 respectively is

(a) $x^2 - y^2 = 4$ (b) $x^2 - y^2 = 5$
(c) $x^2 + y^2 = 4$ (d) $x^2 + y^2 = 5$

Short-cut solution :

Using T-1 Here, $l_1 = 3$ and $l_2 = 2$

$\Rightarrow$ Locus is $x^2 - y^2 = 9 - 4 = 5$ **Ans. (b)**

TIPS AND TRICKS: (T-2)

The locus of mid point of chord of circle $(x - a)^2 + (y - b)^2 = r^2$ which subtends an angle 'θ' at centre of circle is

$$(x-a)^2 + (y-b)^2 = r^2 \cos^2 \frac{\theta}{2}$$

Illustration 2

Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtend an angle $\frac{2\pi}{3}$ at its centre is **[AIEEE 2006]**

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = \frac{27}{4}$
(c) $x^2 + y^2 = \frac{9}{4}$ (d) None of these

Short-cut solution :

Using T-2 $a = 0$, $b = 0$ and $r = 3$, $\theta = \frac{2\pi}{3}$

$\Rightarrow$ Locus is $x^2 + y^2 = 9\cos^2\left(\frac{2\pi}{3}\right) = \frac{9}{4}$ **Ans. (c)**

Illustration 3

The locus of the mid point of a chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin, is

(a) $x + y = 2$ (b) $x^2 + y^2 = 1$

(c) $x^2 + y^2 = 2$ (d) $x + y = 1$

Short-cut solution :

Using T-2 Here, $a = b = 0$, $r = 2$ and $\theta = 90°$

$\Rightarrow$ Locus is $x^2 + y^2 = 4\cos^2\left(\frac{90°}{2}\right) = 2$ **Ans. (c)**

TIPS AND TRICKS: (T-3)

Use of substitution method. Substitute the co-ordinate(s) as the requirement of problem.

Illustration 4

Tangents drawn from the point P(1, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of triangle PAB is,

(a) $x^2 + y^2 + 4x - 6y + 19 = 0$ (b) $x^2 + y^2 - 4x - 10y + 19 = 0$

(c) $x^2 + y^2 - 2x + 6y - 29 = 0$ (d) $x^2 + y^2 - 6x - 4y + 19 = 0$

Short-cut solution :

Using T-3 Since, circumcircle passes through (1, 8), so this point will satisfy equation of circle.

Checking options (a), (b), (c), (d)

$\because x^2 + y^2 - 4x - 10y + 19 = 1^2 + 8^2 - 4 \times 1 - 10 \times 8 + 19 = 84 - 84 = 0$

Ans. (b)

Illustration 5

The equations of the tangents drawn from the point (0, 1) to the circle $x^2 + y^2 - 2x + 4y = 0$ are

(a) $2x - y + 1 = 0, x + 2y + 2 = 0$ (b) $2x - y + 1 = 0, x + 2y - 2 = 0$

(c) $2x - y - 1 = 0, x + 2y + 2 = 0$ (d) $2x - y - 1 = 0, x + 2y - 2 = 0$

Short-cut solution :

Using T-3 Point (0, 1) must satisfy both the equations of tangent.

Checking options (a), (b), (c), (d)

$\because 2x - y + 1 = 2 \times 0 - 1 + 1 = 0$ and $x + 2y - 2 = 0 + 2 \times 1 - 2 = 0$

Ans. (b)

TECHNIQUE

The x-coordinates of two points A and B are roots of equation $px^2 + qx + r = 0$ and y-coordinates are roots of equation $ay^2 + by + c = 0$, then to find the equation of circle with AB as a diameter in both the quadratic equation first makes the coefficient of x^2 and y^2 equal to one and then add.

$$\left(x^2 + \frac{q}{p}x + \frac{r}{p}\right) + \left(y^2 + \frac{b}{a}y + \frac{c}{a}\right) = 0$$

Illustration 6

The x-coordinates of two points A and B are roots of equation $x^2 + 2x - a^2 = 0$ and y-coordinate are roots of equation $y^2 + 4y - b^2 = 0$ then equation of the circle which has diameter AB is-

(a) $(x-1)^2 + (y-2)^2 = 5 + a^2 + b^2$ (b) $(x+1)^2+(y+2)^2 = \sqrt{(5+a^2+b^2)}$

(c) $(x+1)^2 + (y+2)^2 = (a^2 + b^2)$ (d) $(x+1)^2 + (y+2)^2 = 5 + a^2 + b^2$

Short-cut solution :

$\boxed{\text{Using Tech.}}$

$p = 1, q = 2, r = -a^2$

$a = 1, b = 4, c = -b^2$

Equation of circle

$(x^2 + 2x - a^2) + (y^2 + 4y - b^2) = 0$

$(x + 1)^2 + (y + 2)^2 = 5 + a^2 + b^2$ **Ans. (d)**

Concept Booster Exercise

1. The locus of centre of the circle if intercepts of x and y axis are 8 and 6 respectively is,
   (a) $x^2 - y^2 = 5$ (b) $x^2 - y^2 = 9$
   (c) $x^2 - y^2 = 4$ (d) $x^2 - y^2 = 7$

2. Find the locus of mid point of chords of circle $x^2 + y^2 = 36$ which subtends right angle at origin is
   (a) $x^2 + y^2 = 36$ (b) $x^2 + y^2 = 18$
   (c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 16$

3. Find the locus of mid point of chord of circle $x^2 + y^2 - 8x + 2y + 8 = 0$ which subtends a right angle at the centre of circle
   (a) $(x-4)^2 + (y+1)^2 = \frac{9}{4}$ (b) $(x-4)^2 + (y-1)^2 = \frac{9}{4}$
   (c) $(x+4)^2 + (y+1)^2 = \frac{9}{2}$ (d) $(x-4)^2 + (y+1)^2 = \frac{9}{2}$

4. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. The equation of pair of tangents is
   (a) $2x^2 + 2y^2 + 5xy + 2 = 0$
   (b) $x^2 - y^2 = 5$
   (c) $2x^2 + 2y^2 + 5xy = 0$
   (d) $x^2 + y^2 + 5xy - 2 = 0$

5. The abscissa of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $y^2 + 2py - q^2 = 0$. The radius of the circle with AB as a diameter will be-
   (a) $\sqrt{a^2 + b^2 + p^2 + q^2}$
   (b) $\sqrt{b^2 + q^2}$
   (c) $\sqrt{a^2 + b^2 - p^2 - q^2}$
   (d) $\sqrt{a^2 + p^2}$

6. The equation of the locus of the mid points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is ______

(a) $(x + 3)^2 + (y + 1)^2 = \frac{9}{4}$

(b) $(x - 3)^2 + (y + 1)^2 = \frac{9}{16}$

(c) $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{16}$

(d) $\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{16}$

Solutions

1. **(d)** $\boxed{\text{Using T-1}}$ Here, $l_1 = 4$ and $l_2 = 3$

$\Rightarrow$ Locus is $x^2 - y^2 = 16 - 9 = 7$

2. **(b)** $\boxed{\text{Using T-2}}$ Here, $a = b = 0$, $r = 6$ and $\theta = 90°$

$\Rightarrow$ Locus is $x^2 + y^2 = 36\cos^2\left(\frac{90°}{2}\right) = 18$

3. **(d)** $\boxed{\text{Using T-2}}$ Here, $a = 4$, $b = -1$, $r = 3$, $\theta = 90°$

$\Rightarrow$ Locus is $(x - 4)^2 + (y + 1)^2 = 9\cos^2\left(\frac{90°}{2}\right) = \frac{9}{2}$

4. **(c)** $\boxed{\text{Using T-3}}$ Tangents will always passes through origin.

Checking options (a), (b), (c), (d)

$\because 2x^2 + 2y^2 + 5xy = 2 \times 0 + 2 \times 0 + 5 \times 0 \times 0 = 0$

5. **(a)** $\boxed{\text{Using Tech.}}$ Equation of circle

$x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

$C(-a, -p), r = \sqrt{a^2 + p^2 + b^2 + q^2}$

6. **(c)** $\boxed{\text{Using T-2}}$

Rewrite as, $x^2 + y^2 - 3x + y + \frac{1}{4} = 0$ (Divide by 4)

Hence, $a = \frac{3}{2}$, $b = \frac{-1}{2}$, $r = \frac{3}{2}$, $\theta = \frac{2\pi}{3}$

$\Rightarrow$ Locus is $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{4}\cos^2\left(\frac{\pi}{3}\right) = \frac{9}{16}$

24 Conic Sections

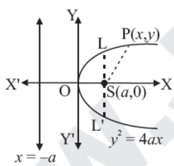
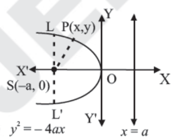
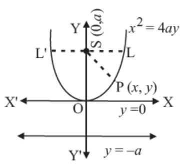
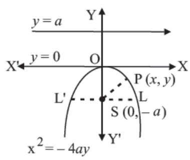
Review of Key Notes and Formulae

1. Parabola

A parabola is a set of all points in a plane that are equidistance from a fixed line & a fixed point in a plane.

The fixed line is called the directrix of the parabola and the fixed point is called the focus.

2. Four Standard Forms of the Parabola

Standard Equation	$y^2 = 4ax$ $(a > 0)$	$y^2 = -4ax$ $(a > 0)$	$x^2 = 4ay$ $(a > 0)$	$x^2 = -4ay$ $(a > 0)$
Shape of Parabola				
Vertex	O (0, 0)	O (0, 0)	O (0, 0)	O (0, 0)
Focus	S $(a, 0)$	S $(-a, 0)$	S $(0, a)$	S $(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	$4a$	$4a$	$4a$	$4a$

3. Equations of Tangents of all Standard Parabola

Equations of parabola	Tangent at (x_1, y_1)
$y^2 = 4ax$	$yy_1 = 2a\ (x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

4. Equations of Normals of all Standard Parabolas at (x_1, y_1)

Equations of parabola	Normal at (x_1, y_1)
$y^2 = 4ax$	$y - y_1 \frac{-y_1}{2a}(x - x_1)$
$y^2 = -4ax$	$y - y_1 \frac{y_1}{2a}(x - x_1)$
$x^2 = 4ay$	$y - y_1 - \frac{2a}{x_1}(x - x_1)$
$x^2 = -4ay$	$y - y_1 \frac{2a}{x_1}(x - x_1)$

5. Important Point of Parabola

(i) A point (x_1, y_1) lies inside, on or outside of the region of the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 <, = \text{ or } > 0$

(ii) The equation of chord of contact of tangents drawn from a point (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

(iii) Length of the chord of contact is $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$

(iv) The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.

(v) The semi latus rectum of a parabola is the H. M. between the segments of any focal chord of a parabola i.e. if PQR is a focal chord, then

$$2a = \frac{2PQ.QR}{PQ + QR}$$

(vi) If the tangent and normal at any point P of parabola meet the axes in T and G respectively and S is the focus of the parabola then

(a) ST = SG = SP

(b) $\angle$ PSK is a right angle, where K is the point where the tangent at P meets the directrix.

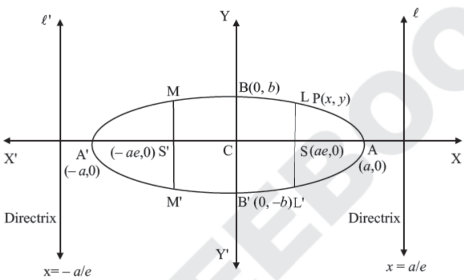
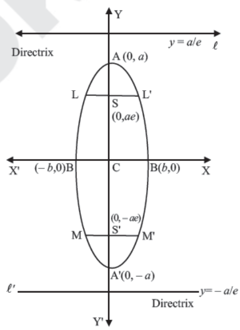
(vii) The area of triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

6. Ellipse

An ellipse is the set of all points in a plane, the sum of whose distances, form two fixed points in the plane is a constant.

The two fixed points are called the 'foci' of the ellipse.

7. Two Standard Forms of the Ellipse

Standard equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **(*a* > *b*), where a and b are constants** **(Horizontal Form of an Ellipse)**	$\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ **(*a* > *b*), where a and b are constants (Vertical Form of an Ellipse)**
Shape of the Ellipse		
Centre (c)	(0, 0)	(0, 0)
Equation of major axis (AA')	$y = 0$	$x = 0$
Equation of minor axis (BB')	$x = 0$	$y = 0$
Length of major axis (= AA')	$2a$	$2a$
Length of minor axis (= BB')	$2b$	$2b$
Foci (S and S')	$(\pm ae, 0)$	$(0, \pm ae)$
Vertices (A and A')	$(\pm a, 0)$	$(0, \pm a)$
Equation of directrices (ℓ and ℓ')	$x = \pm a/e$	$x = \pm a/e$

Eccentricity (e)	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{a^2}}$
Length of latus rectum (LL' or MM')	$2b^2/a$	$2b^2/a$

8. Important Point of Ellipse

(i) The point P (x_1, y_1) lies outside, on or inside the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ according as } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=0 \text{ or } <0$$

(ii) The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a_2}+\frac{yy_1}{b_2}=1$

(iii) The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

(iv) The equation of chord of contact of tangent drawn from a

point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0

where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

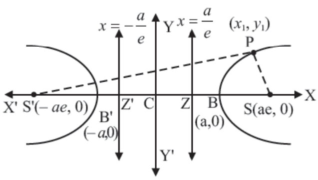
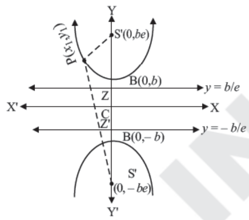
(v) *Number of tangent drawn from a point :* Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct, coincident or imaginary according as the given point lies outside, on or inside the ellipse.

9. Hyperbola

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant.

The two fixed points are called foci of the hyperbola.

10. Hyperbola and its Conjugate

	Hyperbola	Conjugate Hyperbola
		
Standard Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Eq. of transverse axis	$y = 0$	$x = 0$
Eq. of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$y = \pm a/e$	$y = \pm b/e$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$	$e=\sqrt{\frac{a^2-b^2}{b^2}}$
Length of latus rectum	$2b^2/a$	$2a^2/b$

11. Important Point of Hyperbola

(i) The point P (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0, =0$ or <0

(ii) The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

(iii) The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

(iv) The equation of chord of contact of tangent drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$ where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

(v) If asymptotes of the standard hyperbola are perpendicular to each other, then it is known as Rectangular Hyperbola.

Then

$$2\tan^{-1}\frac{b}{a}=\frac{\pi}{2} \Rightarrow b = a \text{ or } x^2 - y^2 = a^2$$

is general form of the equation of the rectangular hyperbola.

(vi) *Number of tangents from a point :* Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct, coincident or imaginary according as the given point lies outside, on or inside the hyperbola.

(a) The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}(b/a)$.

(b) A hyperbola and its conjugate hyperbola have the same asymptotes.

(c) The asymptotes pass through the centre of the hyperbola.

(d) The bisector of the angle between the asymptotes are the coordinate axes.

(vii) (a) The equation of asymptotes of the rectangular hyperbola are $y = \pm x$.

(b) The transverse and conjugate axes of a rectangular hyperbola are equal in length.

TIPS & TRICKS

TIPS AND TRICKS: (T-1)

Length of latus rectum of the parabola

(i) Length of latus rectum of the parabola $ax^2 + by + cx + d = 0$ is

$$=\left|\frac{\text{coefficient of } y}{\text{coefficient of } x^2}\right|=\left|\frac{b}{a}\right|$$

(ii) Length of latus rectum of the parabola

$py^2 + qy + rx + s = 0$ is $=\left|\frac{\text{coefficient of } x}{\text{coefficient of } y^2}\right|=\left|\frac{r}{p}\right|$

Illustration 1

The length of the latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is ____

(a) 4 (b) 6

(c) 8 (d) 10

Short-cut solution :

$\boxed{\text{Using T-1(i)}}$ Length $= \left|\dfrac{\text{coefficient of } y}{\text{coefficient of } x^2}\right| = \left|\dfrac{-8}{1}\right| = 8$ **Ans. (c)**

TIPS AND TRICKS: (T-2)

Length of latus rectum of an ellipse

Let the equation of ellipse $ax^2 + by^2 + cx + dx + e = 0$

(i) Length of latus rectum $= \dfrac{2\left(\text{coefficient of } y^2\right)}{\sqrt{\text{coefficient of } x^2}}, (a > b)$

(ii) Length of latus rectum $= \dfrac{2\left(\text{coefficient of } x^2\right)}{\sqrt{\text{coefficient of } y^2}}, (b > a)$

Illustration 2

The length of the latus rectum of ellipse $4x^2 + 9y^2 - 8x - 36y + 4 = 0$ is ___

(a) 8/3 (b) 4/3

(c) $\sqrt{5}/3$ (d) 16/3

Short-cut solution :

$\boxed{\text{Using T-2(ii)}}$

Since $b > a$

$\therefore$ length of latus rectum $= \dfrac{2\left(\text{coefficient of } x^2\right)}{\sqrt{\text{coefficient of } y^2}}$

$= \dfrac{2(4)}{\sqrt{9}} = \dfrac{8}{3}$

Ans. (a)

Illustration 3

The length of the latus rectum of ellipse $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{16} = 1$ is _____

(a) 8/3 (b) 9/2

(c) 4/3 (d) 3/4

Short-cut solution :

$\boxed{\text{Using T-2(i)}}$ $16(x-2)^2 + 9(y-3)^2 = 144$

Here, a = 16 and b = 9

$$\therefore \text{ length of latus rectum} = \frac{2\left(\text{coefficient of } y^2\right)}{\sqrt{\text{coefficient of } x^2}} \quad (\because a > b)$$

$$= \frac{2(9)}{\sqrt{16}} = \frac{18}{4} = \frac{9}{2}$$

Ans. (b)

TIPS AND TRICKS: (T-3)

Angle θ between two tangents drawn from point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$\tan\theta = \left|\frac{\sqrt{s_1}}{x_1 + a}\right|, \left(\text{where } s_1 = y_1^2 - 4ax_1\right)$$

Illustration 4

Tangents are drawn from the point (–2, –1) to the parabola $y^2 = 4x$. If α is the angle between these tangents then $\tan\alpha$ equals ______ **[JEE M 2014]**

(a) 3 (b) 2

(c) 1/3 (d) 1/2

Short-cut solution :

$\boxed{\text{Using T-3}}$

$s_1 = y_1^2 - 4ax_1 = (-1)^2 - 4(-2)$

$= 1 + 8 = 9$

$$\because \tan\alpha = \left|\frac{\sqrt{s_1}}{x_1 + a}\right|$$

$$= \left|\frac{\sqrt{9}}{-2+1}\right| = \left|\frac{3}{-1}\right| = 3$$

TIPS AND TRICKS-4

Eccentricity of an ellipse

Eccentricity of an ellipse $ax^2 + by^2 + cx + dy + e = 0$ is ______

(i) If $a > b$, then $e = \sqrt{1 - \dfrac{\text{coefficient of } y^2}{\text{coefficient of } x^2}}$

(ii) If $b > a$, then $e = \sqrt{1 - \dfrac{\text{coefficient of } x^2}{\text{coefficient of } y^2}}$

Illustration 5

The equation $x^2 + 4y^2 + 2x + 16y + 13 = 0$ represents an ellipse then eccentricity is ______

(a) 5 (b) $\dfrac{\sqrt{3}}{2}$

(c) $3\sqrt{5}$ (d) $\dfrac{2}{\sqrt{3}}$

Short-cut solution :

$\boxed{\text{Using T-4(ii)}}$

$$e = \sqrt{1 - \frac{\text{coefficient of } x^2}{\text{coefficient of } y^2}} \qquad (\because b > a)$$

$$\therefore\ e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

Ans. (b)

Illustration 6

The eccentricity of the ellipse $4x^2 + y^2 - 8x + 2y + 1 = 0$ is ______

(a) $\sqrt{3}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{2}{\sqrt{3}}$

Short-cut solution :

Using T-4(i)

$$e=\sqrt{1-\frac{\text{coefficient of } y^2}{\text{coefficient of } x^2}} \quad (\because a>b)$$

$$\therefore e=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{2}}$$

Ans. (c)

TIPS AND TRICKS: (T-5)

Eccentricity of hyperbola

Let the equation of hyperbola is $ax^2 - by^2 + cx + dy + e = 0$ then

$$e=\sqrt{1+\frac{\text{coefficient of } x^2}{\left|\text{coefficient of } y^2\right|}}=\sqrt{1+\frac{a}{|b|}}$$

Illustration 7

The eccentricity of hyperbola $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ is ___

(a) 3/16 (b) 3/4

(c) $\frac{4}{\sqrt{3}}$ (d) $\sqrt{\frac{19}{3}}$

Short-cut solution :

Using T-5

$$e=\sqrt{1+\frac{\text{coefficient of } x^2}{\left|\text{coefficient of } y^2\right|}}$$

$$=\sqrt{1+\frac{16}{3}}=\sqrt{\frac{19}{3}}$$

Ans. (d)

TIPS AND TRICKS: (T-6)

Area of triangle formed by tangent of parabola $y^2 = 4ax$ drawn from (x_1, y_1) and their chord of contact is $\frac{1}{2a}\left(y_1^2 - 4ax_1\right)^{3/2}$

Illustration 8

The area of triangle formed by tangents of parabola $y^2 = 4x$ drawn from the point $(-2, -1)$ and their chord of contact is ____

(a) 5 (b) 27/2

(c) 27/8 (d) $\frac{\sqrt{3}}{8}$

Short-cut solution :

$\boxed{\text{Using T-6}}$

Area of triangle $= \frac{1}{2a}\left(y_1^2 - 4ax_1\right)^{3/2}$

$= \frac{1}{2(1)}\left[(-1)^2 - 4(-2)\right]^{3/2}$

$= \frac{1}{2}[1+8]^{3/2} = \frac{1}{2}(9)^{3/2} = \frac{27}{2}$

Ans. (b)

TIPS AND TRICKS: (T-7)

The product of the length of perpendicular from the foci to any tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to b^2. Where $a > b$. i.e. b is length of semi minor axis.

Note: Same result apply in hyperbola.

Illustration 9

The product of the perpendiculars drawn from the foci of the ellipse,
$\frac{x^2}{9} + \frac{b^2}{25} = 1$ upon the tangent to it at the point $\left(\frac{3}{2}, \frac{5\sqrt{3}}{2}\right)$ is

(a) 25 (b) 18

(c) 9 (d) 20

Short-cut solution :

$\boxed{\text{Using T-7}}$

$\because \frac{x^2}{9} + \frac{b^2}{25} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{b^2}{5^2} = 1$

$\because 3 < 5$

$\therefore$ length of semi minor axis $b = 3$.

So, product of perpendicular is $3^2 = 9$.

Ans. (c)

SHORTCUTS: (SC-1)

Equations of tangent of parabola in slope form

If equation of parabola is $y^2 = 4ax$ and equation of tangent is $y = mx + c$ then

(i) point of contact $= \left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$

(ii) condition of tangency is $c = \dfrac{a}{m}$

(iii) equation of tangent is $y = mx + \dfrac{a}{m}$

Illustration 10

Let L_1 be a tangent to the parabola $y^2 = 4(x + 1)$ and L_2 be a tangent to the parabola $y^2 = 8(x + 2)$ such that L_1 and L_2 intersect at right angles. Then L_1 and L_2 meet on the straight line : **[JEE M 2020]**

(a) $x + 3 = 0$ (b) $2x + 1 = 0$

(c) $x + 2 = 0$ (d) $x + 2y = 0$

Short-cut solution :

$\boxed{\text{Using SC-1(iii)}}$

$L_1 : y = m_1(x+1) + \dfrac{1}{m_1}$ [Tangent to $y^2 = 4(x + 1)$]

$L_2 : y = m_2(x+2) + \dfrac{2}{m_2}$ [Tangent to $y^2 = 8(x + 2)$]

$m_1^2(x+1) - ym_1 + 1 = 0$ ...(i)

$m_2^2(x+2) - ym_2 + 2 = 0$ ...(ii)

$\because m_2 = -\dfrac{1}{m_1}$ $(\because L_1 \perp L_2)$

[From (ii)]

$\Rightarrow 2m_1^2 + ym_1 + (x+2) = 0$ ...(iii)

From (i) and (iii),

$\dfrac{x+1}{2} = \dfrac{-y}{y} = \dfrac{1}{x+2} \Rightarrow x + 3 = 0$

Ans. (a)

SHORTCUTS: (SC-2)

Equation of tangent of parabola in slope form

If equation of parabola is $x^2 = 4ay$ and equation of tangent is $y = mx + c$, then

(i) point of contact = $(2am, am^2)$

(ii) condition of tangency is $c = -am^2$

(iii) equation of tangent is $y = mx - am^2$

Illustration 11

If line $y = 2x - c$ touches the parabola $x^2 = 8y$ then value of c is

(a) -8 (b) 8

(c) 4 (d) 16

Short-cut solution :

Using SC-2(ii)

Here $a = 2$ and $m = 2$

$\because$ The line touches the parabola if $c = -am^2$

$\therefore c = -2(2)^2 \Rightarrow c = -8$ **Ans. (a)**

SHORTCUTS: (SC-3)

Equation of normal of parabola in slope form

If equation of parabola is $y^2 = 4ax$ and equation of normal is $y = mx + c$, then

(i) point of foot of normal = $(am^2, -2am)$

(ii) condition of normality is $c = -2am - am^3$

(iii) equation of normal is $y = mx - 2am - am^3$

Illustration 12

If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c)? **[JEE M 2019]**

(a) $\left(\frac{1}{2}, 2, 3\right)$ (b) $(1, 1, 3)$

(c) $\left(\frac{1}{2}, 2, 0\right)$ (d) $(1, 1, 0)$

Short-cut solution :

Using SC-3(iii)

Normal to $y^2 = 8ax$ is

$y = mx - 4am - 2am^3$ ...(i)

and normal to $y^2 = 4b(x - c)$ with slope m is

$y = m(x - c) - 2bm - bm^3$ ...(ii)

Since, both parabolas have a common normal.

$\therefore \quad 4am + 2am^3 = cm + 2bm + bm^3$

$\Rightarrow \quad 4a + 2am^2 = c + 2b + bm^2$ or $m = 0$

$\Rightarrow \quad (4a - c - 2b) = (b - 2a)\, m^2$

or (X-axis is common normal always)

Since, x-axis is a common normal.

Hence all the options are correct for $m = 0$. **Ans. (a, b, c, d)**

SHORTCUTS: (SC-4)

Equation of normal of parabola in slope form

If equation of parabola is $x^2 = 4ay$ and equation of normal is $y = mx + c$, then

(i) point of foot of normal = $\left(\frac{-2a}{m}, \frac{a}{m^2}\right)$

(ii) condition of normality is c = $2a + \frac{a}{m^2}$

(iii) equation of normal is y = $mx + 2a + \frac{a}{m^2}$

Illustration 13

If a line y = mx + 15 is equation of normal of parabola $x^2 = 12y$ then value of m is –

(a) $\sqrt{3}$ (b) $-\sqrt{3}$

(c) $\frac{1}{\sqrt{3}}$ (d) 3

Short-cut solution :

Using SC-4(ii)

Here $a = 3$ and $c = 15$

$\because$ A line y = mx + 5 is normal of parabola

$$\therefore\ c = 2a + \frac{a}{m^2}$$

$$\Rightarrow 15 = 2 \times 3 + \frac{3}{m^2}$$

$$\Rightarrow \frac{3}{m^2} = 9$$

$$\Rightarrow m^2 = \frac{1}{3} \Rightarrow m = \pm\frac{1}{\sqrt{3}}$$

Ans. (c)

SHORTCUTS: (SC-5)

Focal chord of a parabola

Focal chord of a parabola is a chord which passes through the focus of the parabola. Let $y^2 = 4ax$ be the equation of a parabola and $P(at^2, 2at)$ a point on it, then

(i) equation of focal chord through P is $y = \dfrac{2t}{t^2 - 1}(x - a)$

(ii) length of focal chord $= a\left(t + \dfrac{1}{t}\right)^2$

Illustration 14

If the point P(4, –2) is one end of the focal chord PQ of the parabola $y^2 = x$, then the length of focal chord is –

(a) $\dfrac{9}{4}$ (b) $\dfrac{81}{4}$

(c) $\dfrac{81}{64}$ (d) $\dfrac{8}{64}$

Short-cut solution :

Using SC-5(ii)

Here $a = \dfrac{1}{4}$ and one end of focal chord is $(at^2, 2at)$ but given that (4, –2)

$$2 \times \frac{1}{4} \times t = -2 \Rightarrow t = -4$$

$\therefore$ Length of focal chord = $a\left(t+\frac{1}{t}\right)^2$

$=\frac{1}{4}\left(-4-\frac{1}{4}\right)^2=\frac{1}{4}\left(\frac{-9}{4}\right)^2=\frac{81}{64}$ **Ans. (c)**

SHORTCUTS: (SC-6)

Equation of tangent of ellipse in slope form

If equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and equation of tangent is y = mx + c, then

(i) points of contacts = $\left(\pm a^2m/c, \pm b^2/c\right)$

(ii) condition of tangency is $c^2 = a^2m^2 + b^2$

(iii) equation of tangent is $y = mx \pm \sqrt{a^2m^2+b^2}$

Illustration 15

If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to: **[JEE M 2019]**

(a) $\sqrt{2}-1$ (b) $2\sqrt{2-1}$

(c) $2\sqrt{2}+1$ (d) $\sqrt{2}+1$

Short-cut solution :

Using SC-6

Let tangent to parabola at point $\left(\frac{1}{4m^2}, -\frac{1}{2m}\right)$ is $y = mx+\frac{1}{4m}$

and tangent to ellipse is, $y = mx \pm \sqrt{m^2+\frac{1}{2}}$

Now, condition for common tangency,

$\frac{1}{4m}=\pm\sqrt{m^2+\frac{1}{2}} \Rightarrow \frac{1}{16m^2}=m^2+\frac{1}{2}$

$\Rightarrow 16m^4 + 8m^2 - 1 = 0 \Rightarrow m^2=\frac{-8\pm\sqrt{64+64}}{2(16)}$

$=\frac{-8\pm 8\sqrt{2}}{2(16)}=\frac{\sqrt{2}-1}{4} \Rightarrow \alpha=\frac{1}{4m^2}=\frac{1}{4\frac{\sqrt{2}-1}{4}}=\sqrt{2}+1$

Illustration 16

The number of values of c such that the straight line $y = 4x + c$ touches the curve $(x^2/4) + y^2 = 1$ is

(a) 0 (b) 1

(c) 2 (d) infinite

Short-cut solution :

Using SC-6(ii)

The given curve is $\frac{x^2}{4}+\frac{y^2}{1}=1$ (an ellipse) and given line is $y = 4x + c$.

We know that $y = mx + c$ touches the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c=\pm\sqrt{a^2m^2+b^2}$

Hence the given line touches the given ellipse if

$c=\pm\sqrt{4\times 16+1}=\pm\sqrt{65}$

$\therefore$ There are two values of c exist. **Ans. (c)**

Illustration 17

Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. **[AIEEE 2005]**

Short-cut solution :

Using SC-6(iii)

Let the common tangent to circle $x^2 + y^2 = 16$ and ellipse $x^2/25 + y^2/4 = 1$ be $y=mx+\sqrt{25m^2+4}$ ..(i)

Since it is tangent to circle $x^2 + y^2 = 16$.

$$\therefore \quad \frac{\sqrt{25m^2+4}}{\sqrt{m^2+1}}=4$$

[Since length of perpendicular from centre of the circle to the tangent is equal to the radius of the circle.]

$\Rightarrow \quad 25m^2 + 4 = 16m^2 + 16 \quad \Rightarrow \quad 9m^2 = 12$

$\therefore \quad m = \dfrac{-2}{\sqrt{3}}$

[Since, the slope of any tangent to the given circle at any point in the 1st quadrant will be positive.]

$\therefore$ Equation of common tangent is

$$y = -\frac{2}{\sqrt{3}}x + \sqrt{25.\frac{4}{3}+4} \Rightarrow y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$$

This tangent meets the axes at $A(2\sqrt{7},0)$ and $B\left(0,4\sqrt{\dfrac{7}{3}}\right)$

$\therefore$ Length of intercepted portion of tangent between the axes

$$= AB = \sqrt{(2\sqrt{7}) + \left(4\sqrt{\frac{7}{3}}\right)^2} = 14/\sqrt{3}$$

SHORTCUTS: (SC-7)

Equation of normal of ellipse in slope form

If equation of ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and equation of normal is y = mx + c, then

(i) points of foots of normal $= \left(\dfrac{\pm a^2}{\sqrt{a^2+b^2m^2}}, \dfrac{\pm mb^2}{\sqrt{a^2+b^2m^2}}\right)$

(ii) condition of normality is $c^2 = \dfrac{m^2\left(a^2-b^2\right)^2}{a^2+b^2m^2}$

(iii) equation of normal is $y = mx \pm \dfrac{m\left(a^2-b^2\right)}{\sqrt{a^2+b^2m^2}}$

Illustration 18

The line lx + my + n =0 is a normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ if

(a) $\dfrac{a^2}{m^2}+\dfrac{b^2}{l^2}=\dfrac{(a^2-b^2)^2}{n^2}$

(b) $\dfrac{a^2}{l^2}+\dfrac{b^2}{m^2}=\dfrac{(a^2-b^2)^2}{n^2}$

(c) $\dfrac{a^2}{l^2}-\dfrac{b^2}{m^2}=\dfrac{(a^2-b^2)^2}{n^2}$

(d) none of these

Short-cut solution :

Using SC-7

Since lx + my + n = 0 ⇒ y = $\frac{-l}{m}x-\frac{n}{m}$ is normal of ellipse

$$\therefore\ c^2=\frac{m^2\left(a^2-b^2\right)^2}{a^2+b^2m^2}$$

$$\Rightarrow\left(\frac{n}{m}\right)^2=\frac{\left(\frac{-l}{m}\right)^2\left(a^2-b^2\right)^2}{a^2+b^2\left(\frac{-l}{m}\right)^2}$$

$$\Rightarrow\ \frac{n^2}{m^2}=\frac{l^2\left(a^2-b^2\right)^2}{a^2m^2+b^2l^2}$$

$$\Rightarrow\ \frac{a^2m^2+b^2l^2}{m^2l^2}=\frac{\left(a^2-b^2\right)^2}{n^2}$$

$$\Rightarrow\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{\left(a^2-b^2\right)^2}{n^2}$$

Ans. (b)

SHORTCUTS: (SC-8)

Equation of tangent of hyperbola in slope form

If equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and equation of tangent is

y = mx + c, then

(i) points of contacts = $\left(\pm\frac{a^2m}{c},\pm\frac{b^2}{c}\right)$

(ii) condition of tangency is $c^2 = a^2m^2 - b^2$

(iii) equation of tangent is $y=mx\pm\sqrt{a^2m^2-b^2}$

Illustration 19

If the line $y = mx + c$ is a common tangent to the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ and the circle $x^2 + y^2 = 36$, then which one of the following is true?

[JEE M 2020]

(a) $c^2 = 369$ (b) $5m = 4$

(c) $4c^2 = 369$ (d) $8m + 5 = 0$

Short-cut solution :

Using SC-8(iii)

General tangent to hyperbola in slope form is

$y = mx \pm \sqrt{100m^2 - 64}$

and the general tangent to the circle in slope form is

$y = mx \pm 6\sqrt{1 + m^2}$

For common tangent,

$36(1 + m^2) = 100m^2 - 64$

$\Rightarrow 100 = 64m^2 \Rightarrow m^2 = \frac{100}{64}$

$\therefore c^2 = 36\left(1 + \frac{100}{64}\right) = \frac{164 \times 36}{64} = \frac{369}{4}$

$\Rightarrow 4c^2 = 369$

Ans. (c)

SHORTCUTS: (SC-9)

Equation of normal of hyperbola in slope form

If equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and equation of normal is y = mx + c, then

(i) points of foot of normal = $\left(\pm\frac{a^2}{\sqrt{a^2 - b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2 - b^2m^2}}\right)$

(ii) condition of normality is $c^2 = \frac{m^2\left(a^2 + b^2\right)^2}{a^2 - b^2m^2}$

(iii) equation of normal is $y = mx \pm \frac{m\left(a^2 + b^2\right)}{\sqrt{a^2 - b^2m^2}}$

Illustration 20

If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$, then a value of m is : **[JEE M 2019]**

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{15}}{2}$

(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{3}{\sqrt{5}}$

Short-cut solution :

Using SC-9(iii)

Since, lx + my + n = 0 is a normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$

but it is given that $mx - y + 7\sqrt{3}$ is normal to hyperbola $\frac{x^2}{24} - \frac{y^2}{18} = 1$

then $\frac{24}{m^2} - \frac{18}{(-1)^2} = \frac{(24+18)^2}{(7\sqrt{3})^2} \Rightarrow m = \frac{2}{\sqrt{5}}$

TECHNIQUE

Find equation of axis, if the parabola is in the form

(i) $ax^2 + bx + cy + d = 0$, then differentiate given equation w.r.t. x keeping y as constant

(ii) $py^2 + qy + rx + s = 0$, then differentiate given equation w.r.t. y keeping x as constant

Illustration 21

The equation of the axis of the parabola $x^2 - 4x - 3y + 10 = 0$ is

(a) y + 2 = 0 (b) x + 2 = 0

(c) x – 2 = 0 (d) y – 2 = 0

Short-cut solution :

Using Tech.

Differentiating given equation w.r.t. x keeping y as constant, we get

$2x - 4 = 0 \Rightarrow x - 2 = 0$

$\therefore$ Equation of axis is $x - 2 = 0$ **Ans. (c)**

Illustration 22

The equation of the axis of the parabola $4y^2 - 6x - 4y = 5$ is -

(a) $y - 1 = 0$ (b) $2y + 1 = 0$

(c) $2y - 1 = 0$ (d) $y - 2 = 0$

Short-cut solution :

Using Tech.

Differentiating given equation w.r.t. y keeping x as constant, we get

$8y - 4 = 0 \Rightarrow 2y - 1 = 0$

$\therefore$ equation of axis is $2y - 1 = 0$ **Ans. (c)**

Concept Booster Exercise

1. Find the latus rectum of the parabola $3x^2 - 6x - y + 6 = 0$ is –

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) $\frac{2}{3}$ (d) $\frac{2}{7}$

2. The length of the latus rectum of ellipse $9x^2 + 16y^2 - 36x + 96y + 36 = 0$

(a) $\frac{9}{2}$ (b) $\frac{7}{2}$ (c) $\frac{2}{3}$ (d) $\frac{5}{4}$

3. Tangents are drawn from the point $(-1, 1)$ to the parabola $y^2 = 8x$. If α is the angle between these tangents then $\tan\alpha$ equals -

(a) 2 (b) 3 (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{3}}$

4. The eccentricity of the ellipse $25x^2 + 16y^2 - 150x - 175 = 0$ is -

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{5}{4}$ (d) $\frac{2}{3}$

5. The eccentricity of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$ is -

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{3}$ (d) 3

6. The area of triangle formed by tangents of parabola $y^2 = 12x$ drawn from the point (4, 1) and their chord of contact is -

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

7. The product of the perpendiculars from the foci to any tangent to the ellipse $5x^2 + 8y^2 = 40$ is

(a) 8 (b) 5 (c) 3 (d) $\sqrt{13}$

8. If the common tangent to the parabolas, $y^2 = 4x$ and $x^2 = 4y$ also touches the circle, $x^2 + y^2 = c^2$, then c is equal to: **[JEE M 2020]**

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

9. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to: **[JEE M 2020]**

(a) –32 (b) –64 (c) –128 (d) 128

10. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is **[JEE M 2014]**

(a) $\frac{1}{8}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

11. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is: **[JEE M 2016]**

(a) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
(b) $x^2 + y^2 - 4x + 9y + 18 = 0$
(c) $x^2 + y^2 - 4x + 8y + 12 = 0$
(d) $x^2 + y^2 - x + 4y - 12 = 0$

12. If a line $y = \sqrt{3}x + 14$ is equation of normal of parabola $x^2 = 4ay$ then value of a is

(a) 6 (b) 3 (c) 24 (d) 12

13. The focal chord to $y^2 = 16x$ is tangent to $(x - 6)^2 + y^2 = 2$, then the possible values of the slope of this chord, are **[AIEEE 2003S]**

(a) $\{-1, 1\}$ (b) $\{-2, 2\}$ (c) $\{-2, -1/2\}$ (d) $\{2, -1/2\}$

14. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is **[JEE M 2014]**

(a) $\left(x^2+y^2\right)^2 = 6x^2 + 2y^2$ (b) $\left(x^2+y^2\right)^2 = 6x^2 - 2y^2$

(c) $\left(x^2-y^2\right)^2 = 6x^2 + 2y^2$ (d) $\left(x^2-y^2\right)^2 = 6x^2 - 2y^2$

15. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

(a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$

(c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) $\left(\frac{2}{5}, -\frac{1}{5}\right)$

16. If the line $2x - \frac{8}{3}\lambda y = -3$ is a normal to the conic $x^2 + \frac{y^2}{4} = 1$, then λ equals to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{3}{8}$

17. The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is: **[JEE M 2019]**

(a) $x - y + 1 = 0$ (b) $x - y + 7 = 0$

(c) $x - y + 9 = 0$ (d) $x - y - 3 = 0$

18. If $y = mx + 7\sqrt{3}$ is normal to $\frac{x^2}{18} - \frac{y^2}{24} = 1$, then the value of m can be

(a) $\frac{2}{\sqrt{5}}$ (b) $\frac{4}{\sqrt{5}}$

(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$

19. The equation of the axis of the parabola $y^2 - 2y - 4x + 5 = 0$ is

(a) $y + 1 = 0$ (b) $y - 1 = 0$

(c) $2y - 1 = 0$ (d) $2y + 1 = 0$

Numerical Value Problems

20. Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q, If area $(\Delta OPQ) = 4$ sq. units, then m is equal to ________. **[JEE M 2020]**

21. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is **[AIEEE 2010]**

22. If the eccentricity of the two ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equal, then the value of $\frac{a}{b}$ is :

23. The eccentricity of ellipse $12x^2 + 4y^2 + 24x - 16y + 25 = 0$ is $\sqrt{k}$, then k is _____.

24. The eccentricity of the hyperbola of $x^2 - 3y^2 = 2x + 8$ is $\frac{2}{\sqrt{k}}$, then k is ______.

25. Equation of the tangent to the hyperbola $2x^2 - 3y^2 = 6$ is $y = 3x \pm k$, then value of k is __ .

Solutions

1. **(a)** Using T-1

Length of latus rectum $= \left|\dfrac{\text{coefficient of } y}{\text{coefficient of } x^2}\right| = \left|\dfrac{-1}{3}\right| = \dfrac{1}{3}$

2. **(a)** Using T-2

Length of latus rectum $= \dfrac{2\left(\text{coefficient of } x^2\right)}{\sqrt{\text{coefficient of } y^2}}, (\because b > a)$

$= \dfrac{2(9)}{\sqrt{16}} = \dfrac{18}{4} = \dfrac{9}{2}$

3. **(b)** Using T-3

$S_1 = y_1^2 - 4ax_1 = (1)^2 - 8(-1) = 9$

$\because \tan\alpha = \left|\dfrac{\sqrt{S_1}}{x_1 + a}\right| = \left|\dfrac{\sqrt{9}}{-1+2}\right| = 3$

4. **(a)** Using T-4 $e = \sqrt{1 - \dfrac{\text{coefficient of } y^2}{\text{coefficient of } x^2}} \; (\because a > b)$

$\therefore \; e = \sqrt{1 - \dfrac{16}{25}} = \dfrac{3}{5}$

5. **(a)** Using T-5

$e = \sqrt{1 + \dfrac{\text{coefficient of } x^2}{\left|\text{coefficient of } y^2\right|}}$

$= \sqrt{1 + \dfrac{1}{3}} = \dfrac{2}{\sqrt{3}}$

6. **(c)** Using T-6

Area of triangle $= \dfrac{1}{2a}\left(y_1^2 - 4ax_1\right)^{3/2}$

$= \dfrac{1}{2\times 3}\left[4^2 - 4\times 3(1)\right]^{3/2} = \dfrac{1}{6}\times 8 = \dfrac{4}{3}$

7. **(b)** $\boxed{\text{Using T-7}}$ $\because 5x^2+8y^2=40 \Rightarrow \frac{x^2}{8}+\frac{y^2}{5}=1$

$\because 8>5 \therefore b^2=5$

So, product of perpendicular is $b^2 = 5$.

8. **(b)** $\boxed{\text{Using SC-1(iii)}}$ Equation tangent to parabola $y^2 = 4x$ with slope m be:

$y = mx + \frac{1}{m}$ ...(i)

$\boxed{\text{Using SC-2(iii)}}$

$\because$ Equation of tangent to $x^2 = 4y$ with slope m be :

$y = mx - am^2$ ...(ii)

From eq. (i) and (ii),

$\frac{1}{m} = -m^2 \Rightarrow m = -1$

$\therefore$ Equation tangent : $x + y + 1 = 0$

It is tangent to circle $x^2 + y^2 = c^2$

$\Rightarrow c = \frac{1}{\sqrt{2}}$

9. **(c)** $\boxed{\text{Using SC-1(iii)}}$ $y = mx + 4$ ...(i)

Tangent of $y^2 = 4x$ is

$\Rightarrow \quad y = mx + \frac{1}{m}$ ...(ii)

[$\because$ Equation of tangent of $y^2 = 4\,ax$ is $y = mx + \frac{a}{m}$]

From (i) and (ii)

$4 = \frac{1}{m} \Rightarrow m = \frac{1}{4}$

So, line $y = \frac{1}{4}x + 4$ is also tangent to parabola

$x^2 = 2by$, so solve both equations.

$x^2 = 2b\left(\frac{x+16}{4}\right)$

$\Rightarrow \quad 2x^2 - bx - 16b = 0$

$\Rightarrow \quad D = 0$[For tangent]

$\Rightarrow \quad b^2 - 4 \times 2 \times (-16b) = 0$

$\Rightarrow \quad b^2 + 32 \times 4b = 0$

$\quad b = -128, b = 0$ (not possible)

10. **(c)** $\boxed{\text{Using SC-1(iii)}}$ Let tangent to $y^2 = 4x$ be $y = mx + \frac{1}{m}$

Since this is also tangent to $x^2 = -32y$

$$\therefore \quad x^2 = -32\left(mx + \frac{1}{m}\right)$$

$$\Rightarrow \quad x^2 + 32mx + \frac{32}{m} = 0$$

Now, D = 0

$$(32)^2 - 4\left(\frac{32}{m}\right) = 0$$

$$\Rightarrow m^3 = \frac{4}{32} \quad \Rightarrow m = \frac{1}{2}$$

11. **(c)** $\boxed{\text{Using SC-3(iii) Put m = –t}}$

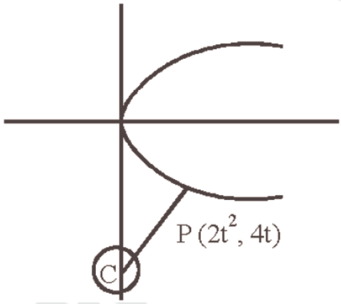
Minimum distance ⇒ perpendicular distance

Eq^n of normal at $p(2t^2, 4t)$

$y = -tx + 4t + 2t^3$

It passes through C(0, –6)

$t^3 + 2t + 3 = 0 \Rightarrow t = -1$



Centre of new circle = $P(2t^2, 4t) = P(2, -4)$

Radius = PC $= \sqrt{(2-0)^2 + (-4+6)^2} = 2\sqrt{2}$

∴ Equation of circle is :

$$(x-2)^2 + (y+4) = \left(2\sqrt{2}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 8y + 12 = 0$$

12. **(a)** $\boxed{\text{Using SC-4}}$ Here $m = \sqrt{3}$ and c = 14

∵ A line $y = \sqrt{3}x + 14$ is normal of parabola

$$\therefore c = 2a + \frac{a}{m^2}$$

$$14 = 2a + \frac{a}{3} \Rightarrow 14 = \frac{7a}{3}$$

$$\Rightarrow a = 6.$$

13. **(a)** Given parabola $y^2 = 16x$, its focus $= (4, 0)$. Let m be the slope of focal chord then its equation is

$y = m(x - 4)$ ...(i)

But it is given that equation (i) is a tangent to the circle

$(x - 6)^2 + y^2 = 2$ with centre, $C(6, 0)$ and radius (r) $= \sqrt{2}$

$\therefore$ Length of perpendicular from (6, 0) to (i) $= \sqrt{2}$

$$\Rightarrow \frac{6m - 4m}{\sqrt{m^2+1}} = \sqrt{2} \Rightarrow 2m = \sqrt{2(m^2+1)}$$

$$\Rightarrow 2m^2 = m^2 + 1 \qquad \Rightarrow m = \pm 1$$

14. **(a)** $\boxed{\text{Using SC-6(iii)}}$ Given equation of ellipse can be written as $\frac{x^2}{6} + \frac{y^2}{2} = 1$

$\Rightarrow a^2 = 6, b^2 = 2$

Now, equation of any variable tangent is

$y = mx \pm \sqrt{a^2m^2 + b^2}$ ...(i)

where m is slope of the tangent

So, equation of perpendicular line drawn from centre to tangent is

$y = \frac{-x}{m}$ ...(ii)

Eliminating m, we get

$$(x^4 + y^4 + 2x^2y^2) = a^2x^2 + b^2y^2$$

$$\Rightarrow (x^2 + y^2)^2 = a^2x^2 + b^2y^2$$

$$\Rightarrow (x^2 + y^2)^2 = 6x^2 + 2y^2$$

15. **(b, d)** $\boxed{\text{Using SC-6(ii)}}$ Let $y = \frac{8}{9}x + c$ be the tangent to $\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$

where $c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{\frac{1}{4} \times \frac{64}{81} + \frac{1}{9}} = \pm\frac{5}{9}$

and points of contact are $\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$

$$= \left(\frac{2}{5}, \frac{-1}{5}\right) \text{ or } \left(\frac{-2}{5}, \frac{1}{5}\right)$$

16. **(a)** $\boxed{\text{Using SC-7}}$

$\because 2x - \frac{8}{3}\lambda y = -3 \Rightarrow y = \frac{3}{4}\lambda x + \frac{9}{8}\lambda$ is normal of ellipse $\frac{x^2}{1} + \frac{y^2}{4} = 1$

$\therefore\ a = 1, b = 2, m = \dfrac{3}{4\lambda}$ and $c = \dfrac{9}{8\lambda}$

Now, for normality

$$c^2 = \frac{m^2\left(a^2 - b^2\right)^2}{a^2 + b^2m^2}$$

$$\frac{81}{64\lambda^2} = \frac{\frac{9}{16\lambda^2}(1-4)^2}{1+4\cdot\frac{9}{16\lambda^2}}$$

$$\frac{81}{64} = \frac{\frac{9}{16}\times 9}{\frac{16\lambda^2+36}{16\lambda^2}}$$

$16\lambda^2 + 36 = 64\lambda^2$

$48\lambda^2 = 36 \Rightarrow \lambda = \dfrac{\sqrt{3}}{2}$

17. **(a)** Given, the equation of line,

$x - y = 2 \Rightarrow y = x - 2$

$\therefore$ its slope $= m = 1$

Equation of hyperbola is:

$\dfrac{x^2}{5} - \dfrac{y^2}{4} = 1 \Rightarrow a^2 = 5, b^2 = 4$

The equation of tangent to the hyperbola is,

$y = mx \pm \sqrt{a^2m^2 - b^2} = x \pm \sqrt{5-4} \Rightarrow y = x \pm 1$

18. **(a)** $\boxed{\text{Using SC-9}}$

Here c = $7\sqrt{3}$; $a^2 = 24$ and $b^2 = 18$

By condition of normality.

$$c^2 = \frac{m^2\left(a^2+b^2\right)^2}{b^2 - a^2m^2}$$

$$\left(7\sqrt{3}\right)^2 = \frac{m^2(18+24)^2}{24-18m^2}$$

$$237 = \frac{m^2 \times (42)^2}{24-18m^2}$$

$24 - 18\,m^2 = 12\,m^2$

$24 = 30m^2$

$m^2 = \frac{4}{5} \Rightarrow m = \pm\frac{2}{\sqrt{5}}$

19. **(b)** $\boxed{\text{Using Tech.}}$

Differentiating given equation w.r. to y keeping x as constant, we get

$2y - 2 = 0 \Rightarrow y - 1 = 0.$

$\therefore$ equation of axis is $y - 1 = 0$

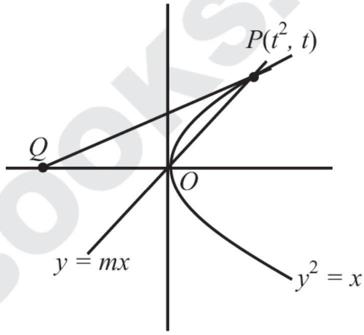
20. **(0.5)** $\boxed{\text{Using Key Notes}}$ Let the coordinates of $P = P(t^2, t)$

Tangent at $P(t^2, t)$ is $ty = \frac{x + t^2}{2}$

$\Rightarrow 2ty = x + t^2$

$Q(-t^2, 0), O(0, 0)$

$\therefore$ Area of $\Delta OPQ = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ t^2 & t & 1 \\ -t^2 & 0 & 1 \end{vmatrix} = 4$



$\Rightarrow |t|^3 = 8$

$\Rightarrow t = \pm 2\ (t > 0)$

$\therefore$ $4y = x + 4$ is a tangent

$\therefore$ P is $(4, 2)$; Now, $y = mx$ $\quad\therefore m = \frac{1}{2}$

21. **(2)** $\boxed{\text{Using Key Notes}}$

Intersection point of nearest directrix $x = \frac{a}{e}$ and x-axis is $\left(\frac{a}{e}, 0\right)$

Since $2x + y = 1$ passes through $\left(\frac{a}{e}, 0\right)$

$\therefore \frac{2a}{e} = 1 \Rightarrow a = \frac{e}{2}$

Also $y = -2x + 1$ is a tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore 1 = a^2(-2)^2 - b^2 \Rightarrow 4a^2 - b^2 = 1 \Rightarrow 4a^2 - a^2(e^2 - 1) = 1$

$\Rightarrow 4 \times \frac{e^2}{4} - \frac{e^2}{4}(e^2 - 1) = 1$

$\Rightarrow e^2 = 4\ as\ e > 1$ for hyperbola $\Rightarrow e = 2$

22. (2.6)

Here a = 13, b = 5

$\because$ Both ellipse have same eccentricity therefore values of a and b are same

$$\therefore \frac{a}{b} = \frac{13}{5} = 2.6$$

23. (0.67) $\boxed{\text{Using T-4}}$ Here $a > b$

$$\therefore e = \sqrt{1 - \frac{\text{coefficient of } y^2}{\text{coefficient of } x^2}}$$

$$= \sqrt{1 - \frac{4}{12}} = \sqrt{\frac{8}{12}} = \sqrt{\frac{2}{3}} = \sqrt{k}$$

$$\therefore k = \frac{2}{3} = 0.67$$

24. (3) $\boxed{\text{Using T-5}}$ Here, $x^2 - 3y^2 - 2x - 8 = 0$

$$e = \sqrt{1 + \frac{\text{coefficient of } x^2}{|\text{coefficient of } y^2|}}$$

$$= \sqrt{1 + \frac{1}{3}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{k}} \Rightarrow k = 3.$$

25. (5) $\boxed{\text{Using SC-8(iii)}}$ We have the equation of hyperbola as

$$2x^2 - 3y^2 = 6 \Rightarrow \frac{x^2}{3} - \frac{y^2}{2} = 1$$

On comparing this equation to the standard equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ we get

$a^2 = 3, b^2 = 2$

Given equation of line is $y = 3x + 4$

We know that $y = mx + c \Rightarrow m = 3$

Equation of tangent to the hyperbola :

$$y = mx \pm \sqrt{a^2m^2 - b^2} = 3x \pm \sqrt{3 \times 3^2 - 2}$$

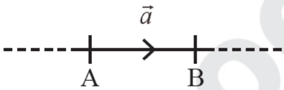
$$= 3x \pm \sqrt{27 - 2} = 3x \pm \sqrt{25}$$

$$\Rightarrow \quad y = 3x \pm 5 \Rightarrow k = 5.$$

25 Vector Algebra

Review of Key Notes and Formulae

1. **Vector Quantity:** A quantity which has magnitude & also a direction in space is called a vector quantity.



The direct line segment AB is a vector denoted as $\overrightarrow{AB}$ or $\vec{a}$. The point A from where the vector $\overrightarrow{AB}$ starts is called its initial point, & the point B where it ends is called its terminal point. The distance between these two points is called the magnitude of the vector denoted as $|\overrightarrow{AB}|$ or $|\vec{a}|$ or a.

2. **Position Vector:** Let O be the origin & P be a point in space having coordinates (x, y, z) with respect to the origin O. Then the vector $\overrightarrow{OP}$ is called the position vector of the point P with respect to O.

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

Properties of magnitude

$$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

$$|\vec{a}+\vec{b}| \ge |\vec{a}|-|\vec{b}|$$

$$|\vec{a}-\vec{b}| \le |\vec{a}|+|\vec{b}|$$

$$|\vec{a}-\vec{b}| \ge |\vec{a}|-|\vec{b}|$$

3. **Direction Cosines:** The angles made by $\overrightarrow{OP}$ with positive direction of x, y & z-axes (say α, β & γ respectively) are called its direction angles, and the cosine value of these angles i.e., cos α, cos β & cos γ are called direction cosines of $\overrightarrow{OP}$, denoted by *l*, *m* & *n* respectively.

4. **Types of Vectors:**

   (i) **Zero vector :** A vector whose initial and terminal points coincide, is called a zero vector (or null vector) denoted as $\vec{O}$. It has zero magnitude.

(ii) **Unit vector :** A vector whose magnitude is unity (i.e., 1 unit) is called unit vector. The unit vector in the direction of $\vec{a}$ is denoted as $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

(iii) **Coinitial vectors :** Two or more vectors having the same initial point are called coinitial vectors.

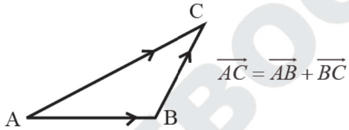
(iv) **Collinear vectors :** Two or more vectors are called collinear, if they are parallel to the same line, irrespective of their magnitude.

(v) **Equal vectors :** Two vectors are said to be equal, if they have same magnitude & direction regardless of the position of their initial points.

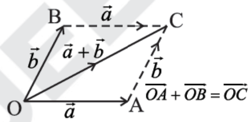
(vi) **Negative of a vector :** A vector whose magnitude is the same as that of the given vector, but the direction is opposite to that of it, is called negative of the given vector.

5. Addition of Vector:

(i) Triangle Law of Vector Addition



(ii) Parallelogram Law of Vector Addition



Properties of Vector Addition :

(i) For any two vectors $\vec{a}$ & $\vec{b}$,
$\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative property)

(ii) For any three vectors $\vec{a}, \vec{b}$ & $\vec{c}$,
$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (Associative property)

6. Section Formulae: The position vector of a point R dividing a line segment joining the points P & Q whose position vectors are $\vec{a}$ & $\vec{b}$ respectively, in the ratio $m : n$

(i) Internally, is given by $\frac{m\vec{b}+n\vec{a}}{m+n}$

(ii) Externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$

The position vector of the middle point of PQ is given by $\frac{1}{2}(\vec{a}+\vec{b})$.

7. Scalar (or Dot) Product of Two Vectors: Let $\vec{a}\ \&\ \vec{b}$ be the two non-zero vectors inclined at an angle θ, then scalar product is defined as:

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \le \theta \le \pi$

Properties of dot product:

(i) $\vec{a}\cdot\vec{b}$ is a real number.

(ii) $\vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

(iii) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

(iv) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1\ \&\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(v) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ or $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$

(vi) The scalar product is commutative i.e., $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

8. Projection of Vector Along a Directed Line:

Projection of a vector $\vec{a}$ on other vector $\vec{b}$, is given by

$$\vec{a}\cdot\hat{b} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|}$$

9. Vector (or Cross) Product of Two Vectors: Let $\vec{a}\ \&\ \vec{b}$ be two non-zero vectors inclined at an angle θ.

Then, vector product is defined as : $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$

where, $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}\ \&\ \vec{b}$ such that $\vec{a}, \vec{b}\ \&\ \hat{n}$ form a right handed system.

Properties of cross product:

(i) $\vec{a}\times\vec{b}$ is a vector which is perpendicular to both $\vec{a}$ and $\vec{b}$.

(ii) $\vec{a}\times\vec{b} = \vec{0} \Rightarrow \vec{a} \parallel \vec{b}$

(iii) $\vec{a}\cdot\vec{a} = 0$

(iv) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

(v) $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

(vi) $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

(vii) Vector product is not commutative.

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}\ \&\ \hat{i}\times\hat{k} = -\hat{j}$

(viii) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a triangle, then its area is given by $\frac{1}{2}|\vec{a}\times\vec{b}|$

(ix) If $\vec{a}$ & $\vec{b}$ represent the adjacent sides of a parallelogram then its area is given by $|\vec{a}\times\vec{b}|$

(x) Cross product of vectors in component form

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.

Then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

10. Scalar Triple Product : Scalar triple product of three vectors $\vec{a},\vec{b}$ and $\vec{c}$ is denoted by $(\vec{a}\times\vec{b})\cdot\vec{c}$ or $(\vec{b}\times\vec{c})\cdot\vec{a}$ or $(\vec{c}\times\vec{a})\cdot\vec{b}$.

$(\vec{a}\times\vec{b})\cdot\vec{c},(\vec{b}\times\vec{c})\cdot\vec{a}$ and $(\vec{c}\times\vec{a})\cdot\vec{b}$ can be also written as $[\vec{a}\,\vec{b}\,\vec{c}],[\vec{b}\,\vec{c}\,\vec{a}]$ and $[\vec{c}\,\vec{a}\,\vec{b}]$ respectively.

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$,

then $(\vec{a}\times\vec{b})\cdot\vec{c}=[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

Properties of scalar triple product:

(i) $[\vec{a}\,\vec{b}\,\vec{c}]=[\vec{b}\,\vec{c}\,\vec{a}]=[\vec{c}\,\vec{a}\,\vec{b}]$ but $[\vec{a}\,\vec{b}\,\vec{c}]=-[\vec{a}\,\vec{c}\,\vec{b}]$ etc.

i.e. change of any two vector in scalar triple product changes the sign of the scalar triple product.

(ii) The position of dots and cross in a scalar triple product can be interchanged. Hence $(\vec{a}\times\vec{b}).\vec{c}=\vec{a}.(\vec{b}\times\vec{c})$.

(iii) $[\vec{a}\,\vec{b}\,\vec{c}]=$ volume of the parallelopiped whose coterminous edges are formed by $\vec{a},\vec{b},\vec{c}$.

(iv) If any two of the vectors $\vec{a},\vec{b},\vec{c}$ are equal, then $[\vec{a}\,\vec{b}\,\vec{c}]=0$.

(v) The value of a scalar triple product is zero, if two of its vectors are parallel.

(vi) Four points A, B, C, D with position vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ respectively are coplanar if and only if $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=0$ i.e. if and only if $[\vec{b}-\vec{a}\ \ \vec{c}-\vec{a}\ \ \vec{d}-\vec{a}]=0$

(vii) Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

(viii) Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

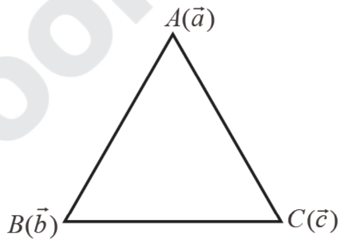
11. Vector Triple Product : If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are known as vector triple product. Vector triple product of three vectors is a vector quantity.

(a) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\,\vec{c})\vec{b} - (\vec{b}.\,\vec{c})\vec{a}$

(b) The vector triple product is not associative i.e., $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

12. Centroid of a Triangle : If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ be P.V.'s of the vertices A, B, C of a triangle ABC respectively, then the P.V. of the centroid G of the triangle is $\frac{\overrightarrow{a} + \overrightarrow{b} + \overrightarrow{c}}{3}$.



Also, the P.V. of incentre I of ΔABC is $\frac{(BC)\overrightarrow{a} + (CA)\overrightarrow{b} + (AB)\overrightarrow{c}}{BC + CA + AB}$ and the P.V. of orthocentre of ΔABC is $\frac{\overrightarrow{a}(\tan A) + \overrightarrow{b}(\tan B) + \overrightarrow{c}(\tan C)}{\tan A + \tan B + \tan C}$

13. Linear Combination of Vectors : A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$ ...., etc, if there exist scalars x, y, z,…, etc., such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$

14. Linearly Independent Vectors : A set of non-zero vectors $\vec{a}_1, \vec{a}_2, ...\vec{a}_n$ is said to be linearly independent, if $x_1\vec{a}_1 + x_2\vec{a}_2 + ... + x_n\vec{a}_n = 0$

$\Rightarrow x_1 = x_2 = ... = x_n = 0.$

15. Collinearity of Three Points : The necessary and sufficient condition that three points with P.V.'s $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are collinear is that there exist three scalars x, y, z not all zero such that $x\overrightarrow{a} + y\overrightarrow{b} + z\overrightarrow{c} = 0 \Rightarrow x + y + z = 0$

16. Reciprocal System of Vectors: The two system of vectors are called reciprocal system of vectors if by taking dot product we get unity.

Thus, if $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and if

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left|\vec{a}\ \vec{b}\ \vec{c}\right|}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{\left|\vec{a}\ \vec{b}\ \vec{c}\right|} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{\left|\vec{a}\ \vec{b}\ \vec{c}\right|}$$

Then $\vec{a}', \vec{b}'$ and $\vec{c}'$ are said to be reciprocal system of vectors for the vector $\vec{a}, \vec{b}$ and $\vec{c}$.

TIPS AND TRICKS: (T-1)

The unit vector $\hat{n}$ perpendicular to both $\vec{a}$ and $\vec{b}$ is given by $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Illustration 1

If $\vec{a} = \hat{i} + \hat{j} - \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c}$ is a unit vector perpendicular to the vector $\vec{a}$ and coplanar with $\vec{a}$ and $\vec{b}$, then a unit vector $\vec{d}$ perpendicular to both $\vec{a}$ and $\vec{c}$ is

(a) $\frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} + \hat{k})$ (b) $\frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$

(c) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (d) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

Short-cut solution :

$\boxed{\text{Using T-1}}$ $\vec{c} = x\vec{a} + y\vec{b} = (x+y)\hat{i} + (x-y)\hat{j} - (x-y)\hat{k}$

$\vec{c}\cdot\vec{a} = 0 \Rightarrow x + y + x - y + x - y = 0 \quad \Rightarrow 3x + y = 0$ ...(1)

Also $(x+y)^2 + (x-y)^2 + (x-y)^2 = 1$ ... (2)

Solving (1) and (2), $x = \pm\frac{1}{6},\ y = \pm\frac{1}{2}$

$\therefore \quad \vec{c} = \frac{1}{3}(-\hat{i} + 2\hat{j} - 2\hat{k})$ or $\vec{c} = \frac{1}{3}(i - 2j + 2k)$

So, the desired vector $\vec{\ }$ is $\vec{d} = \pm\dfrac{\vec{a}\times\vec{c}}{|\vec{a}\times\vec{c}|} = \pm\dfrac{\hat{j}+\hat{k}}{\sqrt{2}}$ **Ans. (b)**

TIPS AND TRICKS: (T-2)

If $\vec{a}$ and $\vec{b}$ are any two non-collinear vectors and x, y are scalars, then $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$.

If $\vec{a}$ and $\vec{b}$ are any collinear vectors then $\vec{a} = \lambda\vec{b}$ or $\vec{a} + \lambda\vec{b} = 0$ where λ is scalar.

Illustration 2

If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, then the value of α for which the vectors $\vec{u} = (\alpha - 2)\vec{a} + \vec{b}$ and $\vec{v} = (2 + 3\alpha)\vec{a} - 3\vec{b}$ are collinear is : **[JEE M 2013]**

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$

Short-cut solution :

$\boxed{\text{Using T-2}}$ Since, $\vec{u}$ and $\vec{v}$ are collinear, therefore $k\vec{u} + \vec{v} = 0$

$\Rightarrow [k(\alpha - 2) + 2 + 3\alpha]\ \vec{a}\ + (k - 3)\ \vec{b}\ = 0$ ...(i)

Since $\vec{a}$ and $\vec{b}$ are non-collinear, then for some constant m and n,

$m\vec{a} + n\vec{b} = 0 \Rightarrow$ m = 0, n = 0

Hence from equation (i)

$k - 3 = 0 \Rightarrow k = 3$

And $k(\alpha - 2) + 2 + 3\alpha = 0$

$\Rightarrow 3(\alpha - 2) + 2 + 3\alpha = 0 \Rightarrow \alpha = \frac{2}{3}$ **Ans. (b)**

TIPS AND TRICKS: (T-3)

Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

Illustration 3

The sum of the distinct real values of μ, for which the vectors, $\mu\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + \mu\hat{j} + \hat{k}$, $\hat{i} + \hat{j} + \mu\hat{k}$, are co-planar, is : **[JEE M 2019]**

(a) –1 (b) 0

(c) 1 (d) 2

Short-cut solution :

$\boxed{\text{Using T-3}}$ $\because$ Three vectors $(\mu\hat{i} + \hat{j} + \hat{k}), (\hat{i} + \mu\hat{j} + \hat{k})$ and $(\hat{i} + \hat{j} + \mu\hat{k})$ are copalnar.

$$\therefore \quad \begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$$

$\Rightarrow \quad \mu(\mu^2 - 1) + 1 - \mu + 1 - \mu = 0$

$\Rightarrow \quad (1 - \mu)[2 - \mu(\mu + 1)] = 0$

$\Rightarrow \quad (1 - \mu)\,[\mu^2 + \mu - 2] = 0$

$\Rightarrow \quad \mu = 1, -2$

Therefore, sum of all real values = 1 – 2 = –1 **Ans. (a)**

SHORTCUTS: (SC-1)

Vector along the bisector of two given vectors $\vec{a}$ and $\vec{b} = \lambda\left(\hat{a} + \hat{b}\right)$.

Illustration 4

Find a unit vector $\vec{c}$ if $-\hat{i} + \hat{j} - \hat{k}$ bisect the angle between vector $\vec{c}$ and $3\hat{i} + 4\hat{j}$.

Short-cut solution :

$\boxed{\text{Using SC-1}}$ Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$.

Given that $|\vec{c}| = \sqrt{x^2 + y^2 + z^2} = 1$ ...(i)

$$\therefore \ -\hat{i} + \hat{j} - \hat{k} = \lambda\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{1} + \frac{3\hat{i} + 4\hat{j}}{5}\right]$$

$$-\hat{i} + \hat{j} - \hat{k} = \frac{\lambda}{5}\left[(5x + 3)\hat{i} + (5y + 4)\hat{j} + 5z\hat{k}\right]$$

$$\Rightarrow x = \frac{-5 + 3\lambda}{5\lambda}, y = \frac{5 - 4\lambda}{5\lambda}, z = \frac{-1}{\lambda}$$

Putting in (i), we get

$(5 + 3\lambda)^2 + (5 - 4\lambda)^2 + 25 = 25\lambda^2$

$$\Rightarrow \lambda = \frac{15}{2}$$

$$\therefore \vec{c} = \frac{1}{15}\left(-11\hat{i} + 10\hat{j} - 2\hat{k}\right).$$

SHORTCUTS: (SC-2)

Three vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ will be linearly dependent vectors if

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Illustration 5

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}|=\sqrt{3}$, then

(a) $\alpha=1, \beta=-1$ (b) $\alpha=1, \beta=\pm 1$

(c) $\alpha=-1, \beta=\pm 1$ (d) $\alpha=\pm 1, \beta=1$

Short-cut solution :

Using SC-2 Since $\vec{a}, \vec{b}$ and $\vec{c}$ are linearly dependent vectors

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0 \Rightarrow \beta = 1 \text{ and } |\vec{c}| = \sqrt{3} \Rightarrow 1+\alpha^2+\beta^2 = 3 \Rightarrow \alpha = \pm 1 \quad \textbf{Ans. (d)}$$

SHORTCUTS: (SC-3)

The components of $\vec{a}$ along and perpendicular to $\vec{b}$ are

$\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ and $\vec{a}-\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ respectively.

Illustration 6

Express the vector $\vec{a}=5\hat{i}-2\hat{j}+5\hat{k}$ as sum of two vectors such that one is parallel to the vector $\vec{b}=3\hat{i}+\hat{k}$ and the other is perpendicular to $\vec{b}$.

Short-cut solution :

Using SC-3 Required two vectors are $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ and $\vec{a}-\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

$$\therefore \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = 2\left(3\hat{i}+\hat{k}\right) = 6\hat{i}+2\hat{k}$$

$$\text{and } \vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} = \left(5\hat{i}-2\hat{j}+5\hat{k}\right)-\left(6\hat{i}+2\hat{k}\right)$$

$= -\hat{i} - 2\hat{j} + 3\hat{k}.$

such that $\left(6\hat{i} + 2\hat{k}\right) + \left(-\hat{i} - 2\hat{j} + 3\hat{k}\right) = 5\hat{i} - 2\hat{j} + 5\hat{k} = \vec{a}.$

SHORTCUTS: (SC-4)

Method of substitution

Substitute vectors $\hat{i}, \hat{j}$ or $\hat{k}$ in place of any unit, mutually perpendicular or non-coplanar vector.

Illustration 7

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of equal magnitude, then the angle between $\vec{a}$ and $\vec{a} + \vec{b} + \vec{c}$ is

(a) $\cos^{-1}\left(1/\sqrt{3}\right)$ (b) $\sin^{-1}\left(1/\sqrt{3}\right)$

(c) $\cos^{-1}(1/3)$ (d) $\sin^{-1}(1/3)$

Short-cut solution :

Using SC-4 Let $\vec{a} = \hat{i}, \vec{b} = \hat{j}$ and $\vec{c} = \hat{k}$

$$\therefore \cos\theta = \frac{\hat{i} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right)}{\left|\hat{i}\right|\left|\hat{i} + \hat{j} + \hat{k}\right|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(1/\sqrt{3}\right)$$

Ans. (a)

TECHNIQUE

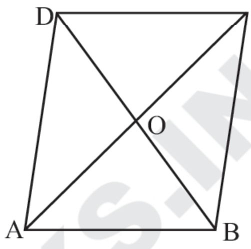
To prove the geometrical properties or solve the some geometrical problem, take one vertex at origin.

Illustration 8

Show that the diagonals of a rhombus bisect each other at right angles.

Short-cut solution :

Using Tech. Let ABCD be a rhombus whose diagonals [AC] and [BD] intersect at O. Take A as origin and let $\vec{b}$ and $\vec{d}$ be the position vectors of vartices B and D respectively. Then $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AD} = \vec{d}$.



Since ABCD is a rhombus, [BC] is equal and parallel to [AD], therefore, $\overrightarrow{BC} = \overrightarrow{AD} = \vec{d}$.

From ΔABC, we get

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{b} + \vec{d}.$$

∴ the P.V. of the mid-point of [AC] $= \frac{\vec{0} + (\vec{b} + \vec{d})}{2} = \frac{\vec{b} + \vec{d}}{2}$.

Also P.V. of the mid-point of [BD] $= \frac{\vec{b} + \vec{d}}{2}$.

Hence the mid-points of the diagonals [AC] and [BD] coincide, therefore the diagonals [AC] and [BD] bisect each other.

Again, as ABCD is a rhombus, $|AB| = |AD|$

$\Rightarrow \quad |AB|^2 = |AD|^2 \qquad \Rightarrow \quad (\overrightarrow{AB})^2 = (\overrightarrow{AD})^2$

$\Rightarrow \quad (\overrightarrow{AB})^2 - (\overrightarrow{AD})^2 = 0 \qquad \Rightarrow \quad (\overrightarrow{AB} + \overrightarrow{AD}) \cdot (\overrightarrow{AB} - \overrightarrow{AD}) = 0$

$\Rightarrow \quad (\vec{b} + \vec{d}) \cdot (\vec{b} - \vec{d}) = 0 \qquad \Rightarrow \quad \overrightarrow{AC} \cdot \overrightarrow{DB} = 0$

$\Rightarrow$ $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are perpendicular. (∵ $\overrightarrow{AC}$ and $\overrightarrow{DB}$ are proper vecotrs)

Hence the diagonals of a rhombus bisect each other at right angles.

Illustration 9

Using vector algebra, prove that angle in a semi-circle is a right angle.

Short-cut solution :

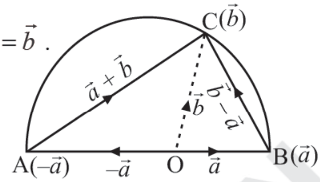
Using Tech. Let O be the centre of a circle and [AB] be the diameter. Let C be a point on the circle, then we want to prove that $\angle ACB = 90°$.

Take O as origin and let $\overrightarrow{OB} = \vec{a}$ and $\overrightarrow{OC} = \vec{b}$,

then $\quad \overrightarrow{OA} = -\overrightarrow{OB} = -\vec{a} \qquad$ (∵ $|OA| = |OB|$ = radius)

$\therefore$ P.V. of A = $-\vec{a}$,

P.V. of B = $\overrightarrow{OB} = \vec{a}$ and P.V. of C = $\overrightarrow{OC} = \vec{b}$.



Now, $\overrightarrow{AC}$ = P.V. of C – P.V. of A

$= \vec{b} - (-\vec{a}) = \vec{a} + \vec{b}$

and $\overrightarrow{BC}$ = P.V. of C – P.V. of B = $\vec{b} - \vec{a}$.

$$\therefore \quad \overrightarrow{AC} \cdot \overrightarrow{BC} = (\vec{a}+\vec{b}) \cdot (\vec{b}-\vec{a}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a} = \vec{b}\cdot\vec{b} - \vec{a}\cdot\vec{a}$$

$$(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$$

$= |\vec{b}|^2 - |\vec{a}|^2 = |OC|^2 - |OB|^2$

$= 0$ $\quad (\because |OC| = |OB| = \text{radius})$

Thus $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0 \Rightarrow \overrightarrow{AC} \perp \overrightarrow{BC} \Rightarrow \angle ACB = 90°$.

Hence, the angle in a semi-circle is a right angle.

Concept Booster Exercise

1. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ be two vectors. If a vector perpendicular to both the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ has the magnitude 12 then one such vector is : **[JEE M 2019]**

   (a) $4\left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)$ (b) $4\left(2\hat{i} - 2\hat{j} - \hat{k}\right)$

   (c) $4\left(2\hat{i} + 2\hat{j} - \hat{k}\right)$ (d) $4\left(-2\hat{i} - 2\hat{j} + \hat{k}\right)$

2. Let $\vec{\alpha} = (\lambda - 2)\,\vec{a} + \vec{b}$ and $\vec{\beta} = (4\lambda - 2)\,\vec{a} + 3\vec{b}$ be two given vectors where vectors $\vec{a}$ and $\vec{b}$ are non-collinear. The value of λ for which vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear, is: **[JEE M 2019]**

   (a) -4 (b) -3 (c) 4 (d) 3

3. Let $\vec{a} = \hat{i} + 2\hat{j} + 4\hat{k}, \vec{b} = \hat{i} + \lambda\hat{j} + 4\hat{k}$ and $\vec{c} = 2\hat{i} + 4\hat{j} + \left(\lambda^2 - 1\right)\hat{k}$ be coplanar vectors. Then the non-zero vector $\vec{a} \times \vec{c}$ is : **[JEE M 2019]**

   (a) $-10\hat{i} - 5\hat{j}$ (b) $-14\hat{i} - 5\hat{j}$ (c) $-14\hat{i} + 5\hat{j}$ (d) $-10\hat{i} + 5\hat{j}$

4. The vector $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$ lies in the plane of the vectors $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{j} + \hat{k}$ and bisects the angle between $\vec{b}$ and $\vec{c}$. Then which one of the following gives possible values of α and β? **[AIEEE 2008]**

   (a) $\alpha = 2,\ \beta = 2$ (b) $\alpha = 1,\ \beta = 2$ (c) $\alpha = 2,\ \beta = 1$ (d) $\alpha = 1, \beta = 1$

5. If the vectors $\vec{a} = 2\hat{i} + \hat{j} + 4\hat{k}, \vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = 2\hat{i} - 3\hat{j} - \lambda\hat{k}$ are linearly dependent vectors, then find the value of λ.

   (a) 1 (b) -1 (c) 2 (d) -2

6. What is the component of $\left(3\hat{i} + 4\hat{j}\right)$ along $\left(\hat{i} + \hat{j}\right)$?

   (a) $\frac{1}{2}\left(\hat{i} + \hat{j}\right)$ (b) $\frac{3}{2}\left(\hat{i} + \hat{j}\right)$ (c) $\frac{5}{2}\left(\hat{i} + \hat{j}\right)$ (d) $\frac{7}{2}\left(\hat{i} + \hat{j}\right)$

7. For any vector $\vec{a}, \left|\vec{a} \times \hat{i}\right|^2 + \left|\vec{a} \times \hat{j}\right|^2 + \left|\vec{a} \times \hat{k}\right|^2$ is equal to **[AIEEE 2005]**

   (a) $\left|\vec{a}\right|^2$ (b) $2\left|\vec{a}\right|^2$ (c) $3\left|\vec{a}\right|^2$ (d) $4\left|\vec{a}\right|^2$

8. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

[AIEEE 2011]

(a) $\hat{i} - 3\hat{j} + 3\hat{k}$ (b) $-3\hat{i} - 3\hat{j} - \hat{k}$

(c) $3\hat{i} - \hat{j} + 3\hat{k}$ (d) $\hat{i} + 3\hat{j} - 3\hat{k}$

9. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors and

$$\vec{b}_1 = \vec{b} - \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}, \vec{b}_2 = \vec{b} + \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\vec{a},$$

$$\vec{c}_1 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2}\vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2}\vec{b}_1, \vec{c}_2 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{b}_1 \cdot \vec{c}}{|\vec{b}_1|^2}\vec{b}_1,$$

$$\vec{c}_3 = \vec{c} - \frac{\vec{c} \cdot \vec{a}}{|\vec{c}|^2}\vec{a} + \frac{\vec{b} \cdot \vec{c}}{|\vec{c}|^2}\vec{b}_1, \vec{c}_4 = \vec{c} - \frac{\vec{c} . \vec{a}}{|\vec{c}|^2}\vec{a} = \frac{\vec{b} . \vec{c}}{|\vec{b}|^2}\vec{b}_1$$

then the set of orthogonal vectors is **[AIEEE 2005S]**

(a) $(\vec{a}, \vec{b}_1, \vec{c}_3)$ (b) $(\vec{a}, \vec{b}_1, \vec{c}_2)$

(c) $(\vec{a}, \vec{b}_1, \vec{c}_1)$ (d) $(\vec{a}, \vec{b}_2, \vec{c}_2)$

10. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i} + \hat{j} + \hat{k}$, $\hat{i} - \lambda^2\hat{j} + \hat{k}$, and $\hat{i} + \hat{j} - \lambda^2\hat{k}$ are coplanar, is **[AIEEE 2007]**

(a) zero (b) one (c) two (d) three

11. The lengths of the diagonals of a parallelogram whose adjacent sides are $\vec{a} = 2\vec{m} + \vec{n}$ and $\vec{b} = \vec{m} - 2\vec{n}$ where $\vec{m}$ and $\vec{n}$ are unit vectors inclined at an angle of 60° are $\sqrt{m}$ and $\sqrt{n}$ then value of $m + n$ is

(a) 13 (b) 7 (c) 20 (d) 6

Numerical Value Problems

12. Let $\vec{\alpha} = (\lambda - 2)\vec{a} + \vec{b}$ and $\vec{\beta} = (4\lambda - 2)\vec{a} + 3\vec{b}$ be two given vectors where vectors $\vec{a}$ and $\vec{b}$ are non-collinear. The value of $|\lambda|$ for which vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear, is ________.

13. If the vectors, $\vec{p} = (a+1)\hat{i} + a\hat{j} + a\hat{k}$, $\vec{q} = a\hat{i} + (a+1)\hat{j} + a\hat{k}$ and $\vec{r} = a\hat{i} + a\hat{j} + (a+1)\hat{k}$ $(a \in \mathrm{R})$ are coplanar and $3(\vec{p}.\vec{q})^2 - \lambda|\vec{r} \times \vec{q}|^2 = 0$, then the value of λ is _____. **[JEE M 2020]**

14. Let $\vec{u} = \hat{i} + \hat{j}, \vec{v} = \hat{i} - \hat{j}$ and $\vec{w} = \hat{i} + 2\hat{j} + 3\hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{u} \cdot \hat{n} = 0$ and $\vec{v} \cdot \hat{n} = 0$, then $|\vec{w} \cdot \hat{n}|$ is equal to ______. **[AIEEE 2003]**

15. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be such that $|\vec{a}| = 2, |\vec{b}| = 4$ and $|\vec{c}| = 4$. If the projection of $\vec{b}$ on $\vec{a}$ is equal to the projection of $\vec{c}$ on $\vec{a}$ and $\vec{b}$ is perpendicular to $\vec{c}$, then the value of $|\vec{a} + \vec{b} - \vec{c}|$ is _______. **[JEE M 2020]**

16. Consider the set of eight vectors $V = \left\{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\right\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is ______. **[JEE M 2013]**

Solutions

1. **(b)** Using T-1

Let vector be $\lambda[(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})]$

Given, $a = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore$ $a + b = 4\hat{i} + 4\hat{j}$ and $\vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$

$\therefore$ vector $= \lambda[(4\hat{i} + 4\hat{j})\times(2\hat{i} + 4\hat{k})]$

$= \lambda[16\hat{i} - 16\hat{j} - 8\hat{k}] = 8\lambda[2\hat{i} - 2\hat{j} - \hat{k}]$

Given that magnitude of the vector is 12.

$\therefore$ $12 = 8|\lambda|\sqrt{4+4+1} \Rightarrow |\lambda| = \dfrac{1}{2}$

$\therefore$ required vector is $\pm 4\ (2\hat{i} - 2\hat{j} - \hat{k})$

2. **(a)** Using T-2

Let $\vec{\alpha}$ and $\vec{\beta}$ are collinear for same k

i.e., $\vec{\alpha} = k\vec{\beta}$

$(\lambda - 2)\ \vec{a} + \vec{b} = k((4\lambda - 2)\ \vec{a} + 3\vec{b})$

$(\lambda - 2)\ \vec{a} + \vec{b} = k(4\lambda - 2)\ \vec{a} + 3k\vec{b}$

$(\lambda - 2 - k(4\lambda - 2))\ \vec{a} + \vec{b}\ (1 - 3k) = 0$

But $\vec{a}$ and $\vec{b}$ are non-collinear, then

$\lambda - 2 - k(4\lambda - 2) = 0,\ 1 - 3k = 0$

$\Rightarrow k = \dfrac{1}{3}$ and $\lambda - 2 - \dfrac{1}{3}(4\lambda - 2) = 0$

$3\lambda - 6 - 4\lambda + 2 = 0$

$\lambda = -4$

3. **(d)** Using T-3

$\because$ $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar

$$\therefore \begin{vmatrix} 1 & 2 & 4 \\ 1 & \lambda & 4 \\ 2 & 4 & (\lambda^2 - 1) \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda^3 - \lambda - 16 + 2(8 - \lambda^2 + 1) + 4(4 - 2\lambda) = 0$

$\Rightarrow \quad \lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$

i.e., $(\lambda - 2)(\lambda - 3)(\lambda + 3) = 0$

For $\lambda = 2, \ \vec{c} = 2\hat{i} + 4\hat{j} + 3\hat{k}$

$$\therefore \quad \vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 4 & 3 \end{vmatrix} = -10\hat{i} + 5\hat{j}$$

For $\lambda = 3$ or -3, $\overline{c} = 2\overline{a} \Rightarrow \overline{a} \times \overline{c} = 0$ (Rejected)

4. **(d)** $\boxed{\text{Using SC-1}}$

$\because \ \vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\therefore \ \vec{a} = \lambda(\hat{b} + \hat{c})$

$$\Rightarrow \alpha\hat{i} + 2\hat{j} + \beta\hat{k} = \lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} + \frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)$$

$$\Rightarrow \alpha\,\hat{i} + 2\hat{j} + \beta\,\hat{k} = \frac{\lambda}{\sqrt{2}}\hat{i} + \sqrt{2}\lambda\,\hat{j} + \frac{\lambda}{\sqrt{2}}\hat{k}$$

$\Rightarrow \alpha = 1, \beta = 1$

5. **(a)** $\boxed{\text{Using SC-2}}$ Since $\vec{a}, \vec{b}$ and $\vec{c}$ are linearly dependent vectors

$$\therefore \begin{vmatrix} 2 & 1 & 4 \\ 4 & -2 & 3 \\ 2 & -3 & -\lambda \end{vmatrix} = 0 \Rightarrow 8\lambda - 8 = 0 \Rightarrow \lambda = 1$$

6. **(d)** $\boxed{\text{Using SC-3}}$ $\left(3\hat{i} + 4\hat{j}\right)\cdot\left(\hat{i} + \hat{j}\right) = 7, \left|\hat{i} + \hat{j}\right| = \sqrt{2}$

$$\text{Components} = \frac{7}{\left(\sqrt{2}\right)^2}\left(\hat{i} + \hat{j}\right) = \frac{7}{2}\left(\hat{i} + \hat{j}\right)$$

7. **(b)** $\boxed{\text{Using SC-4}}$ Let $\vec{a} = \hat{i}$

$$\therefore \left|\vec{a} \times \hat{i}\right|^2 + \left|\vec{a} \times \hat{j}\right|^2 + \left|\vec{a} \times \hat{k}\right|^2$$

$$= \left|\hat{i} \times \hat{i}\right|^2 + \left|\hat{i} \times \hat{j}\right|^2 + \left|\hat{i} \times \hat{k}\right|^2 = \left|\hat{k}\right|^2 + \left|-\hat{j}\right|^2$$

$$= 1 + 1 = 2 = 2\left|\hat{i}\right|^2 = 2\left|\vec{a}\right|^2$$

8. (c) **Trick:** Projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

We know that vector in the plane of $\vec{a}$ and $\vec{b}$ is

$= \vec{v} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{v} = (1+\lambda)\hat{i} + (1-\lambda)\hat{j} + (1+\lambda)\hat{k}$

$\because$ Projection of $\vec{v}$ on $\vec{c}$ is $\dfrac{1}{\sqrt{3}}$

$$\therefore \frac{\vec{v}.\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}} \Rightarrow \frac{(1+\lambda)-(1-\lambda)-(1+\lambda)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\Rightarrow 1 - \lambda = -1$

$\Rightarrow \lambda = 2 \Rightarrow \lambda = 2$

$\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$

9. (b) [**Trick:** If $\vec{a}, \vec{b}$ and $\vec{c}$ are orthogonal vectors then $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$]

We observe that

$$\vec{a}.\vec{b}_1 = \vec{a}.\vec{b} - \left(\frac{\vec{b}.\vec{a}}{|\vec{a}|^2}\right)\left|\vec{a}\right|^2 = \vec{a}.\vec{b} - \vec{a}.\vec{b} = 0 \quad \text{...(i)}$$

$$\vec{a}.\vec{c}_2 = \vec{a}.\left(\vec{c} - \frac{\vec{c}.\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{c}.\vec{b_1}}{|\vec{b_1}|^2}\vec{b}_1\right)$$

$$= \vec{a}.\vec{c} - \frac{\vec{a}.\vec{c}}{|\vec{a}|^2}|\vec{a}|^2 - \frac{\vec{c}.\vec{b_1}}{|\vec{b_1}|^2}(\vec{a}.\vec{b}_1)$$

$$= \vec{a}.\vec{c} - \vec{a}.\vec{c} - 0 = 0 \quad \text{[from (i)]}$$

And $\vec{b}_1.\vec{c}_2 = \vec{b}_1.\left(\vec{c} - \dfrac{\vec{c}.\vec{a}}{|\vec{a}|^2}\vec{a} - \dfrac{\vec{c}.\vec{b_1}}{|\vec{b_1}|^2}\vec{b}_1\right)$

$$= \vec{b}_1.\vec{c} - \frac{(\vec{c}.\vec{a})(\vec{b_1}.\vec{a})}{|\vec{a}|^2} - \frac{\vec{c}.\vec{b_1}}{|\vec{b_1}|^2}\left|\vec{b}_1\right|^2$$

$$= \vec{b}_1.\vec{c} - 0 - b_1.\vec{c} = 0 \quad \text{(from (i))}$$

Hence $\vec{a}.\vec{b}_1 = \vec{a}.\vec{c_2} = \vec{b}_1.\vec{c}_2 = 0$

$\Rightarrow (\vec{a}, \vec{b}_1, \vec{c}_2)$ is a set of orthogonal vectors.

10. **(c)** $\boxed{\text{Using T-3}}$ Since given three vectors are coplanar.

$$\therefore \begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

Applying, $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} 2-\lambda^2 & 2-\lambda^2 & 2-\lambda^2 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

Applying $R_2 = R_2 - R_1$ and $R_3 = R_3 - R_1$

$$\Rightarrow (2-\lambda^2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -(1+\lambda^2) & 0 \\ 0 & 0 & -(1+\lambda^2) \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda^2)(1+\lambda^2)^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$ $\qquad [1 + \lambda^2 \neq 0]$

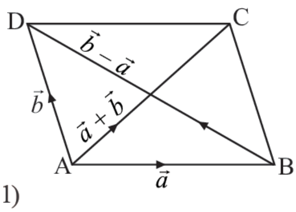
$\therefore$ Two real solutions.

11. **(c)** $\boxed{\text{Using Tech.}}$ Let the parallelogram be ABCD with $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AD} = \vec{b}$, then $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

$\Rightarrow \quad |AC| = |\overrightarrow{AC}| = |\vec{a} + \vec{b}|$

and $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2 = (2\vec{m} + \vec{n} + \vec{m} - 2\vec{n})^2$

$= (3\vec{m} - \vec{n}) \cdot (3\vec{m} - \vec{n})$



$= 9\vec{m} \cdot \vec{m} - 6\vec{m} \cdot \vec{n} + \vec{n} \cdot \vec{n}$

$= 9|\vec{m}|^2 - 6|\vec{m}||\vec{n}|\cos 60° + |\vec{n}|^2$

$= 9 - 6 \times \frac{1}{2} + 1 \qquad (\because |\vec{m}| = |\vec{n}| = 1)$

$= 7.$

Hence, $|AC| = |\vec{a}+\vec{b}| = \sqrt{7}$.

Further $\overrightarrow{BD} = \overrightarrow{BA}+\overrightarrow{AD} = -\vec{a}+\vec{b} = -(2\vec{m}+\vec{n})+(\vec{m}-2\vec{n}) = -\vec{m}-3\vec{n}$

$\Rightarrow \quad |BD| = |\overrightarrow{BD}| = |-\vec{m}-3\vec{n}| = |\vec{m}+3\vec{n}|$.

Also, $\quad |\vec{m}+3\vec{n}|^2 = (\vec{m}+3\vec{n})\cdot(\vec{m}+3\vec{n}) = \vec{m}\cdot\vec{m}+6\vec{m}\cdot\vec{n}+9\vec{n}\cdot\vec{n}$

$= |\vec{m}|^2 + 6|\vec{m}||\vec{n}|\cos 60° + 9|\vec{n}|^2$

$= 1+6\times\frac{1}{2}+9 = 13 \qquad (\because |\vec{m}| = |\vec{n}| = 1)$

$\Rightarrow \quad |\vec{m}+3\vec{n}| = \sqrt{13} \quad \Rightarrow \quad |BD| = \sqrt{13}$.

Hence, the length of the diagonals are $\sqrt{13}$ and $\sqrt{7}$.

12. **(4)** $\boxed{\text{Using T-2}}$ Let $\vec{a}$ and $\vec{b}$ are collinear for same k

i.e., $\qquad \vec{a} = k\vec{b}$

$(\lambda-2)\ \vec{a} + \vec{b} = k((4\lambda-2)\ \vec{a} + 3\,\vec{b}$

$(\lambda-2)\ \vec{a} + \vec{b} = k(4\lambda-2)\ \vec{a} + 3k\vec{b}$

$(\lambda-2-k(4\lambda-2))\ \vec{a} + \vec{b}\ (1-3k) = 0$

But $\vec{a}$ and $\vec{b}$ are non-collinear, then

$\lambda-2-k(4\lambda-2) = 0,\ 1-3k = 0$

$\Rightarrow \quad k = \frac{1}{3}$ and $\lambda-2-\frac{1}{3}\ (4\lambda-2) = 0$

$3\lambda-6-4\lambda+2 = 0$

$\lambda = -4 \Rightarrow |\lambda| = 4$

13. **(1)** $\boxed{\text{Using T-3}}$ $\begin{vmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{vmatrix} = 0$

$\Rightarrow \quad 3a+1 = 0 \quad \Rightarrow \quad a = -\frac{1}{3}$

The given vectors

$\vec{p} = \frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}-\frac{1}{3}\hat{k} = \frac{1}{3}(2\hat{i}-\hat{j}-\hat{k})$

$$\vec{q}=\frac{1}{3}(-\hat{i}+2\hat{j}-\hat{k})$$

$$\vec{r}=\frac{1}{3}(-\hat{i}-\hat{j}+2\hat{k})$$

Now, $\vec{p}.\vec{q}=\frac{1}{9}(-2-2+1)=-\frac{1}{3}$

$$\vec{r}\times\vec{q}=\frac{1}{9}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & -1\\ -1 & -1 & 2\end{vmatrix}$$

$$=\frac{1}{9}(\hat{i}(4-1)-\hat{j}(-2-1)+\hat{k}(1+2))$$

$$=\frac{1}{9}(3\hat{i}+3\hat{j}+3\hat{k})=\frac{\hat{i}+\hat{j}+\hat{k}}{3}$$

$$|\vec{r}\times\vec{q}|=\frac{1}{3}\sqrt{3} \quad \Rightarrow \quad |\vec{r}\times\vec{q}|^2=\frac{1}{3}$$

$$3(\vec{p}.\vec{q})^2-\lambda|\vec{r}\times\vec{q}|^2=0$$

$$\Rightarrow \quad 3.\frac{1}{9}-\lambda.\frac{1}{3}=0 \quad \Rightarrow \quad \lambda=1$$

14. **(3)** $\boxed{\text{Using SC-4}}$ Since $\vec{u}=\hat{i}+\hat{j}$ and $\vec{v}=\hat{i}-\hat{j}$

Given that $\vec{u}\cdot\hat{n}=0$ and $\vec{v}\cdot\hat{n}=0$

So, let $\hat{n}=\hat{k}$

$\therefore\ |\vec{w}\cdot\hat{n}|=3.$

15. **(6)** $\boxed{\text{Using SC-3}}$

$\because$ Projection of $\vec{b}$ on $\vec{a}$ = Projection of $\vec{c}$ on $\vec{a}$

$\therefore \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$

Given, $\vec{b}\cdot\vec{c}=0$

$\because |\vec{a}+\vec{b}-\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\vec{a}\cdot\vec{b}-2\vec{b}\cdot\vec{c}-2\vec{a}\cdot\vec{c}$

$= 4 + 16 + 16 = 36.$

$\Rightarrow |\vec{a}+\vec{b}-\vec{c}|^2=6$

16. **(5)** [**Trick:** Use permutation and combination]

Given 8 vectors are

(1, 1, 1), (–1, –1, –1); (–1, 1, 1), (1, –1, –1); (1, –1, 1),

(–1, 1, –1); (1, 1, –1), (–1, –1, 1)

These are 4 diagonals of a cube and their opposites.

For 3 non-coplanar vectors first we select 3 groups of diagonals and its opposite in ${}^4C_3$ ways. Then one vector from each group can be selected in $2 \times 2 \times 2$ ways.

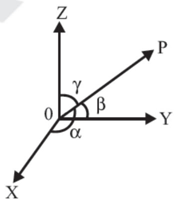
$\therefore$ Total ways $= {}^4C_3 \times 2 \times 2 \times 2 = 32 = 25$

$\therefore$ $p = 5$

26 Three Dimensional Geometry

Review of Key Notes and Formulae

Straight Line



1. Direction Cosines of a Line (DC's)

The direction cosines are generally denoted by l, m, n.

Hence, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Note that $l^2 + m^2 + n^2 = 1$

2. Direction Ratios of a Line (DR's)

Any three numbers a, b and c proportional to the direction cosines l, m and n, respectively are called direction ratios of the line.

- The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$
- $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$
- $l = \pm\dfrac{a}{\sqrt{a^2 + b^2 + c}}$, $m = \pm\dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$ and $n = \pm\dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$

3. Equation of a Line

(i) Equation of a line through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda\vec{b}$

In cartesian form, $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$

Here, a, b, c are also the direction ratios of the line.

(ii) Equation of a line passing through two given points with position vectors $\vec{a}$ and $\vec{b}$:

In vector form, $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

In cartesian form, $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ where, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

& $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

4. Angle between Two Lines

In vector form,

The angle between two lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$$

In cartesian form,

The angle between two lines :

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is :

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

- If two lines are perpendicular, then $\vec{b}_1 \cdot \vec{b}_2 = 0$ or $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
- If two lines are parallel, then $\vec{b}_1 = \lambda\vec{b}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

5. Shortest Distance Between Two Lines

(i) Distance between parallel lines

The shortest distance between parallel lines

$L_1 : \vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $L_2 : \vec{r} = \vec{a}_2 + \mu\vec{b}$ is

$$d = \left|\frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|}\right|$$

(ii) Distance between two skew lines

In vector form,

The distance between two skew lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ & $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given as:

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

In cartesian form,

The distance between two skew lines:

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$

and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is :

$$d = \left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

Plane

6. Equation of a Plane in Normal Plane Form

Vector form

$\vec{r}\cdot\hat{n}=d$

Here $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$\hat{n}$ is the unit vector along the normal from origin to the plane.

d is perpendicular distance of the plane from the origin.

Cartesian form

$lx + my + nz = d$

where l, m, n are the direction cosines of $\hat{n}$ (unit vector along the normal from origin to the plane).

7. Equation of a Plane Perpendicular to a Given Vector and Passing Through a Given Point

Vector form

Let a plane pass through a point with position vector $\vec{a}$ and perpendicular to the vector $\vec{N}$.

Then its equation is given as: $(\vec{r}-\vec{a})\cdot\vec{N}=0$

Cartesian form

Let a plane pass through a point $(x_1, y_1\ z_1)$ & the direction ratio of the vector perpendicular to the plane be A, B, C. Then its equation is given as:

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

8. Equation of a Plane Passing Through Three Non-Collinear Points

Vector form

$[\vec{r}\,\vec{b}\,\vec{c}]+[\vec{r}\,\vec{a}\,\vec{b}]+[\vec{r}\,\vec{c}\,\vec{a}]=[\vec{a}\,\vec{b}\,\vec{c}]$

or $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$

where, $\vec{a},\vec{b},\vec{c}$ are the position vector of three given non-collinear points through which the plane passes.

Cartesian form

The equation of plane passing through three non-collinear points Y with coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) & (x_3, y_3, z_3) is given as:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

9. Intercept Form of the Equation of a Plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

where a, b, c are the intercepts made by the plane on x, y & z axes respectively.

10. Plane Passing Through the Intersection of Two Given Planes

Vector form

Equation of plane passing through the line of intersection of two planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is given as:

$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$

Cartesian form

Let $\vec{n}_1 = A_1\hat{i}+B_1\hat{j}+C_1\hat{k}$

$\vec{n}_2 = A_2\hat{i}+B_2\hat{j}+C_2\hat{k}$

and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$,

therefore its cartesian equation is:

$(A_1x + B_1y + C_1z - d_1) + \lambda(A_2x + B_2y + C_2z - d_2) = 0$

11. Coplanarity of Two Lines

Vector form

Two lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$

are coplanar, if $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$

and equation of coplanar plane is $\left(\vec{r}-\overrightarrow{a_1}\right)\cdot\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right)=0$

Cartesian form

Two lines $\dfrac{x-x_1}{a_1}=\dfrac{y-y_1}{b_1}=\dfrac{z-z_1}{c_1}$

and $\dfrac{x-x_2}{a_2}=\dfrac{y-y_2}{b_2}=\dfrac{z-z_2}{c_2}$ are coplanar,

if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$ and equation of coplanar plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

12. Angle Between Two Planes

Vector form : The angle between two planes

$\vec{r}\cdot\vec{n}=d_1$ & $\vec{r}\cdot\vec{n}=d_2$ is given as:

$$\cos\theta=\left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right|$$

Cartesian form : The angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- If two planes are perpendicular, then
  $\vec{n}_1\cdot\vec{n}_2=0$ or $a_1a_2+b_1b_2+c_1c_2=0$
- If two planes are parallel, then
  $\vec{n}_1=\lambda\vec{n}_2$ or $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

13. Distance of a Point from a Plane

Vector form

Distance of a point with position vector $\vec{a}$ from a plane

$\vec{r}\cdot\vec{n} = d$ is given as:

$$\frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$$

Cartesian form

Distance of a point (x_1, y_1, z_1) from a plane : $ax + by + cz = d$ is given as :

$$\left|\frac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

14. Angle Between a Line and a Plane

Vector form

Angle between a line

$\vec{r} = \vec{a} + \lambda\vec{b}$ and a plane $\vec{r}\cdot\vec{n} = d$ is

$$\cos\theta = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|$$

Cartesian form

Angle between a line $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$

and a plane $a_2x + b_2y + c_2z = d$ is given as:

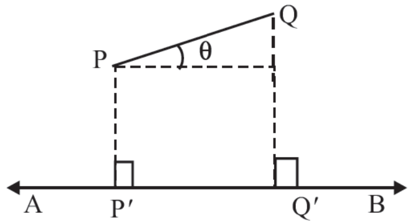
$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

- If line is perpendicular to the plane,
  then $\vec{n} = \lambda\vec{b}$ or $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$
- If line is parallel to the plane, then
  $\vec{n}\cdot\vec{b} = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

15. Projection of a Line Segment Joining two Points on a Line

Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$; and AB be a given line with dc's as ℓ, m, n.

If P' and Q' are the foot of perpendicular from P and Q to the line AB, then P'Q' is the projection of PQ on the line AB.



If the line segment PQ makes angle θ with the line AB, then projection of PQ on line AB is $P'Q' = PQ \cos\theta$. On replacing the value of $\cos\theta$ in this, we shall get the following value of P'Q'.

$P'Q' = \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$, where l, m, n are direction cosines of line AB.

16. Bisectors of the Angles Between two Lines

The equation of the bisectors of the angle between the straight lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + r\vec{c}$ are given by $\vec{r} = \vec{a} + t\left\{\frac{\vec{b}}{|\vec{b}|} \pm \frac{\vec{c}}{|\vec{c}|}\right\}$, where $t \in R$.

17. Distance Between Two Parallel Planes : Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$$

18. Ratio of Division of a Line Segment by a Plane : The ratio in which the line segment PQ, joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, is divided by plane $ax + by + cz + d = 0$ is $-\left(\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$.

TIPS! & TRICKS

TIPS AND TRICKS: (T-1)

Direction vector of line

(i) Direction vector of line which is perpendicular to two given lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\vec{b}_1 \times \vec{b}_2$

(ii) Direction vector of line which is parallel to two given planes

$\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{n}_1 \times \vec{n}_2$.

Illustration 1

Find the vector equation of the line passing through the point (1, 2, –4) and perpendicular to the two lines:

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

Short-cut solution :

Using T-1(i) $\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 24\hat{i} + 36\hat{j} + 72\hat{k}$

$\therefore$ Equation of line is

$$\vec{r} = \left(i + 2\hat{j} - 4\hat{k}\right) + \mu\left(24\hat{i} + 36\hat{j} + 72\hat{k}\right)$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Illustration 2

Find the equations of the line passing through the point (3, 0, 1) parallel to the plane $x + 2y = 0$ and $3y - z = 0$.

Short-cut solution :

Using T-1(ii)

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix} = -2\hat{i} + \hat{j} + 3\hat{k}$$

$\therefore$ Equation of line is $\dfrac{x-3}{-2} = \dfrac{y}{1} = \dfrac{z-1}{3}$.

TIPS AND TRICKS: (T-2)

Normal of Plane

(i) Normal of plane perpendicular to two given planes

$\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is $\vec{n}_1 \times \vec{n}_2$.

(ii) Normal of plane parallel to two given lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $\vec{b}_1 \times \vec{b}_2$.

Illustration 3

Find the equation of the plane passing through the point (–1, 3, 2) and perpendicular to each of the planes x + 2y + 3z = 5 and 3x + 3y + z = 0.

Short-cut solution :

Using T-2(i) Here, $\vec{n}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{n}_2 = 3\hat{i} + 3\hat{j} + \hat{k}$

$$\therefore \vec{n} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{vmatrix} = \hat{i}(2-9) - \hat{j}(1-9) + \hat{k}(3-6) = -7\hat{i} + 8\hat{j} - 3\hat{k}$$

$\therefore$ Equation of plane is $-7(x + 1) + 8(y - 3) - 3(z - 2) = 0$

$\Rightarrow 7x - 8y + 3z - 25 = 0$

Illustration 4

The equation of the plane through the point (–1, 2, 0) and parallel to the lines $\frac{x}{3}=\frac{y+1}{0}=\frac{z-2}{-1}$ and $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$ is

(a) $2x+3y+z-4=0$ (b) $x-2y+3z+5=0$

(c) $x+2y+3z-3=0$ (d) $x+y+3z-1=0$

Short-cut solution :

Using T-2(ii) Here, $\vec{b}_1=3\hat{i}-\hat{k}$ and $\vec{b}_2=\hat{i}+\hat{j}-\hat{k}$

$$\left(\because \frac{x-1}{1}=\frac{y+1/2}{1}=\frac{z+1}{-1}\right)$$

$$\therefore\ \vec{n}=\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&0&-1\\1&1&-1\end{vmatrix}=\hat{i}(0+1)-\hat{j}(-3+1)+\hat{k}(3-0)$$

$=\hat{i}+2\hat{j}+3\hat{k}.$

$\therefore$ Equation of plane is $(x+1)+2(y-2)+3(z-0)=0$

$\Rightarrow x+2y+3z-3=0$ **Ans. (c)**

TIPS AND TRICKS: (T-3)

Bisectors of angles between two planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Illustration 5

Equation of the plane bisecting the acute angle between the planes $x-2y+2z+3=0$ and $3x-6y-2z+2=0$

(a) $2(8x-16y+4z)+27=0$ (b) $8x-16y+4z+27=0$

(c) $16x-32y+8z-27=0$ (d) $16x+32y+8z-27=0$

Short-cut solution :

Using T-3 Equation of the plane bisecting the angle between given planes are

$$\frac{x-2y+2z+3}{\sqrt{1^2+2^2+2^2}}=\pm\frac{3x-6y-2z+2}{\sqrt{3^2+6^2+2^2}}$$

$\Rightarrow 2x-4y-20z-15=0$ and $16x-32y+8z+27=0$

Angle between $16x - 32y + 8z + 27 = 0$ and $x - 2y + 2z + 3 = 0$ is

$$\cos\theta = \frac{16+64+16}{24\sqrt{21}} = \frac{4}{\sqrt{21}} > \frac{1}{\sqrt{2}}$$

$\Rightarrow \theta < 45°$ (acute angle)

$\therefore$ Required equation is $16x - 32y + 8z + 27 = 0$. **Ans. (a)**

TIPS AND TRICKS: (T-4)

The lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are intersecting lines, if $(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1) = 0$.

Illustration 6

If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ are intersect, then k is equal to: **[AIEEE 2012]**

(a) -1 (b) $\frac{2}{9}$ (c) $\frac{9}{2}$ (d) 0

Short-cut solution :

$\boxed{\text{Using T-4}}$ Given lines are $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$

$\therefore a_1 = \hat{i} - \hat{j} + \hat{k},\ \vec{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$a_2 = 3\hat{i} + k\hat{j},\ \vec{b}_2 = \hat{i} + 2\hat{j} + \hat{k}$

Given lies are intersect if

$(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0 \Rightarrow (\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$

$$\Rightarrow \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(3 - 8) - (k + 1)(2 - 4) - 1(4 - 3) = 0$

$\Rightarrow 2(-5) - (k + 1)(-2) - 1(1) = 0$

$\Rightarrow -10 + 2k + 2 - 1 = 0 \quad \Rightarrow \quad k = \frac{9}{2}$ **Ans. (c)**

TIPS AND TRICKS: (T-5)

Centre and radius of sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Centre: $(-u, -v, -w)$

Radius: $\sqrt{u^2 + v^2 + w^2 - d}$

Illustration 7

If (2, 3, 5) is one end of a diameter of the sphere $x^2+y^2+z^2-6x-12y-2z+20=0$, then the cooordinates of the other end of the diameter are **[AIEEE 2007]**

(a) (4, 3, 5) (b) (4, 3, – 3) (c) (4, 9, –3) (d) (4, –3, 3).

Short-cut solution :

Using T-5 We know that centre of sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

is $(-u, -v, -w)$

Given that, $x^2+y^2+z^2-6x-12y-2z+20=0$

$\therefore$ Centre $\equiv (3, 6, 1)$

Coordinates of one end of diameter of the sphere are (2, 3, 5). Let the coordinates of the other end of diameter are (α, β, γ)

$$\therefore \frac{\alpha+2}{2}=3, \frac{\beta+3}{2}=6, \frac{\gamma+5}{2}=1$$

$\Rightarrow \alpha=4, \beta=9$ and $\gamma=-3$

$\therefore$ Coordinate of other end of diameter are (4, 9, –3) **Ans. (c)**

SHORTCUTS: (SC-1)

Distance from a point $P(x_0, y_0, z_0)$ to a line

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Let $Q(x_1, y_1, z_1)$ and $\vec{b}=a\hat{i}+b\hat{j}+c\hat{k}$

$$\therefore d=\frac{\left|\overrightarrow{PQ}\times\vec{b}\right|}{\left|\vec{b}\right|}.$$

Illustration 8

Find the distance between point p(0, 2, 3) and line $\frac{x-3}{2}=\frac{y-1}{1}=\frac{z+1}{2}$

Short-cut solution :

Using SC-1 Here P(0, 2, 3), Q(3, 1, –1) and $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$

$\overrightarrow{PQ}=3\hat{i}-\hat{j}-4\hat{k}$

$$\overrightarrow{PQ}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & -4\\ 2 & 1 & 2\end{vmatrix}=2\hat{i}-14\hat{j}+5\hat{k}$$

$$\therefore d=\frac{\left|\overrightarrow{PQ}\times\vec{b}\right|}{\left|\vec{b}\right|}=\frac{\sqrt{2^2+14^2+5^2}}{\sqrt{2^2+1^2+2^2}}=\frac{15}{3}=5 \text{ units}$$

SHORTCUTS: (SC-2)

Projection of a line segment on a line or plane.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then

(i) projection of PQ on a line having direction vector $\vec{b}$ is $\overrightarrow{PQ}\cdot\hat{b}$.

(ii) projection of PQ on a plane having normal vector $\vec{n}$ is $\left|\overrightarrow{PQ}\times\hat{n}\right|$.

Illustration 9

Find the length of projection of the line segment joining (2, –1, 3) and (4, 2, 5) on a line which makes equal angle with coordinate axes.

Short-cut solution :

$\boxed{\text{Using SC-2(i)}}$ Since given line makes equal angle with coordinate axes

$$\therefore\ l = m = n = \frac{1}{\sqrt{3}} \qquad \text{i.e. } \hat{b} = \frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right) \text{ and } \overrightarrow{PQ} = 2\hat{i}+3\hat{j}+2\hat{k}$$

$$\therefore \text{ length of projection } \overrightarrow{PQ}\,.\,\hat{b} = \frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(2\hat{i}+3\hat{j}+2\hat{k}\right)$$

Illustration 10

Find the length of projection of the line segment joining the points (1, 2, 3) and (4, 5, 6) on the plane $2x + y + z = 1$.

Short-cut solution :

$\boxed{\text{Using SC-2(ii)}}$ Here $\vec{n} = 2\hat{i}+\hat{j}+\hat{k} \Rightarrow \hat{n} = \frac{1}{\sqrt{6}}\left(2\hat{i}+\hat{j}+\hat{k}\right)$

and $\overrightarrow{PQ} = 3\hat{i}+3\hat{j}+3\hat{k}$

$$\overrightarrow{PQ}\times\hat{n} = \frac{1}{\sqrt{6}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix} = \frac{3}{\sqrt{6}}\left(-\hat{j}+\hat{k}\right)$$

$$\therefore \text{ length of projection } = \left|\overrightarrow{PQ}\times\hat{n}\right| = \frac{3}{\sqrt{6}}\times\sqrt{2} = \sqrt{3}.$$

SHORTCUTS: (SC-3)

If a, b and c are the projections of a line segment on coordinate axis, then

(i) the length of the line segment $= \sqrt{a^2+b^2+c^2}$

(ii) direction cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Illustration 11

The direction cosines of a straight line, whose projections on the coordinates axes. OX, OY, OZ are 12, 4, 13 respectively are

(a) $\frac{12}{13}, \frac{4}{13}, 1$ (b) $\frac{12}{\sqrt{329}}, \frac{4}{\sqrt{329}}, \frac{13}{\sqrt{329}}$

(c) $\frac{1}{12}, \frac{1}{4}, \frac{1}{13}$ (d) $\frac{12}{329}, \frac{4}{329}, \frac{13}{329}$

Short-cut solution :

Using SC-3(ii) Here, $\sqrt{a^2+b^2+c^2} = \sqrt{12^2+4^2+13^2} = \sqrt{329}$

$\therefore$ Direction cosines:

$\pm\frac{12}{\sqrt{329}},\ \pm\frac{4}{\sqrt{329}},\ \pm\frac{13}{\sqrt{329}}.$ **Ans. (b)**

Illustration 12

If the projection of a line segment on x, y and z axis are respectively 3, 4 and 5 then the length of the line segment is

(a) $3\sqrt{2}$ (b) $5\sqrt{2}$ (c) $7\sqrt{2}$ (d) None of these

Short-cut solution :

Using SC-3(i)

Length of line segment $= \sqrt{3^2+4^2+5^2} = 5\sqrt{2}$ unit **Ans. (b)**

SHORTCUTS: (SC-4)

Foot of perpendicular of a point on the line: Position vector of foot of perpendicular of a point $P(\vec{P})$ on the line $\vec{r} = \vec{a} + \lambda\vec{b}$

$$= \vec{a} + \frac{(\vec{p}-\vec{a})\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b}$$

Illustration 13

Find the foot of the perpendicular from the point (0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$.

Short-cut solution :

Using SC-4 Here $\vec{P} = 2\hat{j} + 3\hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a} = -3\hat{i} + \hat{j} - 4\hat{k}$

$\therefore$ Position vector of foot of perpendicular

$$= \left(-3\hat{i} + \hat{j} - 4\hat{k}\right) + \frac{\left(3\hat{i} + \hat{j} + 7\hat{k}\right)\cdot\left(5\hat{i} + 2\hat{j} + 3\hat{k}\right)}{5^2 + 2^2 + 3^2}\left(5\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \left(-3\hat{i} + \hat{j} - 4\hat{k}\right) + 1\left(5\hat{i} + 2\hat{j} + 3\hat{k}\right) = \left(2\hat{i} + 3\hat{j} - \hat{k}\right)$$

$\therefore$ Foot of perpendicular = (2, 3, –1)

SHORTCUTS: (SC-5)

Image of point by a given line. Let $Q(\vec{q})$ is the image of $P(\vec{P})$ in the line $\vec{r} = \vec{a} + \lambda\vec{b}$, then image of point $Q(\vec{q})$ $\vec{q} = 2\left(\vec{a} + \frac{(\vec{P} - \vec{a})\cdot\vec{b}}{|\vec{b}|^2}\cdot\vec{b}\right) - \vec{P}$.

Illustration 14

Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Short-cut solution :

$\boxed{\text{Using SC-5}}$ Here, $\vec{P} = \hat{i} + 6\hat{j} + 3\hat{k}, \vec{a} = \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Image of point $\vec{p}$ is

$$\vec{q} = 2\left[\left(\hat{j} + 2\hat{k}\right) + \frac{\left(\hat{i} + 5\hat{j} + \hat{k}\right)\cdot\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)}{1^2 + 2^2 + 3^2}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\right] - \left(\hat{i} + 6\hat{j} + 3\hat{k}\right)$$

$$= 2\left[\left(\hat{j} + 2\hat{k}\right) + 1\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\right] - \left(\hat{i} + 6\hat{j} + 3\hat{k}\right)$$

$$= 2\hat{i} + 6\hat{j} + 10\hat{k} - \hat{i} - 6\hat{j} - 3\hat{k} = \hat{i} + 0.\hat{j} + 7\hat{k}$$

$\therefore$ Image is (1, 0, 7).

SHORTCUTS: (SC-6)

Foot of perpendicular of a point on the plane. The foot (x, y, z) of perpendicular of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

Note: Foot of perpendicular from origin to plane lx + my + nz = d is (ld, md, nd).

Illustration 15

Foot of the perpendicular drawn from the point (1, 3, 4) to the plane $2x - y + z + 3 = 0$

(a) (1, 2, –3) (b) (–1, 4, 3) (c) (–3, 5, 2) (d) (0, –4, –7)

Short-cut solution :

Using SC-6 Foot of perpendicular is given by

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = \frac{-(2-3+4+3)}{4+1+1}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = -1$$

$\Rightarrow x = -1, y = 4, z = 3$. Point (–1, 4, 3). **Ans. (a)**

SHORTCUTS: (SC-7)

Image of point by a given plane. The image or reflection (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

Illustration 16

The image of the point (–1, 3, 4) in the plane $x - 2y = 0$ is **[AIEEE 2006]**

(a) $\left(-\frac{17}{3}, -\frac{19}{3}, 4\right)$ (b) (15, 11, 4)

(c) $\left(-\frac{17}{3}, -\frac{19}{3}, 1\right)$ (d) None of these

Short-cut solution :

Using SC-7 **(d)** $\frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0} = \frac{-2(-1-6)}{1+4}$

$$\Rightarrow \frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0} = \frac{14}{5}$$

$$\Rightarrow x = \frac{9}{5}, y = \frac{-13}{5}, z = 4.$$

Ans. (d)

SHORTCUTS: (SC-8)

Perpendicular distance from point $P(\vec{P})$ to line $\vec{r} = \vec{a} + \lambda\vec{b}$ is

$$d = \frac{\left|(\vec{p} - \vec{a}) \times \vec{b}\right|}{\left|\vec{b}\right|}$$

Illustration 17

To find distance between point P(0, 2, 3) and line

$$\frac{x-3}{2}=\frac{y-1}{1}=\frac{z+1}{2}$$

Short-cut solution :

Using SC-8 Here $\vec{p}=2\hat{j}+3\hat{k}$

$\vec{a}=3\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$

$\vec{p}-\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$

$$(\vec{p}-\vec{a})\times\vec{b}=\begin{vmatrix} i & j & k \\ -3 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix}$$

$= i(2-4)-j(-6-8)+k(-3-2)= -2\hat{i}+14\hat{j}-5\hat{k}$

$$d=\frac{|(\vec{p}-\vec{a})\times\vec{b}|}{|\vec{b}|}=\frac{\sqrt{(-2)^2+(14)^2+(-5)^2}}{\sqrt{2^2+1^2+2^2}}=\frac{\sqrt{225}}{\sqrt{9}}=\frac{15}{3}=5$$

$\therefore$ distance from point to line is equal to 5.

Concept Booster Exercise

1. If the length of the perpendicular from the point (β, 0, β) (β ≠ 0) to the line, $\frac{x}{1}=\frac{y-1}{0}=\frac{z+1}{-1}$ is $\sqrt{\frac{3}{2}}$, then β is equal to: **[JEE M 2019]**

(a) 1 (b) 2 (c) –1 (d) –2

2. If the position vector of A and B are $2\hat{i}+2\hat{j}+\hat{k}$ and $2\hat{i}+4\hat{j}+4\hat{k}$. The length of projection of the line segment AB on a line which have direction vector $2\hat{i}+3\hat{j}+\hat{k}$ is ________.

(a) $\frac{9}{\sqrt{14}}$ (b) $\frac{\sqrt{14}}{9}$ (c) $\frac{\sqrt{14}}{3}$ (d) $\frac{3}{\sqrt{14}}$

3. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x+y+z=7$ is: **[JEE M 2018]**

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{2}{\sqrt{3}}$

4. If the projections of a line segment on the x, y and z-axes in 3-dimensional space are 2, 3 and 6 respectively, then the length of the line segment is : **[JEE M 2013]**

(a) 12 (b) 7 (c) 9 (d) 6

5. The projections of a vector on the three coordinate axis are 6, –3, 2 respectively. The direction cosines of the vector are : **[AIEEE 2009]**

(a) $\frac{6}{5},\frac{-3}{5},\frac{2}{5}$ (b) $\frac{6}{7},\frac{-3}{7},\frac{2}{7}$ (c) $\frac{-6}{7},\frac{-3}{7},\frac{2}{7}$ (d) 6, – 3, 2

6. The coordinates of the foot of perpendicular from the point (1, 0, 0) to the line $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ are **[AIEEE 2012]**

(a) (2, – 3, 8) (b) (1, – 1, – 10)
(c) (5, – 8, – 4) (d) (3, – 4, – 2)

7. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is : **[AIEEE 2011]**

(a) $\sqrt{29}$ (b) $\sqrt{33}$ (c) $\sqrt{53}$ (d) $\sqrt{66}$

8. If (a, b, c) is the image of the point (1, 2, –3) in the line, $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$, then a + b + c is equals to: **[JEE M 2020]**

(a) 2 (b) – 1 (c) 3 (d) 1

9. The coordinates of the foot of the perpendicular from the point (1, –2, 1) on the plane containing the lines,

$\frac{x+1}{6}=\frac{y-1}{7}=\frac{z-3}{8}$ and $\frac{x-1}{3}=\frac{y-2}{5}=\frac{z-3}{7}$, is : **[JEE M 2017]**

(a) (2, –4, 2) (b) (–1, 2, –1) (c) (0, 0, 0) (d) (1, 1, 1)

10. A perpendicular is drawn from a point on the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{1}$ to the plane $x + y + z = 3$ such that the foot of the perpendicular Q also lies on the plane $x - y + z = 3$. Then the co-ordinates of Q are : **[JEE M 2019]**

(a) (1, 0, 2) (b) (2, 0, 1) (c) (–1, 0, 4) (d) (4, 0, –1)

11. Let *P* be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and *R* be any point (2, 1, 6). Then the image of *R* in the plane *P* is: **[JEE M 2020]**

(a) (6, 5, 2) (b) (6, 5, –2) (c) (4, 3, 2) (d) (3, 4, –2)

12. If Q (0, –1, –3) is the image of the point P in the plane 3x – y + 4z = 2 and R is the point (3, –1, –2), then the area (in sq. units) of ΔPQR is : **[JEE M 2019]**

(a) $2\sqrt{13}$ (b) $\frac{\sqrt{91}}{4}$ (c) $\frac{\sqrt{91}}{2}$ (d) $\frac{\sqrt{65}}{2}$

13. The distance of the point (1, –2, 4) from the plane passing through the point (1, 2, 2) and perpendicular to the planes x – y + 2z = 3 and 2x – 2y + z + 12 = 0, is : **[JEE M 2016]**

(a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

14. The distance of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines **[JEE M 2017]**

$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is :

(a) $\frac{10}{\sqrt{74}}$ (b) $\frac{20}{\sqrt{74}}$ (c) $\frac{10}{\sqrt{83}}$ (d) $\frac{5}{\sqrt{83}}$

15. The equation of the line through the point (–1, 2, 0) and perpendicular to the lines $\frac{x}{3}=\frac{y+1}{0}=\frac{z-2}{-1}$ and $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$ is

(a) $\frac{x+1}{1}=\frac{y-2}{2}=\frac{z}{3}$ (b) $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z}{3}$

(c) $\frac{x+1}{2}=\frac{y-2}{1}=\frac{z-1}{3}$ (d) $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z}{3}$

16. Find the equation of the plane passing through the point (1, 2, 3) and parallel to each of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$

(a) $\frac{x+1}{1}=\frac{x-2}{-2}=\frac{z-3}{1}$ (b) $\frac{x-1}{1}=\frac{x-2}{-2}=\frac{z-3}{1}$

(c) $\frac{x-1}{1}=\frac{x-2}{2}=\frac{z-3}{1}$ (d) $\frac{x+1}{1}=\frac{x-2}{2}=\frac{z-3}{1}$

17. A plane which bisects the angle between the two given planes $2x - y + 2z - 4 = 0$ and $x + 2y + 2z - 2 = 0$, passes through the point : **[JEE M 2019]**

(a) (1, –4, 1) (b) (1, 4, –1) (c) (2, 4, 1) (d) (2, –4, 1)

18. If the straight lines $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$ intersect at a point, then the integer k is equal to **[JEE M 2008]**

(a) –5 (b) 5 (c) 2 (d) –2

19. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the line joining the centres of the spheres $x^2+y^2+z^2+6x-8y-2z=13$ and $x^2+y^2+z^2-10x+4y-2z=8$ then a equals **[JEE M 2005]**

(a) – 1 (b) 1 (c) – 2 (d) 2

20. If the lines $x = ay + b, z = cy + d$ and $x = a'z + b', y = c'z + d'$ are perpendicular, then: **[JEE M 2019]**

(a) ab' + bc' + 1 = 0 (b) cc' + a + a' = 0

(c) bb' + cc' + 1 = 0 (d) aa' + c + c' = 0

21. The number of distinct real values of λ for which the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{\lambda^2}$ and $\frac{x-3}{1}=\frac{y-2}{\lambda^2}=\frac{z-1}{2}$ are coplanar is :

[JEE M 2016]

(a) 2 (b) 4 (c) 3 (d) 1

22. The shortest distance between the lines $\frac{x-1}{0}=\frac{y+1}{-1}=\frac{z}{1}$ and $x+y+z+1=0$, $2x-y+z+3=0$ is : **[JEE M 2020]**

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

23. Equation of the plane containing the straight line $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ is **[AIEEE 2010]**

(a) $x+2y-2z=0$ (b) $3x+2y-2z=0$
(c) $x-2y+z=0$ (d) $5x+2y-4z=0$

24. The length of the perpendicular from the point (2, –1, 4) on the straight line, $\frac{x+3}{10}=\frac{y-2}{-7}=\frac{z}{1}$ is : **[JEE M 2019]**

(a) greater than 3 but less than 4 (b) less than 2
(c) greater than 2 but less than 3 (d) greater than 4

25. Two lines $\frac{x-3}{1}=\frac{y+1}{3}=\frac{z-6}{-1}$ and $\frac{x+5}{7}=\frac{y-2}{-6}=\frac{z-3}{4}$ intersect at the point R. The reflection of R in the *xy*- plane has coordinates: **[JEE M 2019]**

(a) (2, –4, –7) (b) (2, 4, 7) (c) (2, –4, 7) (d) (–2, 4, 7)

26. If the angle between the lines, $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{5-x}{-2}=\frac{7y-14}{P}=\frac{z-3}{4}$ is $\cos^{-1}\left(\frac{2}{3}\right)$, then P is equal to **[JEE M 2018]**

(a) $-\frac{7}{4}$ (b) $\frac{2}{7}$ (c) $-\frac{4}{7}$ (d) $\frac{7}{2}$

27. The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ and $\vec{r}=(\hat{i}+\hat{j})+\mu(-\hat{i}+\hat{j}-2\hat{k})$ is : **[JEE M 2019]**

(a) 3 (b) $\frac{1}{3}$ (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$

28. If the plane $2x - y + 2z + 3 = 0$ has the distances $\frac{1}{2}$ and $\frac{2}{3}$ units from the planes $4x - 2y + 4z + \lambda = 0$ and $2x - y + 2z + \mu = 0$, respectively, then the maximum value of $\lambda + \mu$ is equal to : **[JEE M 2019]**

(a) 9 (b) 15 (c) 5 (d) 13

29. The plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to y-axis also passes through the point: **[JEE M 2019I]**

(a) $(-3, 0, -1)$ (b) $(-3, 1, 1)$ (c) $(3, 3, -1)$ (d) $(3, 2, 1)$

30. An angle between the plane, $x + y + z = 5$ and the line of intersection of the planes, $3x + 4y + z - 1 = 0$ and $5x + 8y + 2z + 14 = 0$, is **[JEE M 2018]**

(a) $\cos^{-1}\left(\frac{3}{\sqrt{17}}\right)$ (b) $\cos^{-1}\left(\sqrt{\frac{3}{17}}\right)$

(c) $\sin^{-1}\left(\frac{3}{\sqrt{17}}\right)$ (d) $\sin^{-1}\left(\sqrt{\frac{3}{17}}\right)$

31. If the distance of the point $P\,(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is **[AIEEE 2010]**

(a) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (b) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ (c) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (d) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

Numerical Value Problems

32. If the foot of the perpendicular drawn from the point $(1, 0, 3)$ on a line passing through $(\alpha, 7, 1)$ is $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, then α is equal to ________. **[JEE M 2020]**

33. Let a plane P contain two lines $\vec{r} = \hat{i} + \lambda(\hat{i} + \hat{j}),\ \lambda \in \mathbf{R}$ and $\vec{r} = -\hat{j} + \mu(\hat{j} - \hat{k}),\ \mu \in \mathbf{R}$. If $Q(\alpha, \beta, \gamma)$ is the foot of the perpendicular drawn from the point $M(1, 0, 1)$ to P, then $3(\alpha + \beta + \gamma)$ equals ____________. **[JEE M 2020]**

34. If the distance between the plane, $23x - 10y - 2z + 48 = 0$ and the plane containing the lines

$$\frac{x+1}{2}=\frac{y-3}{4}=\frac{z+1}{3} \text{ and } \frac{x+3}{2}=\frac{y+2}{6}=\frac{z-1}{\lambda}(\lambda \in \mathrm{R})$$

is equal to $\frac{k}{\sqrt{633}}$, then k is equal to ______. **[JEE M 2020]**

35. If the equation of a plane P, passing through the intersection of the planes, $x + 4y - z + 7 = 0$ and $3x + y + 5z = 8$ is $ax + by + 6z = 15$ for some then the distance of the point (3, 2, –1) from the plane P is __________.

[JEE M 2020]

36. Three lines are given by $\vec{r}=\lambda\hat{i}, \lambda \in R$; $\vec{r}=\mu\left(\hat{i}+\hat{j}\right), \mu \in \mathrm{R}$ and

$\vec{r}=v\left(\hat{i}+\hat{j}+\hat{k}\right), v \in R$. Let the lines cut the plane x + y + z = 1 at the points

A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals ____. **[JEE Adv. 2019]**

Solutions

1. **(c)** Using SC-1

Here P(β, 0, β) . Q(0, 1, –1) and $\vec{b} = \hat{i} - \hat{k}$

$$\overrightarrow{PQ} = -\beta\hat{i} + \hat{j} - (\beta+1)\hat{k}$$

$$\overrightarrow{PQ} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\beta & 1 & -(\beta+1) \\ 1 & 0 & -1 \end{vmatrix} = -\hat{i} - \hat{j}(2\beta+1) - \hat{k}$$

$$d = \frac{\sqrt{1+(2\beta+1)^2+1}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$\Rightarrow 4\beta^2 + 4\beta + 3 = 3 \Rightarrow \beta = 0, -1\ (\beta \pm 0)$

$\therefore\ \beta = -1$

2. **(b)** Using SC-2(i) Here $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\hat{b} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k})$$

$$\overrightarrow{AB} = 2\hat{j} + 3\hat{k}$$

$\therefore$ length of projection = $\overrightarrow{AB} \cdot \hat{b}$

$$= \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) \cdot (2\hat{j} + 3\hat{k})$$

$$= \frac{1}{\sqrt{14}}(0+6+3) = \frac{9}{\sqrt{14}}.$$

3. **(c)** Using SC-2(ii) Here $\vec{n} = \hat{i} + \hat{j} + \hat{k} \Rightarrow \hat{n} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

$$\overrightarrow{PQ} = -\hat{i} - \hat{k}$$

$$\overrightarrow{PQ} \times \hat{n} = \frac{1}{\sqrt{3}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{vmatrix} = \frac{1}{\sqrt{3}}\left(-\hat{i} + \hat{k}\right)$$

$\therefore$ length of projection = $\left|\overrightarrow{PQ} \times \hat{n}\right| = \sqrt{\frac{2}{3}}.$

4. **(b)** Using SC-3(i) Length of the line segment $= \sqrt{(2)^2+(3)^2+(6)^2} = 7$

5. **(b)** Using SC-3(ii)

Here, $\sqrt{a^2+b^2+c^2} = \pm\sqrt{6^2+(-3)^2+2^2} = \pm\sqrt{49} = \pm 7$

$\therefore$ Direction cosines, $\pm\dfrac{6}{7},\ \mp\dfrac{3}{7},\ \pm\dfrac{2}{7}.$

6. **(d)** Using SC-4

Here, $\vec{P} = \hat{i}, \vec{a} = \hat{i}-\hat{j}-10\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+8\hat{k}$

$\therefore$ Position vector of foot of perpendicular

$$= \left(\hat{i}-\hat{j}-10\hat{k}\right) + \frac{\left(\hat{j}+10\hat{k}\right)\cdot\left(2\hat{i}-3\hat{j}+8\hat{k}\right)}{2^2+(-3)^2+8^2}\cdot\left(2\hat{i}-3\hat{j}+8\hat{k}\right)$$

$$= \left(\hat{i}-\hat{j}-10\hat{k}\right) + 1\left(2\hat{i}-3\hat{j}+8\hat{k}\right) = 3\hat{i}-4\hat{j}-2\hat{k}$$

$\therefore$ Foot of perpendicular = (3, –4, –2)

7. **(c)** Another from of SC-4

Any point on line $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = \alpha$ is

$(2\alpha, 3\alpha+2, 4\alpha+3)$

Direction ratio of the $\perp$ line is

$2\alpha-3, 3\alpha+3, 4\alpha-8.$ and

Direction ratio of the given line are 2, 3, 4

$\Rightarrow 2(2\alpha - 3) + 3(3\alpha + 3) + 4(4\alpha - 8) = 0$

$\Rightarrow 29\alpha - 29 = 0$

$\Rightarrow \alpha = 1$

Foot of $\perp$ is (2, 5, 7)

Length $\perp$ is $= \sqrt{1^2+6^2+4^2} = \sqrt{53}$

8. Using SC-5

Here $\vec{P} = \hat{i}+2\hat{j}-3\hat{k},\ \vec{a} = -\hat{i}+3\hat{j}$ and $\vec{b} = 2\hat{i}-2\hat{j}-\hat{k}$

Image of point (1, 2, –3) is

$$\vec{q} = 2\left[-\hat{i}+3\hat{j}+\frac{\left(2\hat{i}-\hat{j}-3\hat{k}\right)\cdot\left(2\hat{i}-2\hat{j}-\hat{k}\right)}{2^2+(-2)^2+(-1)^2}\cdot\left(2\hat{i}-2\hat{j}-\hat{k}\right)\right] - \left(\hat{i}+2\hat{j}-3\hat{k}\right)$$

$$= 2\left[-\hat{i}+3\hat{j}+1\left(2\hat{i}-2\hat{j}-\hat{k}\right)\right] - \left(\hat{i}+2\hat{j}-3\hat{k}\right)$$

$$= 2\hat{i}+2\hat{j}-2\hat{k}-\hat{i}-2\hat{j}+3\hat{k} = \hat{i}+0.\hat{j}+\hat{k}$$

$\therefore$ a + b + c = 1 + 0 + 1 = 2

9. **(c)** $\boxed{\text{Using T-2(ii)}}$ $\vec{n} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix} = (9, -18, 9) = (1, -2, 1)$

$\therefore$ Equation of plane is $1(x + 1) -2 (y - 1) + (z - 3) = 0$

$\Rightarrow$ $x - 2y + z = 0$

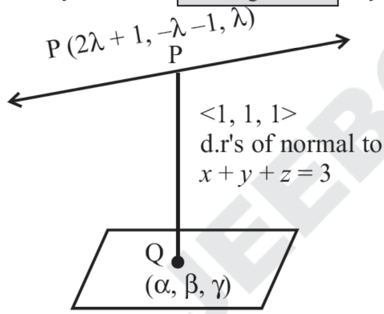
$\boxed{\text{Using SC-6}}$

foot to z

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-1}{1} = -\frac{[1+4+1]}{6}$$

$$\boxed{x = 0, y = 0, z = 0}$$

10. **(b)** It may be solve $\boxed{\text{Using SC-6}}$, try it.



Let co-ordinates of Q be (α, β, γ), then

$\alpha + \beta + \gamma = 3$ ...(i)

$\alpha - \beta + \gamma = 3$ ...(ii)

$\Rightarrow \alpha + \gamma = 3$ and $\beta = 0$

Equating direction ratio's of PQ, we get

$$\frac{\alpha - 2\lambda - 1}{1} = \frac{\lambda + 1}{1} = \frac{\gamma - \lambda}{1}$$

$\Rightarrow \alpha = 3\lambda + 2, \gamma = 2\lambda + 1$

Substituting the values of α and γ in equation (i), we get

$\Rightarrow 5\lambda + 3 = 3 \Rightarrow \lambda = 0$

Hence, point is Q (2, 0, 1)

11. **(b)** $\boxed{\text{Using SC-7}}$ Image of (2, 1, 6,) is optimize by Equation of plane is $x + y - 2z = 3$

$$\Rightarrow \frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2} = \frac{-2(2+1-12-3)}{6}$$

$\Rightarrow$ $(x, y, z) = (6, 5, -2)$

12. **(c)** Using SC-7 Image of Q (0, –1, –3) in plane is,

$$\frac{(x-0)}{3}=\frac{(y+1)}{-1}=\frac{z+3}{+4}=\frac{-2(1-12-2)}{9+1+16}=1$$

$\Rightarrow x = 3, y = -2, z = 1$

$\Rightarrow$ P (3, – 2, 1), Q (0, –1, –3), R (3, – 1, – 2)

$\therefore$ Area of ΔPQR is

$$\frac{1}{2}|\vec{Q}P\times\vec{Q}R|=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & k\\ 3 & -1 & 4\\ 3 & 0 & 1\end{matrix}\right\|$$

$$=\frac{1}{2}|\{\hat{i}(-1)-\hat{j}(3-12)+\hat{k}(3)\}|$$

$$=\frac{1}{2}\sqrt{(1+81+9)}=\frac{\sqrt{91}}{2}$$

13. Using T-2(i)

$$\vec{n}=\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 2 & -2 & 1\end{vmatrix}=3\hat{i}+3\hat{j}$$

Equation of plane is

3(x – 1) + 3(y – 2) + 0(z – 2) = 0

$\Rightarrow$ x + y – 3 = 0

Distance from (1, –2, 4) will be

$$D=\frac{|1-2-3|}{\sqrt{1+1}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$$

14. **(c)** Using T-2(ii) Let the plane be

a (x – 1) + b (y + 1) + c (z + 1) = 0

Normal vector

$$\vec{n}=\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & -1 & -1\end{vmatrix}=5\hat{i}+7\hat{j}+3\hat{k}$$

So plane is 5 (x – 1) + 7 (y + 1) + 3 (z + 1) = 0

$\Rightarrow$ 5x + 7y + 3z + 5 = 0

Distance of point (1, 3, –7) from the plane is

$$\frac{5+21-21+5}{\sqrt{25+49+9}}=\frac{10}{\sqrt{83}}$$

15. **(a)** $\boxed{\text{Using T-1(i)}}$

$$\vec{b} = \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$\therefore$ Equation of line is

$$\frac{x+1}{1} = \frac{y-2}{2} = \frac{z}{3}.$$

16. **(b)** $\boxed{\text{Using T-1(ii)}}$

$$\vec{b} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 4 & 3 & 2 \end{vmatrix} = -5\hat{i} + 10\hat{j} - 5\hat{k}$$

$\therefore$ Equation of line is $\dfrac{x-1}{1} = \dfrac{x-2}{-2} = \dfrac{z-3}{1}$

17. **(d)** $\boxed{\text{Using T-3}}$ The equations of angle bisectors are,

$$\frac{x+2y+2z-2}{3} = \pm\frac{2x-y+2z-4}{3}$$

$\Rightarrow$ $x - 3y - 2 = 0$

or $3x + y + 4z - 6 = 0$

(2, – 4, 1) lies on the second plane.

18. **(a)** $\boxed{\text{Using T-4}}$ the two lines intersect

$\therefore$ $(\vec{a}_2 - \vec{a}_1)\cdot\vec{b}_1 \times \vec{b}_2 = 0$

where, $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b}_1 = k\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$, $\hat{b}_2 = 3\hat{i} + k\hat{j} + 2\hat{k}$

$$\Rightarrow \begin{vmatrix} 1 & 1 & -2 \\ k & 2 & 3 \\ 3 & k & 2 \end{vmatrix} = 0$$

$\Rightarrow$ $1(4 - 3k) - 1(2k - 9) - 2(k2 - 6) = 0$

$\Rightarrow$ $-2k^2 - 5k + 25 = 0 \Rightarrow k = -5$ or $\dfrac{5}{2}$

$\because$ k is an integer, therefore $k = -5$

19. **(c)** $\boxed{\text{Using T-5}}$

Plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid point of the line joining the centres of spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$ respectively centre of spheres are $c_1(-3, 4, 1)$ and $c_2(5, -2, 1)$. Mid point of c_1c_1 is (1, 1, 1).

Satisfying this in the equation of plane, we get

$2a - 3a + 4a + 6 = 0$

$\Rightarrow \quad a = -2.$

20. **(d)** First line is: $x = ay + b, z = cy + d$

[**Trick:** Solve y from both equations then equate to convert standard form of line]

$$\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$$

and another line is: $x = a'z + b', y = c'z + d'$

$$\Rightarrow \quad \frac{x-b'}{a'} = \frac{y-d'}{c'} = \frac{z}{1}$$

$\because$ Both lines are perpendicular to each other

$\therefore\ aa' + c' + c = 0$

21. **(c)** Lines are coplanar [**Trick:** For coplanar use $(\vec{a}_2 \cdot \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$]

$$\begin{vmatrix} 3-1 & 2-2 & 1-(-3) \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & 4 \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(4 - \lambda^4) + 4(\lambda^2 - 2) = 0$

$\Rightarrow \quad 4 - \lambda^4 + 2\lambda^2 - 4 = 0 \Rightarrow \lambda^2(\lambda^2 - 2) = 0$

$\Rightarrow \quad \lambda = 0, \sqrt{2}, -\sqrt{2}$

22. **(b)** For line of intersection of planes $x + y + z + 1 = 0$ and $2x - y + z + 3 = 0$:

$$\vec{b}_2 = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2\hat{i} + \hat{j} - 3\hat{k}$$

[**Trick:** The direction vector of line of intersection of planes is $\vec{n}_1 \times \vec{n}_2$ and for point take the value of one variable to zero and solve other variables.]

Put $y = 0$, we get $x = -2$ and $z = 1$

$L_2 : \bar{r} = (-2\hat{i} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$ and

$L_1 : \bar{r} = (\hat{i} - \hat{j}) + \mu(-\hat{j} + \hat{k})$ (Given)

Now, $\bar{b}_1 \times \bar{b}_2 = -2[\hat{i} + \hat{j} + \hat{k}]$ and $\bar{a}_2 - \bar{a}_1 = -3\hat{i} + \hat{j} + \hat{k}$

$\therefore$ Shortest distance $= \dfrac{1}{\sqrt{3}}$

23. **(c)** [**Trick:** Equation of coplanar plane is $(\vec{r} - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 0$]

Equation of plane containing two lines $\dfrac{x}{3} = \dfrac{y}{4} = \dfrac{z}{2}$ and $\dfrac{x}{4} = \dfrac{y}{2} = \dfrac{z}{3}$ is given by

$$\begin{vmatrix} x & y & z \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 0 \Rightarrow 8x - y - 10z = 0$$

Now equation of plane containing the line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z}{4}$ and perpendicular to the plane $8x - y - 10z = 0$ is given by,

$$\begin{vmatrix} x & y & z \\ 2 & 3 & 4 \\ 8 & -1 & -10 \end{vmatrix} = 0$$

$\Rightarrow -26x + 52y - 26z = 0$ or $x - 2y + z = 0$

24. **(a)** $\boxed{\text{Using SC-8}}$

Here $P = 2\hat{i} - \hat{j} + 4\hat{k}$

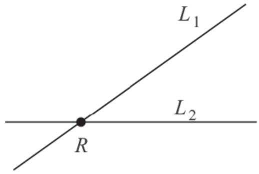
$\vec{a} = -3\hat{i} + 2\hat{j}$ and $\vec{b} = 10\hat{i} - 7\hat{j} + \hat{k}$

$\vec{p} - \vec{a} = 5\hat{i} - 3\hat{j} + 4\hat{k}$

$$(\vec{p} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -3 & 4 \\ 10 & -7 & 1 \end{vmatrix} = 25\hat{i} + 35\hat{j} - 5\hat{k}$$

$$d = \frac{\sqrt{(25)^2 + (35)^2 + 5^2}}{\sqrt{100 + 49 + 1}} = \sqrt{12.5} = 3.54$$

25. **(a)** $\boxed{\text{Using SC-7}}$ Let the coordinate of P with respect to line



$$\frac{x-3}{1}=\frac{y+1}{3}=\frac{z-6}{-1}=\lambda$$

$$\frac{x+5}{7}=\frac{y-2}{-6}=\frac{z-3}{4}=\mu$$

$L_1 = (\lambda + 3, 3\lambda - 1, -\lambda + 6)$
and coordinate of P w.r.t.
line $L_2 = (7\mu - 5, -6\mu + 2, 4\mu + 3)$
$\therefore$ $\lambda - 7\mu = -8$, $3\lambda + 6\mu = 3$, $\lambda + 4\mu = 3$
From above equation : $\lambda = -1$, $\mu = 1$
$\therefore$ Coordinate of point of intersection $R = (2, -4, 7)$.
Image of R w.r.t. xy plane $= (2, -4, -7)$.

26. **(d)** Let θ be the angle between the two lines

Here direction cosines of $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ are 2, 2, 1

Also second line can be written as: $\frac{x-5}{2}=\frac{y-2}{\frac{P}{7}}=\frac{z-3}{4}$

[**Trick:** Convert the equation of line in standard form]

$\therefore$ its direction cosines are $2, \frac{P}{7}, 4$

Also, $\cos\theta = \frac{2}{3}$ (Given)

$$\because \quad \cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

$$\Rightarrow \frac{2}{3} = \left|\frac{(2\times 2)+\left(2\times\frac{P}{7}\right)+(1\times 4)}{\sqrt{2^2+2^2+1^2}\sqrt{2^2+\frac{P^2}{49}+4^2}}\right|$$

$$= \frac{4 + \frac{2P}{7} + 4}{3 \times \sqrt{2^2 + \frac{P^2}{49} + 4^2}}$$

$$\Rightarrow \left(4 + \frac{P}{7}\right)^2 = 20 + \frac{P^2}{49} \Rightarrow 16 + \frac{8P}{7} + \frac{P^2}{49} = 20 + \frac{P^2}{49}$$

$$\Rightarrow \frac{8P}{7} = 4 \Rightarrow P = \frac{7}{2}$$

27. **(c)** [**Trick:** Use formula of coplanar plane]

The equation of plane containing two given lines is,

$$\begin{vmatrix} x-1 & y-1 & z \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix} = 0$$

On expanding, we get $x - y - z = 0$

Now, the length of perpendicular from (2, 1, 4) to this plane

$$= \left|\frac{2-1-4}{\sqrt{1^2+1^2+1^2}}\right| = \sqrt{3}$$

28. **(d)** [**Trick:** Use $d = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}$]

Let, $P_1 : 2x - y + 2z + 3 = 0$

$P_2 : 2x - y + 2z + \frac{\lambda}{2} = 0$

$P_3 : 2x - y + 2z + \mu = 0$

Given, distance between P_1 and P_2 is $\frac{1}{3}$

$$\frac{1}{3} = \frac{\left|3 - \frac{\lambda}{2}\right|}{\sqrt{9}} \Rightarrow \left|3 - \frac{\lambda}{2}\right| = 1 \Rightarrow \lambda_{max} = 8$$

And distance between P_1 and P_3 is $\frac{2}{3}$

$$\frac{2}{3} = \frac{|\mu - 3|}{\sqrt{9}} \Rightarrow \mu_{max} = 5$$

$\Rightarrow (\lambda + \mu)_{max} = 13$

29. **(d)** [**Trick:** Use equation of intersection plane $\Rightarrow \vec{P_1} + \lambda\vec{P_2} = 0$]

Since, equation of plane through intersection of planes

$x + y + z = 1$ and $2x + 3y - z + 4 = 0$ is

$(2x + 3y - z + 4) + \lambda(x + y + z - 1) = 0$

$(2 + \lambda)x + (3 + \lambda)y + (-1 + \lambda)z + (4 - \lambda) = 0$ ...(1)

But, the above plane is parallel to y-axis then

$(2 + \lambda) \times 0 + (3 + \lambda) \times 1 + (-1 + \lambda) \times 0 = 0$

$\Rightarrow \lambda = -3$

Hence, the equation of required plane is $-x - 4z + 7 = 0$

$\Rightarrow x + 4z - 7 = 0$

Therefore, (3, 2, 1) the passes through the point.

30. **(d)** $\boxed{\text{Using T-2(i)}}$ Normal to $3x + 4y + z = 1$ is $3\hat{i} + 4\hat{j} + \hat{k}$.

Normal to $5x + 8y + 2z = -14$ is $5\hat{i} + 8\hat{j} + 2\hat{k}$

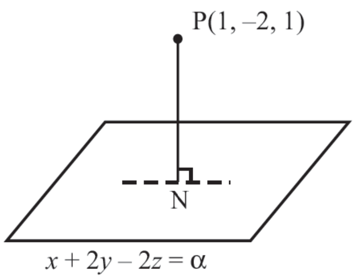
The line of intersection of the planes is perpendicular to both normals, so, direction ratios of the intersection line are directly proportional to the cross product of the normal vectors.

Therefore the direction ratios of the line is $-\hat{j} + 4\hat{k}$.

Hence the angle between the plane $x + y + z + 5 = 0$ and the intersection line is $\sin^{-1}\left(\dfrac{-1+4}{\sqrt{17}\,\sqrt{3}}\right) = \sin^{-1}\left(\sqrt{\dfrac{3}{17}}\right)$

31. **(a)** $\boxed{\text{Using SC-6}}$ Since perpendicular distance of $x + 2y - 2z - \alpha = 0$ from the point $(1, -2, 1)$ is 5

$$\therefore \left|\frac{1-4-2-\alpha}{3}\right| = 5$$



$$\Rightarrow \frac{-5-\alpha}{3} = 5 \text{ or } -5$$

$$\Rightarrow \alpha = -20 \text{ or } 10$$

But $\alpha > 0 \Rightarrow \alpha = 10$

$\therefore$ Equation of plane : $x + 2y - 2z - 10 = 0$

We know that foot of perpendicular from point (x, y, z) to the plane $ax + by + cz + d = 0$ is given by

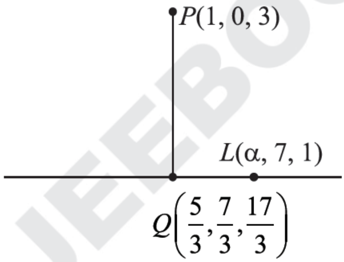
$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{-(ax_1+by_1+cz_1+d)}{(a^2+b^2+c^2)}$$

$$\therefore \frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = \frac{-(1-4-2-10)}{9} = \frac{5}{3}$$

$$\Rightarrow x = \frac{8}{3}, y = \frac{4}{3}, z = -\frac{7}{3}$$

$$\therefore \text{Foot of } \perp^r \equiv \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$$

32. **(4)** Since, PQ is perpendicular to L



[**Trick:** For perpendicular $a_1a_2 + b_1b_2 + c_1c_2 = 0$]

$$\therefore \left(1-\frac{5}{3}\right)\left(\alpha-\frac{5}{3}\right)+\left(\frac{-7}{3}\right)\left(7-\frac{7}{3}\right)+\left(3-\frac{17}{3}\right)\left(1-\frac{17}{3}\right)=0$$

$$\Rightarrow \frac{-2\alpha}{3}+\frac{10}{9}-\frac{98}{9}+\frac{112}{9}=0$$

$$\Rightarrow \frac{2\alpha}{3}=\frac{24}{9} \Rightarrow \alpha = 4$$

33. **(5)** $\boxed{\text{Using SC-6}}$

$$\text{Normal of plane} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\vec{n} = -\hat{i} + \hat{j} + \hat{k}$$

Direction ratios of normal to the plane = <–1, 1, 1>

Equation of plane

$$-1(x-1)+1(y-0)+1(z-0)=0$$

$\Rightarrow x-y-z-1=0$

If (x, y, z) is foot of perpendicular of $M(1, 0, 1)$ on the plane then

$\Rightarrow \frac{x-1}{1}=\frac{y-0}{-1}=\frac{z-1}{-1}=\frac{-(1-0-1-1)}{3}$

$\therefore x=\frac{4}{3}, y=-\frac{1}{3}, z=\frac{2}{3}$

$\alpha+\beta+\gamma=\frac{4}{3}-\frac{1}{3}+\frac{2}{3}=\frac{5}{3}$

$\therefore 3(\alpha+\beta+\gamma)=3\times\frac{5}{3}=5.$

34. **(3)** [**Trick:** Distance between two planes always same]

Since, the line $\frac{x+1}{2}=\frac{y-3}{4}=\frac{z+1}{3}$ contains the point (–1, 3, – 1) and line $\frac{x+3}{2}=\frac{y+2}{6}=\frac{z-1}{\lambda}$ contains the point (–3, –2, 1).

Then, the distance between the plane $23x-10y-2z+48=0$ and the plane containing the lines = perpendicular distance of plane

$23x-10y-2z+48=0$ either from (–1, 3, –1) or (–3, –2, 1).

$=\left|\frac{23(-1)-10(3)-2(-1)}{\sqrt{(23)^2+(10)^2+(-2)^2}}\right|=\frac{3}{\sqrt{633}}$

It is given that distance between the planes

$=\frac{k}{\sqrt{633}} \Rightarrow \frac{k}{\sqrt{633}}=\frac{3}{\sqrt{633}} \Rightarrow k=3$

35. **(3)** [**Trick:** Equation of intersecting plane $P_1+\lambda P_2=0$]

Equation of plane P is

$(x+4y-z+7)+\lambda(3x+y+5z-8)=0$

$\Rightarrow x(1+3\lambda)+y(4+\lambda)+z(-1+5\lambda)+(7-8\lambda)=0$

$\Rightarrow \frac{1+3\lambda}{a}=\frac{4+\lambda}{b}=\frac{5\lambda-1}{6}=\frac{7-8\lambda}{-15}$

From last two ratios, $\lambda=-1$

$\Rightarrow \frac{-2}{a}=\frac{3}{b}=-1$

$\therefore a = 2, b = -3$

$\therefore$ Equation of plane is, $2x - 3y + 6z - 15 = 0$

Distance $= \frac{|6-6-6-15|}{7} = \frac{21}{7} = 3.$

36. **(0.75)** Given that lines are $\vec{r} = \lambda\hat{i}$ ......(1)

$\vec{r} = \mu(\hat{i} + \hat{j})$ ......(2)

$\vec{r} = \nu(\hat{i} + \hat{j} + \hat{k})$ ....(3)

These lines cut the plane $x + y + z = 1$ at points $A\ (\lambda, 0, 0)$, $B\ (\mu, \mu, 0)$ and $C\ (\nu, \nu, \nu)$ respectively

Since, A lies on plane $\Rightarrow \lambda = 1 \Rightarrow A\ (1, 0, 0)$

Since, B lies on plane $\Rightarrow \mu + \mu = 1 \Rightarrow \mu = \frac{1}{2}$

$\Rightarrow B\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

Since, C lies on plane $\Rightarrow \nu + \nu + \nu = 1 \Rightarrow \nu = \frac{1}{3}$

$\Rightarrow C\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$\text{Area } (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \frac{1}{2}\left|\frac{1}{6}\hat{i} + \frac{1}{6}\hat{j} + \frac{1}{6}\hat{k}\right| = \frac{1}{2} \times \frac{1}{6}\sqrt{3} = \frac{\sqrt{3}}{12}$$

$$\therefore (6\Delta)^2 = 36 \times \frac{3}{144} = \frac{3}{4} = 0.75$$