Authentic

SHORTCUTS, TIPS & TRICKS in PHYSICS

for JEE MAIN, ADVANCED and KVPY

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Includes:

- 250+ Chapter-wise & Topic-wise Short-cut Tips & Tricks to solve JEE Level Problems.
- 500+ Illustrations with Shortcut Solutions of JEE Level Questions including JEE Past Years Questions.
- Video Solutions of Selected JEE Level Questions.
- 325+ Chapter-wise JEE Level Questions Exercise with Accurate & Shortest Possible Solutions.
- Chapter-wise & Topic-wise 350+ Important Formulae & Key Theory Concepts

Er. D. C. Gupta
Er. Harsh Gupta

Strategic Book for
Class 11/ 12, Engineering & Medical Exams

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FOREWORD

The competitive exams like JEE test an aspirant's conceptual knowledge & how fast he/ she solve the problems with accuracy. So it becomes necessary that the students should know the short-cut methods in addition to the traditional methods of analysis. Keeping this in mind DISHA Publication brings a unique & innovative book **Authentic SHORTCUTS, TIPS & TRICKS in PHYSICS** for JEE Main, Advanced & KVPY to enable aspirants for advanced abilities to Solve KVPY, JEE Main & Advanced level Questions well within the stipulated time.

An earnest effort has been made to bring the book **AUTHENTIC SHORTCUTS-TIPS and TRICKS in Physics.** We have really worked hard researching for the Best Possible Tips, Tricks and Shortcut Solutions which students must know and can utilize in the examination hall.

- Shortcuts to help you in providing a different perspective to a concept/ problem thus strengthening your conceptual understanding.
- Tips provide you the Most Important Points to remember that aids in Conceptual Understanding & Problem Solving.
- Tricks empower you with magical tools that help you develop unique approaches to solve a problem.
- Shortcut Solutions provides alternate faster methods that save you a lot of time during examination.

The book encompasses 24 Chapters, which start with Review of Important Formulae, followed by Shortcuts, Tips & Tricks which are further followed by Illustrations demonstrating Shortcut Solutions. The book in all contains:

1. 250+ Chapter-wise & Topic-wise Shortcuts, Tips & Tricks to solve JEE Level Problems.
2. 500+ Illustrations with Shortcut Solutions of JEE Level Questions including JEE Past Years Questions.
3. Video Solutions of Selected JEE Level Questions.
4. 325+ Chapter-wise JEE Level Questions Exercise with Accurate & Shortest Possible Solutions.
5. Chapter-wise & Topic-wise 350+ Important Formulae.

This book provides you with hundreds of short-cut methods for the most conceptual and relevant problems. This book can also be used as a REVISION BOOK for various competitive exams. I hope that the book will fulfill the needs of the students for which it has been designed. We have made our best efforts to keep the book error-free but some errors might have crept in by mistake. We request our readers to highlight these errors and their fruitful suggestions so that we can keep on improving this book.

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No Matter where You Prepare from, keep this book as your companion. It would definitely improve your score by 25-30%.

CONTENTS

Units and Measurements 1

TOPIC: *Errors in Measurements and Dimensions.*

Review of Formulae

1. **The true value :** If $a_1, a_2, \ldots\ldots\ldots\ldots\ldots, a_n$ are the observed value of a quantity, then its true value is given by

$$\bar{a} = \frac{a_1 + a_2 + \ldots\ldots + a_n}{n} = \sum_{i=1}^{n} a_i$$

2. Absolute error = true value – observed value

or $\Delta a_i = a_i - \bar{a}$

3. Mean absolute error

$$\Delta\bar{a} = \frac{|\Delta a_1| + |\Delta a_2| + \ldots\ldots + |\Delta a_n|}{n} = \frac{1}{n}\sum_{i=1}^{n}|\Delta a_i|$$

4. Relative error = $\frac{\Delta\bar{a}}{\bar{a}}$, and percentage error

$$= \frac{\Delta\bar{a}}{a} \times 100$$

5. **Error in computed quantity**

(i) If $\pm\Delta x$ and $\pm\Delta y$ be the absolute errors in X and Y respectively and if $Z = X + Y$, then maximum possible error in Z; $\Delta z = \pm(\Delta x + \Delta y)$

(ii) If $Z = X - Y$, then $\Delta z = \pm(\Delta x + \Delta y)$

(iii) If $Z = XY$, then $\frac{\Delta z}{Z} = \pm\left[\frac{\Delta x}{X} + \frac{\Delta y}{Y}\right]$

(iv) If $Z = \frac{X}{Y}$, then $\frac{\Delta z}{Z} = \pm\left[\frac{\Delta x}{X} + \frac{\Delta y}{Y}\right]$

(v) If $Z = X^n$, then $\frac{\Delta z}{Z} = \pm n\left[\frac{\Delta x}{X}\right]$

(vi) If $Z = \frac{KX^aY^b}{W^c}$, then $\frac{\Delta z}{Z} = \pm\left[a\frac{\Delta x}{X} + b\frac{\Delta y}{Y} + c\frac{\Delta w}{W}\right]$

The absolute error has the same unit as the quantity itself, but fractional error has no unit.

6. If $Z = \frac{XY}{X+Y}$, then $\frac{\Delta z}{Z} = \pm\left[\frac{\Delta x}{X} + \frac{\Delta y}{Y} + \frac{(\Delta x + \Delta y)}{X+Y}\right]$

7. **Least count of vernier callipers**

   L.C. = Length of one division of main scale – length of one division of vernier scale

   or L.C. = $\frac{\text{Length of one division of main scale}}{\text{number of divisions on Vernier scale}}$

8. **Least count of a screw gauge**

$$\text{L.C.} = \frac{\text{Pitch}}{[\text{total number of divisions on the circular scale}]}$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. Plane angle and solid angle have units but no dimensions.
2. Resistance R, capacitance C and inductance L then CR, $\sqrt{LC}$ and $\frac{L}{R}$ have dimensions of time.
3. Dimensions of $\sqrt{\frac{1}{\mu_0 \epsilon_0}}$ are of speed. Also if E is the electric field and B is the magnetic field, then dimensions of E/B are of dimensions of speed.
4. Dimensions of $\frac{1}{2}\epsilon_0 E^2$, dimensions of $\frac{B^2}{2\mu_0}$ and $\frac{1}{2}Li^2$ are $ML^{-1}T^{-2}$t, and that of energy density (energy/vol.). Here i is the current.
5. Dimensions of $\sqrt{\frac{\mu_0}{\epsilon_0}}$ are of resistance R.
6. Dimensions of CB^2l^2 are of mass m. (Here C, capacitance, B magnetic field and l, length).
7. Dimensions of qvB, qE and $Bi\ell$ are of force, MLT^{-2}. ($q \to$ charge, $v \to$ velocity, $B \to$ magnetic field, $i \to$ current, $R \to$ length) $E \to$ electric field.

Dimensional Constants:

1. Dimensions of gravitational constant (G) : $[M^{-1}L^3T^{-2}]$
2. Dimensions of Plank's constant (h) : $[ML^2T^{-1}]$
3. Dimensions of gas constant (R) : $[ML^2T^{-2}k^{-1}\ \text{mol}^{-1}]$
4. Dimensions of coefficient of viscosity (η) : $[ML^{-1}T^{-1}]$

Dimensionless constants : Reynolds number (R_N), Mach number (M), Refractive index (μ), Relative density, Relative permittivity ($\in_r$), Relative permeability (μ_r).

Mathematical Ratios are Dimensionless

1. Trigonometric ratio: sin, cos, tan etc.

   [sin θ] → Dimensionless

2. [log x] → Dimensionless

3. [e^x] → Dimensionless

Physical Quantities Having Same Dimensions:

1. Distance, displacement, height, width, radius, wavelength, radius of gyration [L].
2. Force, weight, tension, thrust, energy gradient [MLT^{-2}].
3. Work, energy, moment of force [ML^2T^{-2}].
4. Force constant, surface tension [MT^{-2}].
5. Angular momentum, angular impulse, Plank's constant [ML^2T^{-1}].
6. Angular velocity, frequency, velocity gradient, decay constant [T^{-1}].
7. Stress, pressure, modulus of elasticity, energy density [$ML^{-1}T^{-2}$].
8. Wave number, power of lens, Rydberg's constant [L^{-1}].

Illustration 1

In the expression $\alpha = \frac{Fv^2}{\beta^2}\log_e\left(\frac{2\pi\beta}{v^2}\right)$, *find the dimensions of* α *and* β. *Here F is the force and v is the velocity.*

Short-cut solution :

Given, $\alpha = \frac{Fv^2}{\beta^2}\left[\log_e\left(\frac{2\pi\beta}{v^2}\right)\right]$ → Dimensionless

Dimensions of $\frac{2\pi\beta}{v^2} = 1$

$\therefore$ Dimensions of β = dimensions of v^2

$= [LT^{-1}]^2 = [L^2T^{-2}]$.

Dimensions of $\alpha = \frac{Fv^2}{\beta^2}$

$$= \frac{[MLT^{-2}][LT^{-1}]^2}{[L^2 - T^{-2}]^2}$$

$$= [ML^{-1}T^0]. \qquad \textbf{Ans.}$$

Illustration 2

Find the dimensions of length in terms of L, R, μ_0 and $\in_0$ as fundamental units.

Short-cut solution :

$$\text{Distance} = \text{speed} \times \text{time}$$

$$= \left(\sqrt{\frac{1}{\mu_0 \in_0}}\right) \times \left(\frac{L}{R}\right)$$

$$= [LR^{-1}\mu_0^{-1/2}\in_0^{-1/2}]. \qquad \textbf{Ans.}$$

Illustration 3

If E, the electric field, B the magnetic field, C the capacitance and L the inductance are taken as fundamental units, then find dimensions of radius of gyration in terms of these quantities.

Short-cut solution :

$$\text{Radius of gyration} = \text{Distance} = \text{speed} \times \text{time}$$

$$= (E/B) \times \sqrt{LC}$$

$$= E\,B^{-1}L^{1/2}C^{1/2}. \qquad \textbf{Ans.}$$

Illustration 4

If force 'F', energy 'E' and velocity 'V' are chosen as fundamental quantities, then find dimensions of acceleration.

Short-cut solution :

$$\text{Acceleration} = \frac{F}{m}$$

$$= \frac{F}{\left[\dfrac{\frac{1}{2}mv^2}{\frac{v^2}{2}}\right]} = \frac{F}{E/V^2}$$

$$= [FE^{-1}V^2]. \qquad \textbf{Ans.}$$

Illustration 5

The speed of light 'c', gravitational constant 'G' and Plank's constant 'h' are taken as the fundamental units in a system. Find the dimensions of length in this new system of units.

Short-cut solution :

Multiplying G and h and dividing by c^3, we have

$$\frac{Gh}{c^3} = \frac{[M^{-1}L^3T^{-2}][ML^2T^{-1}]}{[LT^{-1}]^3}$$

$$= L^2$$

$$\therefore \quad L = c^{-3/2}G^{1/2}h^{1/2}. \qquad \textit{Ans.}$$

Illustration 6

If energy E, velocity v and time T are chosen as fundamental units, the dimensions of surface tension will be:

(a) $[Ev^{-2}T^{-2}]$ *(b)* $[Ev^{-2}T^{2}]$ *(c)* $[Ev^{2}T^{-1}]$ *(d)* $[Ev^{-2}T^{-1}]$

Short-cut solution :

$$\text{Surface tension} = \frac{\text{force}}{\text{length}}$$

$$= \frac{(\text{energy/ length})}{\text{length}}$$

$$= (\text{energy/ length}^2)$$

$$= \text{energy/ (speed} \times \text{time})^2$$

$$= \frac{E}{v^2T^2} = [Ev^{-2}T^{-2}]. \qquad \textit{Ans. (a)}$$

Illustration 7

Let l, r, c and v represent inductance, resistance, capacitance and voltage respectively. The dimensions of $\frac{l}{rcv}$ ***in SI units will be :*** ***[JEE Main 2019]***

(a) $[LA^{-2}]$ *(b)* $[A^{-1}]$ *(c)* $[LTA]$ *(d)* $[LT^{2}]$

Short-cut solution :

As we know $\left[\frac{l}{r}\right] = [T]$

and $[cv] = \text{charge} = [AT]$

$$\therefore \quad \frac{l}{rcv} = \frac{T}{AT} = [A^{-1}]. \qquad \textit{Ans. (b)}$$

Illustration 8

If speed (v), acceleration (A) and force (F) are considered as fundamental units, the dimensions of Young's modulus will be: ***[JEE Main 2019]***

(a) $v^{-2}A^2F^{-2}$ ***(b)*** $v^{-2}A^2F^2$ ***(c)*** $v^{-4}A^{-2}F$ ***(d)*** $v^{-4}A^2F$

Short-cut solution :

$$\text{Young's modulus} = \frac{\text{stress}}{\text{strain}} = \frac{(\text{force/area})}{1}$$

$$= \frac{\text{force}}{\text{length}^2} = \frac{\text{force}}{(\text{speed}\times\text{time})^2}$$

$$= \frac{\text{force}}{\text{speed}^2 \times \left(\frac{\text{speed}}{\text{acc}}\right)^2} = \frac{\text{force}\times \text{acc}^2}{\text{speed}^4}$$

$$= [\text{F}v^{-4}\text{A}^2].$$ ***Ans. (d)***

Illustration 9

In terms of potential difference V, electric current I, permeability ε_0 ***permeability*** μ_0 ***and speed of light c, the dimensionally correct equation(s) is/ are:***

[JEE Adv. 2015]

(a) $\mu_0 I^2 = \varepsilon_0 V^2$ ***(b)*** $\mu_0 I^2 = \epsilon_0 V$ ***(c)*** $I = \varepsilon_0 cV$ ***(d)*** $\mu_0 cI = \varepsilon_0 V$

Short-cut solution :

Option (a) $$\frac{\mu_0}{\epsilon_0} = \left(\frac{V}{I}\right)^2$$

As $$\sqrt{\frac{\mu_0}{\epsilon_0}} = \text{R, resistance,}$$

also $$\frac{V}{I} = \text{R, resistance.}$$ So it is correct option.

Option (c) $$\frac{V}{I} = \frac{1}{\epsilon_0 c}$$

$$= \frac{1}{\epsilon_0\sqrt{\frac{1}{\mu_0\,\epsilon_0}}} = \sqrt{\frac{\mu_0}{\epsilon_0}}.$$ ***Ans. (a, c)***

Both sides of above equation are equal to *R*, so this option is also correct.

Illustration 10

If momentum (P), area (A) and time (T) are taken to be fundamental quantities, then energy has the dimensions of

(a) $[PA^{-1}T^{1}]$ (b) $[P^2AT]$ (c) $[PA^{-\frac{1}{2}}T]$ (d) $[PA^{1/2}T^{-1}]$

Short-cut solution :

$$\text{Energy} = \text{force} \times \text{displacement}$$
$$= \frac{\text{momentum}}{\text{time}} \times \sqrt{\text{area}}$$
$$= \frac{P}{T}\sqrt{A} = [PA^{1/2}T^{-1}].$$

Ans. (d)

Illustration 11

The speed of a body is measured with a positive error of (i) 2% (ii) 20%. If the mass of the body is known exactly, then find the error in the calculation of kinetic energy.

Short-cut solution :

(i) kinetic energy, $K = \frac{1}{2}mv^2$

$= c\,v^2$ (m has no error)

For $\frac{\Delta v}{v} \times 100 = 2\%$, we can write

$$\frac{\Delta K}{K} \times 100 = 2\left(\frac{\Delta v}{v} \times 100\right) = 2 \times 2\% = 4\%$$

Ans.

(ii) For $\frac{\Delta v}{v} \times 100 = 20\%$, we can write

$$\frac{\Delta K}{K} \times 100 = \left[2\left(\frac{\Delta v}{v}\right) + \left(\frac{\Delta v}{v}\right)^2\right] \times 100$$
$$= [2 \times 0.2 + 0.2^2] \times 100$$
$$= 44\%.$$

Ans.

Illustration 12

A student uses a simple pendulum of exactly 1 m length to determine g, the acceleration due to gravity. He uses a stop watch with the least count of 1 second for this and records 40 seconds for 20 oscillations. Find the maximum percentage error in the determination of value of g. ***[IIT-JEE - 2010]***

Short-cut solution :

We know that $T = 2\pi\sqrt{\frac{l}{g}}$

or $T^2 = (4\pi)^2 \frac{l}{g}$

or $g = (4\pi)^2 \frac{l}{T^2}$

$\therefore \quad \frac{\Delta g}{g}\times 100 = \frac{\Delta l}{l}\times 100 + 2\frac{\Delta T}{T}\times 100$

As the length of the string is exactly 1m, so $\Delta l = 0$,

also $\frac{\Delta T}{T} = \frac{\Delta t / n}{t / n} = \frac{\Delta t}{t} = \frac{1}{40}$

Now $\frac{\Delta g}{g}\times 100 = 2\frac{\Delta T}{T}\times 100$

$= 2\times\frac{1}{40}\times 100 = 5\%.$ ***Ans.***

Video Solution

Q. *The current voltage relation of diode is given by $I = (e^{1000V/T} - 1)$ mA, where the applied voltage V is in volt and temperature T is in kelvin. If a student makes an error measuring ± 0.01 V while measuring the current of 5mA at 300 K, what will be the error in the value of current in mA?* *[JEE-Main- 2014]*

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=oMddPiyIE04

Illustration 13

While measuring the length of the rod by vernier callipers the reading on main scale is 6.4 cm and the eight division on vernier is in line with marking on main scale division. If the least count of callipers is 0.01 and zero error –0.04 cm, find the length of the rod.

Short-cut solution :

Length of the rod = observed reading – zero error

= (Main scale division + Vernier scale division × *LC*) – Zero error

= (6.4 + 8 × 0.01) – (–0.04)

= 6.4 + 0.08 + 0.04 = 6.52 cm. ***Ans.***

Illustration 14

The circular head of a screw gauge is divided into 200 divisions and move 1 mm ahead in one revolution. Find the pitch and least count of the screw gauge. If the same instrument has a zero error of –0.05 mm and the reading on the main scale in measuring diameter of a wire is 6 mm and that on circular scale is 45, find the diameter of the wire.

Short-cut solution :

Pitch = 1 mm

Number of divisions on circular scale = 200

$$LC = \frac{\text{pitch}}{\text{number of divisions on circular scale}}$$

$$= \frac{1\text{mm}}{200} = 0.005 \text{ mm} = 0.0005 \text{ cm.} \qquad \textbf{\textit{Ans.}}$$

Diameter of the wire = (Main scale reading + Circular scale reading × LC) – Zero error

$$= 6 \text{ mm} + 45 \times 0.005 - (-0.05)$$
$$= 6 \text{ mm} + 0.225 \text{ mm} + 0.05 \text{ mm}$$
$$= 6.275 \text{ mm.} \qquad \textbf{\textit{Ans.}}$$

Concept Booster Exercise

1. Find the dimension of $\frac{B^2}{2\mu_0}$ ***[JEE Main 2020]***

(a) $ML^{-1}T^{-2}$ (b) ML^2T^{-2}
(c) $ML^{-1}T^2$ (d) $ML^{-2}T^{-1}$

2. Expression for time in terms of G (universal gravitational constant), *h* (Planck constant) and *c* (speed of light) is proportional to: ***[JEE Main 2019]***

(a) $\sqrt{\frac{hc^5}{G}}$ (b) $\sqrt{\frac{c^3}{Gh}}$ (c) $\sqrt{\frac{Gh}{c^5}}$ (d) $\sqrt{\frac{Gh}{c^3}}$

3. If L denotes the inductance of an inductor through which a current i is flowing, the dimensions of Li^2 are :

(a) ML^2T^{-2} (b) not expressible in MLT
(c) MLT^{-2} (d) $M^2L^2T^{-2}$

4. Planck's constant h, speed of light c and gravitational constant G are used to form a unit of length L and a unit of mass M. Then the correct option(s) is (are) ***[JEE Adv. 2015]***

(a) $M \propto \sqrt{C}$ (b) $M \propto \sqrt{G}$
(c) $L \propto \sqrt{h}$ (d) $L \propto \sqrt{G}$

5. Let ε_0 be the absolute permittivity, ε_r be the relative permittivity, μ_0 and μ_r be the absolute and relative permeability respectively. If M represents mass, L represents length, T represents time and I represents current, then :

(a) $[\varepsilon_0] = [M^{-1} L^{-3} T^{-2} I]$ (b) $[(\varepsilon_0 \mu_0)^2] = [M^0 L^{-4} T^4]$

(c) $[\varepsilon_r \mu_r] = [M^0 L^{-2} T^2]$ (d) $[\varepsilon_r] = [\mu_r]$

6. The value of resistance is 10.845 Ω and the value of current is 3.23 A. The potential difference is 35.02935 volt. Its value in significant number would be :

Numeric/Integer

(a) 35 V (b) 35.0 V

(c) 35.03 V (d) 35.029 V

7. In an experiment on simple pendulum to determine the acceleration due to gravity, a student measures the length of the thread as 632 cm and diameter of the pendulum bob as 2.256 cm. The student should take the length of the pendulum to be :

Numeric/Integer

[KVPY-2017]

(a) 64.328 cm (b) 64.36 cm

(c) 65.456 cm (d) 65.5 cm

8. A screw gauge advances by 3mm in 6 rotations. There are 50 divisions on circular scale. Find least count of screw gauge ? *[JEE Main 2020]*

Numeric/Integer

(a) 0.002 cm (b) 0.001 cm

(c) 0.01 cm (d) 0.02 cm

9. The following observations were taken for determining surface tension T of water by capillary method :

Diameter of capilary, $D = 1.25 \times 10^{-2}$ m

rise of water, $h = 1.45 \times 10^{-2}$ m

Using $g = 9.80$ m/s^2 and the simplified relation $T = \frac{rhg}{2} \times 10^3$ N/m, the possible error in surface tension is closest to :

Numeric/Integer

(a) 2.4% (b) 10%

(c) 0.15% (d) 1.5%

10. A spectrometer gives the following reading when use to measure the angle of a prism.
Main scale reading : 58.5 degree
Vernier scale reading : 09 divisions

Numeric/Integer

Given that 1 division on main scale corresponds to 0.5 degree. Total divisions on the vernier scale is 30 and match with 29 divisions of the main scale. The angle of the prism from the above data is: *[AIEEE 2012]*

(a) 58.77 degree (b) 58.65 degree

(c) 59 degree (d) 58.59 degree

11. The length and width of a rectangular room are measured to be 3.95 ± 0.05 m and 3.05 ± 0.05 m, respectively, the area of the floor is : *[KVPY -2017]*

Numeric/Integer

(a) 12.05 ± 0.01 m^2 (b) 12.05 ± 0.05 m^2

(c) 12.05 ± 0.34 m^2 (d) 12.05 ± 0.40 m^2

Solutions

1. **(a)** Energy density in magnetic field = $\frac{B^2}{2\mu_0}$

$$= \frac{\text{Force} \times \text{displacement}}{(\text{displacement})^3} = \frac{MLT^{-2} \cdot L}{L^3} = ML^{-1}T^{-2}.$$

2. **(c)** Let $t \propto G^x h^y C^z$

Dimensions of $G = [M^{-1}L^3T^{-2}]$,

$h = [ML^2T^{-1}]$ and $C = [LT^{-1}]$

$[T] = [M^{-1}L^3T^{-2}]^x[ML^2T^{-1}]^y[LT^{-1}]^z$

$[M^0L^0T^1] = [M^{-x+y}\,L^{3x+2y+z}\,T^{-2x-y-z}]$

By comparing the powers of M, L, T both the sides

$-x + y = 0 \Rightarrow x = y$

$3x + 2y + z = 0 \Rightarrow 5x + z = 0$ ..... (i)

$-2x - y - z = 1 \Rightarrow 3x + z = -1$ ..... (ii)

Solving eqns. (i) and (ii),

$x = y = \frac{1}{2}, z = -\frac{5}{2}$ $\quad\therefore t \propto \sqrt{\frac{Gh}{C^5}}$

3. **(a)** The dimensions of Li^2

$= ML^2T^{-2}A^{-2} \times A^2$

$= ML^2T^{-2}$

= dimensions of energy.

4. **(a, c, d)**

$L \propto h^x c^y G^z$

Dimensionally

$[M^0L^1T^0] = [ML^2T^{-1}]^x[LT^{-1}]^y[M^{-1}L^3T^{-2}]^z$

$M^0L^1T^0 = M^{x-z}\,L^{2x+y+3z}\,T^{-x-y-2z}$

$\therefore\; x - z = 0 \Rightarrow x = z$

$\therefore\; 2x + y + 3z = 1$ and $-x - y - 2z = 0$

On solving we get

$x = \frac{1}{2}, y = -\frac{3}{2}, z = \frac{1}{2}$

$\therefore\; L \propto \sqrt{h}$

$L \propto \sqrt{G}$

(c, d) are correct options

$M \propto h^x c^y G^z$

$M^1L^0T^0 \propto [ML^2T^{-1}]^x\,[LT^{-1}]^y\,[M^{-1}L^3T^{-2}]^Z$

$\therefore\; M^1L^0T^0 \propto M^{x-z}\,L^{2x+y+3z}\,T^{-x-y-2z}$

$\therefore\; x - z = 1$

$2x + y + 3z = 0$

$-x - y - 2z = 0$

On solving we get

$$x=\frac{1}{2}, y=\frac{1}{2}, z=-\frac{1}{2}$$

$\therefore\ M \propto \sqrt{C}$

(a) is the correct option.

5. **(a, b, d)** ε_r and μ_r are dimensionless.

6. **(b)** The significant number in the potential, $V = iR$; should be the minimum of either i or R. So corresponding to $i = 3.23\ A$, we have only three significant numbers in $V = 35.02935\ V$. Thus the result is $V = 35.0\ V$.

Error and Instrument

7. **(b)** Effective length should be taken up to centre of mass

$$l_c = 63.2 + \frac{2.256}{2} = 64.328$$

So, student should take the length of pendulum to be 64.3 cm.

By significant figures $l_c = 64.3$ cm.

8. **(b)** Pitch $= \frac{3}{6} = 0.5$ mm

$$\text{L.C.} = \frac{0.5\,\text{mm}}{50} = \frac{1}{100}\,\text{mm} = 0.01\ \text{mm} = 0.001\ \text{cm}$$

9. **(d)** Surface tension, $T = \frac{rhg}{2}\times 10^3$

Relative error in surface tension, $\frac{\Delta T}{T} = \frac{\Delta r}{r} + \frac{\Delta h}{h} + 0$

($\because$ g, 2 and 10^3 are constant)

Percentage error

$$100\times\frac{\Delta T}{T} = \left(\frac{10^{-2}\times 0.01}{1.25\times10^{-2}} + \frac{10^{-2}\times0.01}{1.45\times10^{-2}}\right)100$$

$$= (0.8 + 0.689)$$

$$= (1.489) = 1.489\% \cong 1.5\%$$

10. **(c)** Reading of Vernier = Main scale reading + Vernier scale reading × least count.

Main scale reading = 58.5

Vernier scale reading = 09 division

least count of Vernier = 0.5°/30

Thus $R = 58.5° + 9 \times \frac{0.5°}{30}$

$R = 58.65$

11. **(c)** *Area*, $A = bl = 3.95 \times 3.05 = 12.05\ \text{m}^2$

$$\therefore\ \frac{\Delta A}{A} = \frac{\Delta b}{b} + \frac{\Delta l}{l} = \frac{0.05}{3.05} + \frac{0.05}{3.95} = 0.016 + 0.012 = 0.028$$

$\therefore\ \Delta A = 0.028 \times 12.05$ or $12.05 \pm 0.34\ \text{m}^2$

Motion in a Straight Line

2

TOPIC: *Equations of Uniformly Accelerated Motion, Uniform and Non-uniform Motion.*

Review of Formulae

$$v_{av} = \int_{t_1}^{t_2} \frac{vdt}{(t_2 - t_1)};\ v = u + at,\ s = ut + \frac{1}{2}at^2,\ v^2 = u^2 + 2as,\ s_n = u + \frac{a}{2}(2n-1)$$

Tips and Tricks for Shortcut Solutions

1. Uniform motion is one in which speed is constant, velocity may change. As in case of uniform circular motion.
2. In uniform motion along a straight line

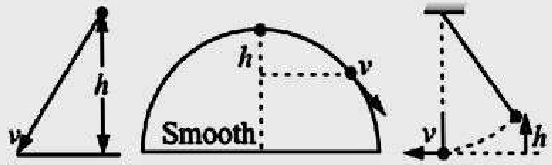
$$\frac{d|\vec{v}|}{dt} = 0 \text{ and } \left|\frac{d\vec{v}}{dt}\right| = 0.$$

3. General equations of motion are;

$$\vec{a} = \frac{d\vec{v}}{dt} \text{ and } \vec{v} = \frac{d\vec{s}}{dt}.$$

4. If particle starts from rest with constant acceleration then distances moved in successive seconds will be 1 : 3 : 5 : ....
5. The displacement in last second of motion of a particle projected vertically upward is independent of its initial velocity. It always be g/2.
6. If a body thrown up from some height with some velocity and another thrown down with same velocity, both will hit the ground with equal velocity.
7. If air resistance is taken into account, then time of descend will be greater than time of ascend.
8. A particle is projected upward, if t_1 and t_2 are the times at which it is at a height 'h' while ascending and descending respectively, then $h = \frac{1}{2}gt_1t_2$.

9. The speed of the body after falling a vertical distance h will be $v = \sqrt{2gh}$. It does not depend on the path followed



10. A particle first accelerates with α and then retards with β along a straight line and stops. If total time of motion is t, then displacement

$$s = \frac{\alpha\beta t^2}{2(\alpha+\beta)}.$$

More Tricks for Short-cut Solutions:

1. Time taken by bolt to hit the floor of the lift going up or down $h = \frac{1}{2}(g \pm a)t^2$.

   Use $+a$, when lift accelerating up or retarding down. If lift moves with constant velocity, then $a = 0$.
2. Time taken to cross or overtake the trains of lengths ℓ_1 and ℓ_2, with initial separation x_0 is:

   $t = \frac{(x_0 + \ell_1 + \ell_2)}{v_2 \pm v_1}$. Use '–' when moves in same direction.
3. If two bodies are thrown vertically upwards with velocities v_1 and v_2, then separation between them at any time; $s = (v_1 \sim v_2)t$, when moving in the same direction.
4. The number of trips of a bird between two oppositely moving cars will always be infinite, irrespective of their speeds.

Illustration 1

A bird flies for 4 s with a velocity $v = (t - 2)$ m/s, in a straight line, where t = time in second. Calculate the distance covered by bird.

Short-cut solution :

$$v = |t - 2|.$$

At $t = 0$, $v = 2$ m/s and $t = 2$s, $v = 0$.

At $t = 4$ s, $v = 2$ m/s, speed-time graph is shown in figure.

Distance = area of speed-time

$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2$

$= 4$ m. *Ans.*

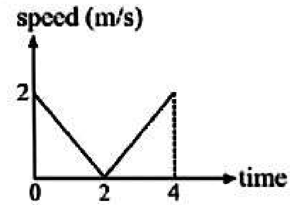


Illustration 2

A bird is tossing (flying to and fro) between two cars moving towards each other on a straight road. One car has a speed of 18 km/h while the other has the speed of 27 km/h. The bird starts moving from first car towards the other and is moving with the speed of 36 km/h and when the two cars were separated by 36 km. What is the total distance covered by the bird till cars collide?

Short-cut solution :

Velocity of approach of cars = 18 + 27 = 45 km/h

Distance between the cars = 36 km

$\therefore$ Time of meeting the cars (t) = $\dfrac{\text{distance between the cars}}{\text{relative speed of car}} = \dfrac{36}{45}$

$= \dfrac{4}{5}\text{h} = 0.8 \text{ h}.$

Speed of the bird (v_b) = 36 km/h

$\therefore$ Distance covered by the bird = $v_b \times t$

= 36 × 0.8

= 28.8 km. ***Ans.***

Illustration 3

Bodies, A and B are thrown vertically upward with velocities 5 m/s and 10 m/s respectively. Find separation between them after one second.

Short-cut solution :

Using, $s = (v_2 - v_1)t$

$= (10 - 5) \times 1 = 5\text{m}.$ ***Ans.***

Illustration 4

The engineer of a train moving at a speed v_1 sights a freight train a distance d ahead of him on the same track moving in the same direction with a slower speed v_2. He puts on the brakes and gives his train a constant deceleration α. Find the minimum value of d at which brakes are applied so as to avoid collision.

Solution :

Collision will be avoided if speed of the train v_1 becomes equal to v_2 in travelling a relative distance d. Therefore final relative speed of trains becomes zero. The initial relative speed $v_{12} = v_1 - v_2$. By third equation of motion, we have

$$v_{12}^2 = u_{12}^2 - 2\,a_{12}s$$

$$0 = (v_1 - v_2)^2 - 2(\alpha - 0)\,d$$

or $$d = \frac{(v_1 - v_2)^2}{2\alpha}.$$

The collision can be avoided if $d \geq \dfrac{(v_1 - v_2)^2}{2\alpha}$. **Ans.**

Video Solution

Illustration 5

A car accelerates from rest at a constant rate 'α' for some time, after which it decelerates at a constant rate 'β' to come to rest. If the total time elapsed is t second, then calculate

(i) the maximum velocity attained by the car, and

(ii) the total displacement travelled by car in terms of α, β and t.

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=TPfnFcCoicU

Solution :

Let car accelerates upto time t_1 and then retards for $t - t_1$.

If v is the maximum velocity attained by the car, then

$$v = u + \alpha t_1$$

$$= 0 + \alpha t_1 \quad \text{...(i)}$$

Also $$0 = v - \beta(t - t_1)$$

or $$v = \beta(t - t_1) \quad \text{...(ii)}$$

From above equations, we get

$$t_1 = \left(\frac{\beta t}{\alpha + \beta}\right) \text{ and } t - t_1 = t - \frac{\beta t}{\alpha + \beta} = \left(\frac{\alpha t}{\alpha + \beta}\right)$$

(i) Maximum velocity, $v_{max} = v = \alpha t_1 = \left(\dfrac{\alpha\beta t}{\alpha + \beta}\right)$

(ii) Now total displacement

$$s = s_1 + s_2$$

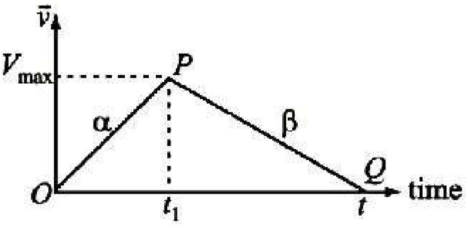
$$= \frac{1}{2}\alpha t_1^2 + \frac{1}{2}\beta(t - t_1)^2$$

$$= \frac{1}{2}\left(\frac{\beta t}{\alpha + \beta}\right)^2 + \frac{1}{2}\beta\left(\frac{\alpha t}{\alpha + \beta}\right)^2$$

$$= \frac{\alpha\beta t^2}{2(\alpha + \beta)}. \quad \textbf{Ans.}$$

Graphical solution :

Let v_{max} be the maximum velocity attained and t_1 be the time at which maximum velocity will occur. The velocity *vs* time graph can be drawn as follows:



The slope of line OP,

$$\alpha = \frac{v_{max}}{t_1}$$

$$\Rightarrow \quad v_{max} = \alpha t_1 \quad \ldots(i)$$

The slope of line PQ,

$$\beta = \frac{v_{max}}{t - t_1}$$

$$\Rightarrow \quad v_{max} = \beta(t - t_1). \quad \ldots(ii)$$

From equations (i) and (ii), we get

$$\alpha t_1 = \beta(t - t_1)$$

which gives $$t_1 = \frac{\beta t}{\alpha + \beta}. \quad \ldots(iii)$$

Substituting value of t_1 in equation (i), we get

(i) $$v_{max} = \frac{\alpha\beta t}{\alpha + \beta}. \quad \textit{Ans.}$$

(ii) Total displacement $$\vec{s} = \text{area of } \vec{v} - t \text{ graph}$$

$$= \frac{1}{2} \times v_{max} \times t$$

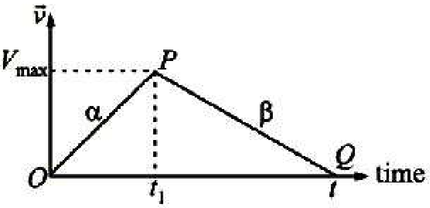
$$= \frac{1}{2} \times \frac{\alpha\beta t}{\alpha + \beta} \times t = \frac{\alpha\beta t^2}{2(\alpha + \beta)} \quad \textit{Ans.}$$

Illustration 6

The distance (s) between two stations is to be covered in minimum time. The maximum value of acceleration or retardation of a car can not exceed α and β respectively. Find the time of motion.

Short-cut solution :

To cover the distance in minimum time the car must get the maximum possible acceleration α and then retard to maximum possible value *b*. Let t_1 is the time up to which car accelerates and *t* is the required time of motion. The velocity-time graph of motion of car can be drawn as in figure.



We know that $$s = \frac{\alpha\beta t^2}{2(\alpha + \beta)}.$$

Solve above equation for *t*, we have

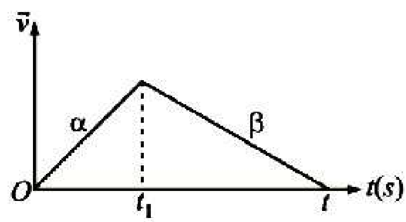
$$t = \sqrt{\frac{2s(\alpha + \beta)}{\alpha\beta}} = \sqrt{2s\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)}. \quad \textit{Ans.}$$

Illustration 7

A particle of mass m moves on x-axis as follows: It starts from rest at $t = 0$ from the point $x = 0$ and comes to rest at $t = 1$ and the point $x = 1$. No other information is available about its motion at intermediate times ($0 \le t \le 1$). Discuss about the acceleration of the particle.

Short-cut solution :

Let α and β are the acceleration and retardation of the particle during the motion. The velocity-time graph of motion of a particle is shown in figure.



We have $s = x = \dfrac{\alpha\beta t^2}{2(\alpha+\beta)}$

Given $x = 1$m, $t = 1s$

let $|\alpha| = |\beta|$, then $1 = \dfrac{\alpha^2(1)^2}{2(\alpha+\alpha)}$

which gives $\alpha = 4$

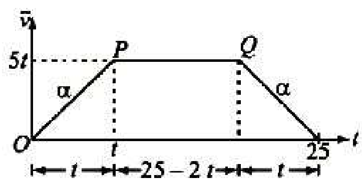
When $|\alpha| > |\beta|$ then $|\alpha|$ will be greater than 4, and $|\beta|$ will be less than 4. *Ans.*

Illustration 8

A car starts moving rectilinearly, first with acceleration $\alpha = 5$ m/s^2 (the initial velocity is equal to zero), then uniformly, and finally, decelerating at the same rate α comes to a stop. The total time of motion equals $\tau = 25$ s. The average velocity during that time is equal to $\langle v \rangle = 72$ km/h. How long does the car move uniformly?

Short-cut solution :

Let t be the time upto which car accelerates or decelerates. The maximum velocity attained in this duration is $5t$. The time upto which car move uniformly $= 25 - 2t$. The velocity – time graph of the motion of car is drawn as in figure.



Given the average velocity in whole time of motion

$$v_{av} = \frac{72 \times 5}{18} = 20 \text{ m/s.}$$

The average velocity from the graph can be obtained as

$$v_{av} = \frac{\text{total displacement}}{\text{total time}}$$

$$= \frac{\text{area of } \vec{v} - t \text{ graph}}{\text{total time}}$$

$$\therefore \quad 20 = \frac{\frac{1}{2}\times[25+(25-2t)]\times 5t}{25}$$

$$= \frac{\frac{1}{2}\times[50-2t)]\times 5t}{25}$$

or $$200 = 50t - 2t^2$$

or $$t^2 - 25t + 100 = 0$$

$$(t-20)(t-5) = 0$$

$$t = 5\text{ s} \quad \text{or} \quad 20\text{ s}$$

But $t = 20$ is not possible

$$\therefore \quad t = 5\text{ s.}$$

The time upto which car moves uniformly

$$= 25 - 2t = 25 - 2 \times 5$$

$$= 15\text{ s.}$$ ***Ans.***

Illustration 9

An elevator, in which a man is standing, is moving upward with a constant acceleration of 1m/s². At some instant when speed of elevator is 10 m/s, the man drops a coin from a height of 2 m. Find the time taken by the coin to reach the floor. (g = 9.8 m/s²)

Short-cut solution :

Using, $$h = \frac{1}{2}(g+a)t^2$$

$$\therefore \quad t = \sqrt{\frac{2h}{g+a}}$$

$$= \sqrt{\frac{2\times 2}{9.8+1}} = 0.61\text{ s.}$$ ***Ans.***

Illustration 10

A balloon is rising vertically upwards with uniform acceleration 15.7 m/s². A stone is dropped from it. After 4 s another stone is dropped from it. Find the distance between the two stones 6 second after the second stone is dropped.

Solution :

Consider motion of stones with respect to the balloon. At the instant of release of stones, the initial velocity of both stones w.r.t. the balloon is zero. The acceleration of stone w.r.t. the balloon

$$[a_{stone}]_{balloon} = g - (-a) = g + a$$

Now $$s_1 = 0 + \frac{1}{2}[a_{stone}]_{balloon}\, t_1^2$$

where $t_1 = (4 + 6) = 10$ s

$$\therefore \quad s_1 = \frac{1}{2}(g + a) \times 10^2$$

$$= \frac{1}{2}(9.8 + 15.7) \times 10^2 \text{ m}$$

and $$s_2 = 0 + \frac{1}{2}(g + a)\, t_2^2$$

where $t_2 = 6s$

$$\therefore \quad s_2 = \frac{1}{2}(9.8 + 15.7) \times 6^2$$

The distance between s_1 and s_2 :

$$s = s_1 - s_2$$

$$= \frac{1}{2}(9.8 + 15.7)\,[10^2 - 6^2]$$

$$= 816 \text{ m.}$$ ***Ans.***

Illustration 11

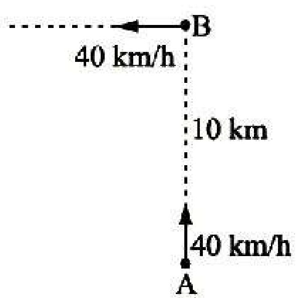
Two ships are 10 km apart on a line from south to north. The one farther north is moving towards west at 40 km/h and other is moving towards north at 40 km/h. What is the distance of closest approach and how long do they take to reach it?

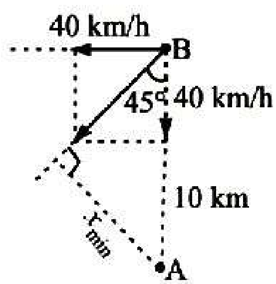
Short-cut solution :

With respect to ship A, velocity of ship B,

$$v_{BA} = \sqrt{40^2 + 40^2}$$

$$= 40\sqrt{2} \text{ km/h}$$





The shortest distance of B from A

$$x_{min} = \frac{10}{\sqrt{2} \text{ km}}$$

Time, $$t = \frac{x_{min}}{v_{BA}} = \frac{10/\sqrt{2}}{40\sqrt{2}} = \frac{1}{8}h.$$ ***Ans.***

Illustration 12

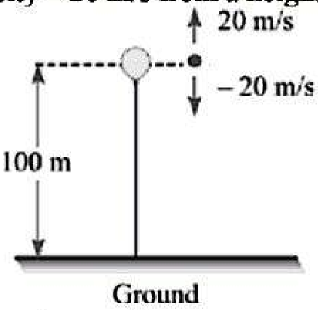
A balloon starts ascending at a constant acceleration of 2m/s². When it was at a height of 100 m from the ground, the food packet is dropped from the balloon. After how much time and with what velocity does it reach the ground ? Take g = 10 m/s².

Short-cut solution :

The velocity of balloon at the height of 100 m;

$$v^2 = 0 + 2 \times 2 \times 100 \text{ or } v = 20 \text{ m/s}$$

Take this velocity – 20 m/s along downward direction. Now consider the food packet is falling with initial velocity – 20 m/s from a height 100 m.



Therefore from second equation, we have

$$h = ut + \frac{1}{2}gt^2$$

or $$100 = -20t + \frac{1}{2} \times 10 \times t^2$$

Solving for t, we get $$t = 2 + \sqrt{24} \text{ s.}$$

The velocity with which it strikes the ground

$$v^2 = u^2 + 2gh = (-20)^2 + 2 \times 10 \times 100$$

or $$v = 49 \text{ m/s}.$$ ***Ans.***

Illustration 13

A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1 m long and requires 1 s. The time when he will fall into the pit 13 m away from him is :

(a) *13 s* *(b)* *37 s* *(c)* *40 s* *(d)* *42 s*

Short-cut solution :

The time taken to move net 2 steps (5 steps forward and 3 steps backward) is 8s, and so for 8 steps he takes 32 s. In last 5 steps he will take 5s and fall into the pit.

$\therefore$ Total time = 32 + 5 = 37 *s*. ***Ans. (b)***

Illustration 14

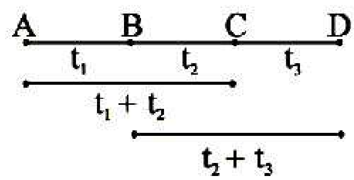
A point moves with uniform acceleration and v_1, v_2 and v_3 denote the average velocities in three successive intervals of time t_1, t_2 and t_3. Which of the following relations is correct?

(a) $v_1 - v_2 : v_2 - v_3 = t_1 - t_2 : t_2 + t_3$ *(b)* $v_1 - v_2 : v_2 - v_3 = t_1 + t_2 : t_2 + t_3$

(c) $v_1 - v_2 : v_2 - v_3 = t_1 - t_2 : t_1 - t_3$ *(d)* $v_1 - v_2 : v_2 - v_3 = t_1 - t_2 : t_2 - t_3$

Short-cut solution :

At point A, we can take velocity v_1 and at point C we can take v_2, so change in velocity in $t_1 + t_2$ becomes $v_2 - v_1$.



The acceleration, $$a = \frac{v_2 - v_1}{t_1 + t_2}$$

Similarly velocity at point B can be taken as v_2 and v_3 at D, so change in velocity becomes $v_3 - v_2$ in $(t_2 + t_3)$.

The acceleration, $a = \dfrac{v_3 - v_2}{t_2 + t_3}$

As acceleration is same throughout, so

$$\frac{v_2 - v_1}{t_1 + t_2} = \frac{v_3 - v_2}{t_2 + t_3}$$

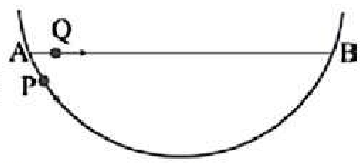
or $$\frac{v_1 - v_2}{t_1 + t_2} = \frac{v_2 - v_3}{t_2 + t_3}.$$ ***Ans. (b)***

Concept Booster Exercise

1. A point traversed half the distance with a velocity v_0. The remaining part of the distance was covered with velocity v_1 for half the time, and with velocity v_2 for the other half of the time. Find the mean velocity of the point averaged over the whole time of motion.
2. If a point moves in a straight line with uniform acceleration and covers successive equal distances in times t_1, t_2, t_3, then show that

$$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}.$$

3. A particle P is sliding down a frictionless hemispherical bowl. If it passes the point A at $t = 0$. At this time, the horizontal component of its velocity is v. A bead Q of the same mass as P is ejected from a long horizontal string AB, with the speed v. Friction between the bead and the string may be neglected. If t_p and t_Q be the respective times taken by P and Q to reach the point B, then :



(a) $t_p < t_Q$ (b) $t_p = t_Q$

(c) $t_p > t_Q$ (d) $\dfrac{t_p}{t_Q} = \dfrac{\text{length of arc ABC}}{\text{length of chord AB}}$

4. The relation between time t and distance x is $t = \alpha x^2 + \beta x$ where α and β are constants. The retardation

(a) 2 αv3 (b) 2 βv3 (c) 2 αβv3 (d) 2 β3v3

where v is the velocity

5. Four persons K, L, M and N are initially at the corners of a square of side of length d. If every person starts moving with velocity v, such that K is always headed towards L, L towards M, M is headed directly towards N and N towards K, then the four persons will meet after :

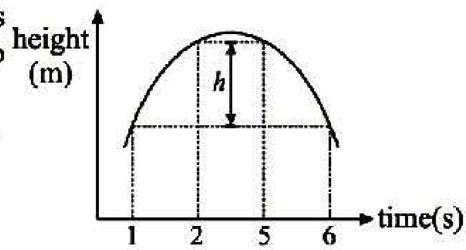
(a) $\dfrac{d}{v}$ sec (b) $\dfrac{\sqrt{2}d}{v}$ sec (c) $\dfrac{d}{\sqrt{2}v}$ sec (d) $\dfrac{d}{2v}$ sec

6. A particle moves with a velocity $(3\hat{i} + 4\hat{j})$ m/s from origin. The displacement of particle along line $x = y$ after two seconds will be: **Numeric/Integer**

(a) 10 m (b) $\frac{7}{\sqrt{2}}$

(c) $7\sqrt{2}$ m (d) none of these

7. A ball is thrown upwards. Its height varies with time as follows. If the acceleration due to gravity is 7.5 m/s^2, then the height h is : **Numeric/Integer**



(a) 10 m
(b) 15 m
(c) 20 m
(d) 25 m

8. Starting from rest a particle moves in a straight line with acceleration $a = \{2 + |t - 2|\}$ m/s^2. Velocity of particle at the end of 4 s will be **Numeric/Integer**

(a) 16 m/s (b) 20 m/s

(c) 8 m/s (d) 12 m/s

9. A particle moves according to law $x = a \cos \pi t$. The distance covered by it in 2.5 s is

(a) $2a$ (b) $3a$

(c) $4a$ (d) $5a$

10. A person walks up a stalled escalator in 90 s. When standing on the same escalator, now moving, he is carried in 60 s. The time it would take him to walk up the moving escalator will be **Numeric/Integer**

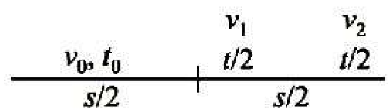
(a) 27 s (b) 72 s

(c) 18 s (d) 36 s

Solutions

1. Let t_0 and t be the time of motion of first half of distance and rest half of the distance respectively, then

$$v_{av} = \frac{s/2 + s/2}{t_0 + t}$$



where $t_0 = \frac{s/2}{v_0}$, and $\frac{v_1 t}{2} + \frac{v_2 t}{2} = \frac{s}{2}$

or $t = \frac{s}{v_1 + v_2}$

$\therefore$
$$v_{av} = \frac{s}{\frac{s/2}{v_0} + \frac{s}{(v_1 + v_2)}}$$

$$= \frac{2v_0(v_1 + v_2)}{2v_0 + v_1 + v_2}.$$

2. Let s be the successive equal distances and v_1, v_2, v_3 the initial velocities for the successive distances and v_4, the final velocity in the third distance. Since the acceleration is constant, so velocities in the three intervals and in the total time are

$\frac{v_1+v_2}{2}, \frac{v_2+v_3}{2}, \frac{v_3+v_4}{2}$ and $\frac{v_1+v_4}{2}$.

We know that average velocity

$$v_{av} = \frac{\text{distance}}{\text{time}},$$

∴ $$\frac{v_1+v_2}{2} = \frac{s}{t_1}, \quad \text{...(i)}$$

$$\frac{v_2+v_3}{2} = \frac{s}{t_2}, \quad \text{...(ii)}$$

$$\frac{v_3+v_4}{2} = \frac{s}{t_3}, \quad \text{...(iii)}$$

and $$\frac{v_1+v_4}{2} = \frac{3s}{t_1+t_2+t_3} \quad \text{...(iv)}$$

Doing (i) – (ii) + (iii), we get

$$\frac{v_1+v_4}{2} = s\left[\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3}\right] \quad \text{...(v)}$$

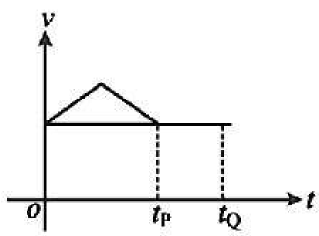
Now from equations (iv) and (v), we get

$$\frac{1}{t_1}-\frac{1}{t_2}+\frac{1}{t_3} = \frac{3}{t_1+t_2+t_3}.$$

3. **(a)** Both the particle has same displacement. We can draw their velocity-time graph as follows:

For the same displacement, area of v-t graph for both are equal, so

$$t_P < t_Q.$$



4. **(a)** Given $t = \alpha x^2 + \beta x$

Differentiating above equation w.r.t. time, we get

$$1 = \alpha \times 2x\frac{dx}{dt} + \beta\frac{dx}{dt}$$

or $$1 = 2\alpha x v + \beta v,$$

∴ $$\frac{1}{v} = 2\alpha x + \beta \quad \text{...(i)}$$

Differentiating again, we get

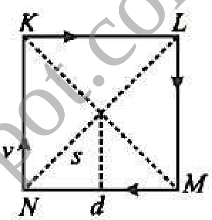
$$0 = 2\alpha(x\frac{dv}{dt} + v\frac{dx}{dt}) + \beta\frac{dv}{dt}$$

or $$0 = 2\alpha(xa + v^2) + \beta a \quad ...(ii)$$

From above equations, we get

$$a = -2\alpha v^3.$$

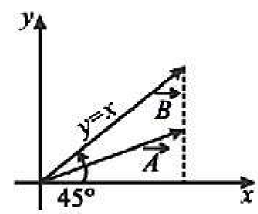
5. (a) $$s = \frac{d/2}{\cos 45°} = \frac{d}{\sqrt{2}}$$



$$v_{\text{effective}} = v\cos 45° = \frac{v}{\sqrt{2}}.$$

Time, $$t = \frac{s}{v_{\text{effective}}} = \frac{d/\sqrt{2}}{v/\sqrt{2}} = \frac{d}{v}.$$

6. (c) If $\vec{A}$ is the displacement along the velocity vector, then



$$\vec{A} = 2\vec{v} = (6\hat{i} + 8\hat{j}).$$

Unit vector along line $y = x$,

$$\vec{B} = \cos 45°\hat{i} + \sin 45°\hat{j}$$

$$= \left(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}\right).$$

Thus the displacement along $\vec{B}$

$$A\cos\theta = \frac{AB\cos\theta}{B} = \frac{\vec{A}\cdot\vec{B}}{B}$$

$$= \frac{(6\hat{i} + 8\hat{j})\cdot(\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}})}{1}$$

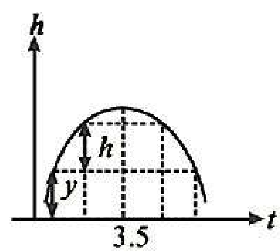
$$= 7\sqrt{2} \text{ m.}$$

7. (b) If u is the velocity of projection, then

$$0 = u - a \times 3.5$$

or $$u = 3.5\,a = 3.5 \times 7.5$$

$$= 26.25 \text{ m/s}$$



Now $$y = u \times 1 - - \times$$

$$= 26.25 - \frac{1}{2} \times 7.5 \times 1^2 = 22.5 \text{ m}$$

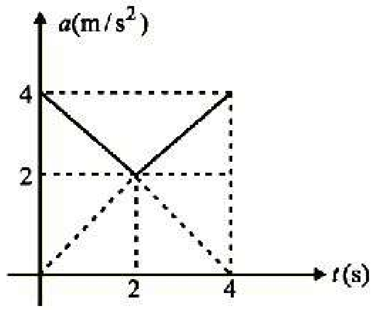
Also $$y + h = u \times 2 - \frac{1}{2} \times a \times 2^2$$

$$= 26.25 \times 2 - \frac{1}{2} \times 7.5 \times 2^2$$

$\therefore \quad h = 15$ m.

8. **(d)** Acceleration can be written as $a = 2 + 2 - t$ or $a = 4 - t$ for $t \le 2s$ and $a = 2 + t - 2$ or $a = t$ for $t \ge 2$ s

Therefore, acceleration time graph of the particle will be as shown below



$$v_f - v_i = \text{area under } (a\text{-}t) \text{ graph}$$

or
$$v_f - 0 = (4 \times 4) - \frac{1}{2}(4)(2)$$
$$= 12 \text{ m/s}$$

9. **(d)** $x = a \cos \pi t$ represents periodic motion with a period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2s$. Thus in 0.5 second, it covers a distance a. So in 2.5 s, it covers a distance 5 a.

10. **(d)** If u and v are the speeds of the person and of the escalator, then

$$\frac{\ell}{u} = 90, \qquad \therefore u = \frac{\ell}{90}$$

and
$$\frac{\ell}{v} = 60, \qquad \therefore v = \frac{\ell}{60}$$

If t is the required time, then

$$t = \frac{\ell}{u+v} = \frac{\ell}{\frac{\ell}{90} + \frac{\ell}{60}} = 36s.$$

Motion in a Plane 3

TOPIC*: *Vector Analysis and Projectile Motion.

Review of Formulae

$$T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g} \; ; \; H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g}$$

$$R = \frac{2u_x u_y}{g} = \frac{u^2\sin 2\theta}{g} ; \; R_{max} = \frac{u^2}{g}, \textit{for } \theta = 45°$$

If T_1 and T_2 are the time of flights for same range R, then $T_1T_2 = \frac{2R}{g}$.

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. If vectors $\vec{A}$ and $\vec{B}$ are perpendicular, then $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$
2. If $\vec{A}+\vec{B}+\vec{C}=0$, the given vectors must be coplanar
3. If $\vec{A}+\vec{B}+\vec{C}=0$, then we may write

   $\vec{A}+\vec{B}=-\vec{C}$ or $|\vec{A}-\vec{B}| \le C \le |\vec{A}+\vec{B}|$
4. For vector $\vec{R} = R_x\hat{i}+R_y\hat{j}+R_z\hat{k}$, if α, β and γ are the angles made by $\vec{R}$ from x, y and z - axis respectively, then

   $\cos\alpha = \frac{R_x}{R}$, $\cos\beta = \frac{R_y}{R}$ and $\cos\gamma = \frac{R_z}{R}$

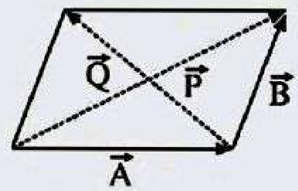
   And $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
5. (i) For perpendicular vectors $\vec{A}\cdot\vec{B}=0$

   (ii) $(\vec{A}+\vec{B})$ and $(\vec{A}\times\vec{B})$ are perpendicular.
6. Unit vector perpendicular to vectors $\vec{A}$ and $\vec{B}$

   $$\hat{n} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}.$$

7. Area of parallelogram

$= |\vec{A}\times\vec{B}| = \left|\frac{\vec{P}+\vec{Q}}{2}\times\frac{\vec{P}-\vec{Q}}{2}\right|$



8. Volume of parallelepiped = $[\vec{A}\,\vec{B}\,\vec{C}]$

9. If scalar triple product $[\vec{A}\,\vec{B}\,\vec{C}] = 0$, the given vectors will be coplanar.

10. If $\vec{A}+\vec{B}=\vec{C}$ and $A^2+B^2=C^2$, then $\vec{A}$ and $\vec{B}$ are perpendicular.

11. If $|\vec{A}|+|\vec{B}|=|\vec{C}|$, then vectors $\vec{A}$ and $\vec{B}$ are parallel vectors.

12. If a particle starts with accelerations in two mutually perpendicular directions, its path will be straight line, $y=\left(\frac{a_y}{a_x}\right)x$.

13. If a particle has constant velocity v in one direction and acceleration in perpendicular direction, then its path will be parabolic,

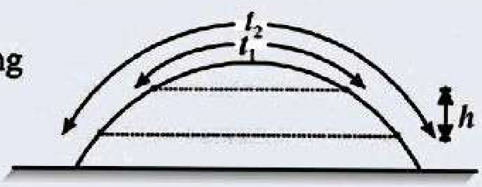
$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$.

14. The maximum height, H and range R are related as: $R = 4H\cot\theta$

15. If horizontal range, $R = H$, the maximum height, then angle of projection $\theta = \tan^{-1}4$.

16. If t_1 and t_2 are the times corresponding to the same levels, then

$h = \frac{g(t_2^2 - t_1^2)}{8}$.



17. If a boy can thrown a ball to a maximum distance 'd', then he can thrown the ball to a maximum height $d/2$.

18. If a particle passes two points situated at equal height y at t_1 and t_2, then $y = \frac{1}{2}gt_1t_2$.

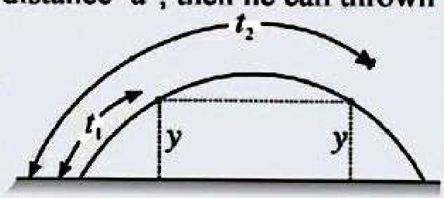


Illustration 1

Which group of forces can give zero resultant?

(a) *5, 10, 16* *(b)* *5, 8, 14* *(c)* *6, 9, 15* *(d)* *10, 20, 31*

Short-cut solution :

If using, $|\vec{A}-\vec{B}| \le |\vec{C}| \le |\vec{A}+\vec{B}|$

In option (c), let A = 6 and B = 9, then $|\vec{A}-\vec{B}| = 3$

and $|\vec{A}+\vec{B}| = 6+9 = 15$

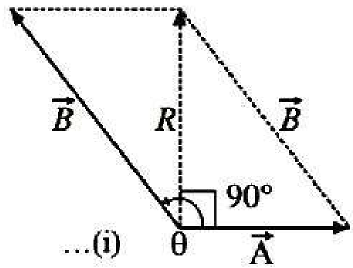
As $|\vec{C}| = 15$ lies from 3 to 15, therefore option (c) is correct. ***Ans. (c)***

Illustration 2

The sum of the magnitudes of two forces at a point is 18N and the magnitude of their resultant is 12 N. If the resultant makes an angle 90° with the force of smaller magnitude, the magnitude of the forces are:

(a) *5N, 13N* *(b)* *6N, 12N* *(c)* *8N, 10N* *(d)* *7N, 11N*

Short-cut solution :



From geometry, we can write

$$B^2 = R^2 + A^2$$

or $$B^2 - A^2 = R^2$$

or $$(B + A)(B - A) = 12^2$$

Given, $$A + B = 18 \quad \text{...(i)}$$

or $$18(B - A) = 144$$

or $$B - A = 8 \quad \text{...(ii)}$$

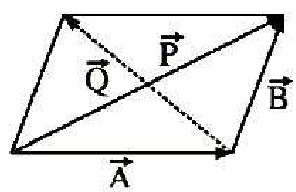
On solving above equations, we get

$A = 5$ N and $B = 13$ N. ***Ans. (a)***

Illustration 3

The diagonals of a parallelogram are given by vectors $3\hat{i} + \hat{j} + 2\hat{k}$ ***and*** $i - 3\hat{j} + 4\hat{k}$. ***Find the area of parallelogram.***

Short-cut solution :



Here, $$\frac{\vec{P}+\vec{Q}}{2} = \frac{[(3\hat{i} + \hat{j} + 2\hat{k}) + (\hat{i} - 3\hat{j} + 4\hat{k})]}{2}$$

$$= \frac{4\hat{i} - 2\hat{j} + 6\hat{k}}{2}$$

$$= 2\hat{i} - \hat{j} + 3\hat{k}$$

and $$\frac{\vec{P}-\vec{Q}}{2} = \frac{(3\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} - 3\hat{j} + 4\hat{k})}{2}$$

$$= \hat{i} + 2\hat{j} - \hat{k}$$

$$\text{Area} = \left|\left(\frac{\vec{P}+\vec{Q}}{2}\right) \times \left(\frac{\vec{P}-\vec{Q}}{2}\right)\right|$$

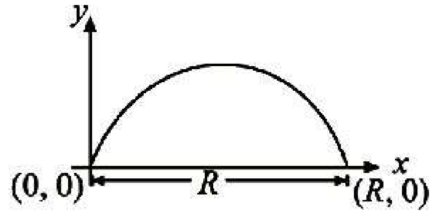
$$= \left|(2\hat{i} - \hat{j} + 3\hat{k}) \times (\hat{i} + 2\hat{j} - \hat{k})\right|$$

$$= 5\sqrt{3} \text{ unit}$$ ***Ans.***

Illustration 4

A particle is projected from the ground, whose equation of trajectory is given by $y = ax - bx^2$, where a and b are constants. Find horizontal range of the particle.

Short-cut solution :



Given, $$y = ax - bx^2$$

Taking point of projection as origin, the coordinates of point of strike are $(R, 0)$.

On substituting $x = R$ and $y = 0$, we get

$$0 = aR - bR^2$$
$$0 = R(a - bR)$$

As $R \neq 0$, ∴ $$a - bR = 0 \Rightarrow R = \frac{a}{b}.$$ ***Ans.***

Illustration 5

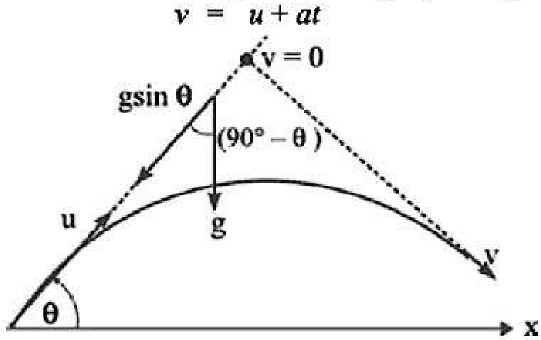
If any point of a parabolic path of a projectile the velocity be u and the direction of motion be at θ with the horizon, show that the particle is moving at right angle to its former direction after an interval of time $t = \frac{u}{g \sin\theta}$.

Short-cut solution :

Consider the motion of particle along the initial line of projection. Along this line initial velocity $= u$, after time t, $v = 0$ and acceleration $a = g \sin\theta$.

Using first equation of motion along the line of projection, we have

$$v = u + at$$



or $$0 = u - g\sin\theta\, t$$

which gives $$t = \frac{u}{g\sin\theta}.$$ ***Ans.***

Illustration 6

Two particles move in a uniform gravitational field with an acceleration g. At the initial moment the particles were located at one point and moved with velocities $u_1 = 3.0$ m/s and $u_2 = 4.0$ m/s horizontally in opposite directions. Find the distance between the particles at the moment when their velocity vectors become mutually perpendicular.

Short-cut solution :

Supposing point of projection as the origin, the velocities of particles at time t after the projection

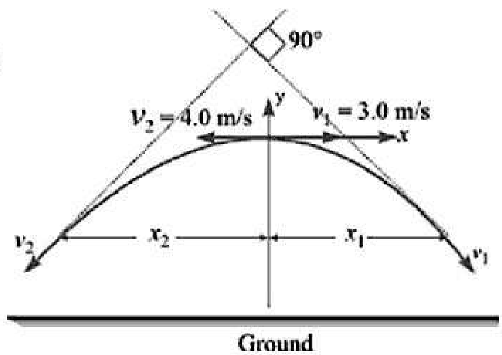
$$\vec{v}_1 = 3.0\hat{i} - gt\hat{j} \quad ....(i)$$

and

$$\vec{v}_2 = -4.0\hat{i} - gt\hat{j} \quad ....(ii)$$

As $\vec{v}_1$ and $\vec{v}_2$ are mutually perpendicular, so $\vec{v}_1 \cdot \vec{v}_2 = 0$

or $(3.0\hat{i} - gt\hat{j}).(-4.0\hat{i} - gt\hat{j}) = 0$



or $3.0 \times 4.0 - g^2 t^2 = 0$

or $t = \frac{\sqrt{12}}{g}$

Both the particles have zero initial velocity in vertical direction. Therefore they fall equal vertical distances. They lie on same horizontal line. Therefore we have

$$x_1 = 3.0\,t \quad \text{and} \quad x_2 = 4.0\,t$$

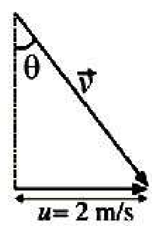
$$\therefore \quad x = x_1 + x_2 = 3.0\,t + 4.0\,t$$

$$= 7.0\,t = \frac{7.0\sqrt{12.0}}{g} \simeq 2.5 \text{ m.} \qquad \textit{Ans.}$$

Illustration 7

A man running along a straight road with uniform velocity u = 2 m/s feels that the rain is falling vertically down. If he double the speed, he finds that rain is coming at an angle θ = 30° with the vertical. Find the actual direction and speed of the rain with respect to the ground

Short-cut solution :



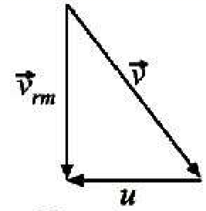
Let velocity of the rain is v and inclined as shown in figure.

The velocity of man $u = 2$ m/s.

The velocity of rain with respect to the man

$$\vec{v}_{rm} = \vec{v}_r - \vec{v}_m$$

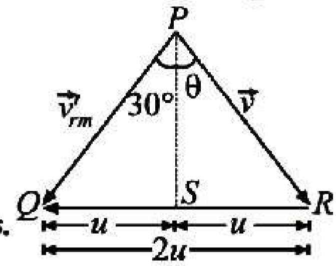
The $\vec{v}_{rm}$ vector is shown in figure,



when he doubles the speed

$$\vec{v}'_{rm} = \vec{v}_r - \vec{v}'_m$$

Now $\vec{v}'_{rm}$ is shown in figure



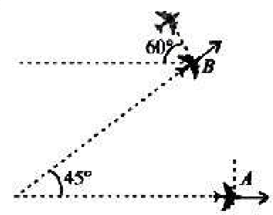
There are two same triangles PSQ and PSR,
so θ = 30°

and $v = QR = 2u$

$= 2 \times 2 = 4$ m/s *Ans.*

Illustration 8

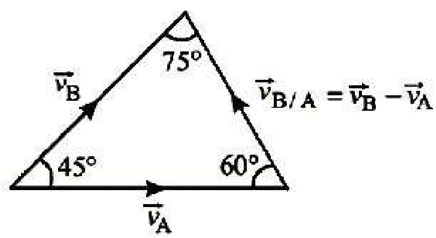
Passengers in the jet transport A flying east at a speed of 800 kmh^{-1} observe a second jet plane B that passes under the transport in horizontal flight. Although the nose of B is pointed in the 45° north east direction, plane B appears to the passengers in A to be moving away from the transport at the 60° angle as shown. The true velocity of B is



(a) *586 kmh^{-1}* *(b)* *$400\sqrt{2}$ kmh^{-1}* *(c)* *717 kmh^{-1}* *(d)* *400 kmh^{-1}*

Solution :

According to law of sine or Lami's Theorem



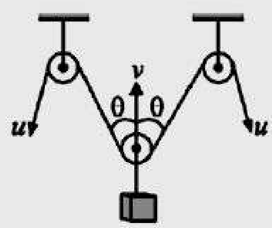
$$\Rightarrow \quad \frac{v_A}{\sin 75°} = \frac{v_B}{\sin 60°} = \frac{v_{B/A}}{\sin 45°}$$

$$\Rightarrow \quad v_B = 717 \text{ kmh}^{-1}.$$ *Ans. (c)*

TIPS! & TRICKS Tips and Tricks for Constraint Relations

Approach - 1 : In the given device 'u' is given and v is unknown.

Equate resolved component of unknown velocity along known velocity



or $v \cos\theta = u$

or $v = \dfrac{u}{\cos\theta}$

Approach - 2 : (i) $\Sigma T v \cos\theta = 0$

(ii) $\Sigma T a \cos\theta = 0$

Here T is the tension in the string.

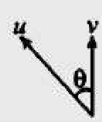
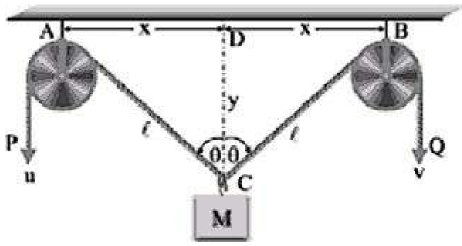


Illustration 9

In the arrangement shown in figure, the ends P and Q of an inextensible string move downwards with uniform speed u and v respectively. Pulley A and B are fixed. Find the velocity of mass M at the instant shown in the figure.



Short-cut solution :

Using, $\Sigma T v \cos\theta = 0$

$T u \cos 180° + T v \cos 180° + 2T\cos\theta \times v_b = 0$

or $v_b = \left[\dfrac{u+v}{2\cos\theta}\right]$. ***Ans.***

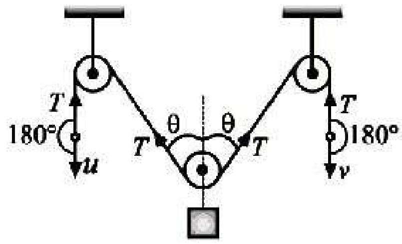
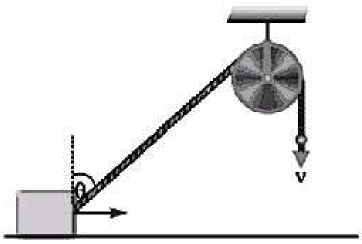


Illustration 10

A block is dragged on a smooth plane with the help of a rope which is pulled with velocity v as shown in figure. Find the horizontal velocity of the block.



Short-cut solution :

Let velocity of block along horizontal direction is vx, then its component along the rope will be $v_x \sin\theta$. Since each point on the rope will move with same velocity v

$\therefore \quad v_x \sin\theta = v$

or $v_x = \dfrac{v}{\sin\theta}$. ***Ans.***

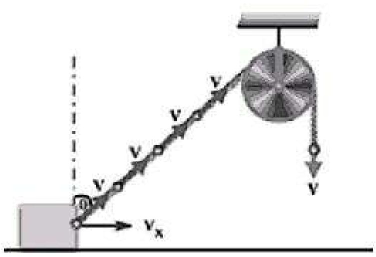
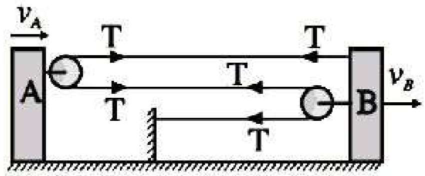


Illustration 11

In the arrangement shown in the figure the block B starts from rest and moves towards right with a constant acceleration. After time t the velocity of A with respect to B become v. Determine the acceleration of A.

Short-cut solution :



Given, $v_A - v_B = v$

or $v_A = (v_B + v)$

Using, $\Sigma Tv\cos\theta = 0$

or $(2T)v_A\cos 0° + (3T)v_B\cos 180° = 0$

or $2Tv_A - 3Tv_B = 0$

or $v_B = \frac{3}{2}v_B$

$= \frac{3}{2}(v_A - v)$

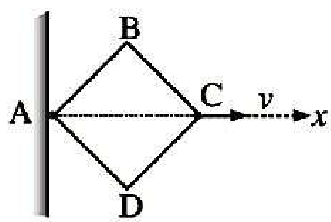
or $v_A = 3v$

Acceleration of block A, $a_A = \frac{v_A - 0}{t}$

$= \frac{3v}{t}$ ***Ans.***

Illustration 12

Four rods, each of length l, have been hinged to form a rhombus. Vertex A is fixed to a rigid support. Vertex C being pulled to the right along x-axis with a uniform speed v as shown. The speed at which vertex B moves at the moment the rhombus takes the shape of a square is :



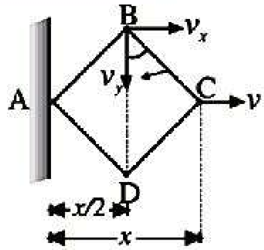
(a) $v/4$ ***(b)*** $v/2$ ***(c)*** $v/\sqrt{2}$ ***(d)*** v

Short-cut solution :

Vertex B has two components of velocities, v_x and v_y

$$v_x = \frac{d(x/2)}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right) = \frac{v}{2}$$

Now $\frac{v_x}{v_y} = \tan 45°$



or $$v_y = v_x = \frac{v}{2}$$

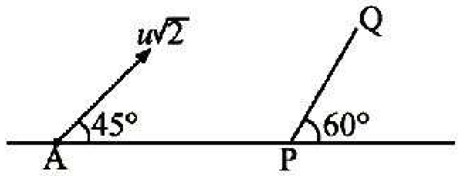
Velocity of vertex B, $$v_B = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{\left(\frac{v}{2}\right)^2 + \left(\frac{v}{2}\right)^2} = \frac{v}{\sqrt{2}}$$

Ans. (c)

Illustration 13

A particle is projected from point A with velocity $u\sqrt{2}$ at an angle of 45° with the horizontal. It strikes the inclined plane PQ at right angle. The velocity of the particle at the time of collision is :

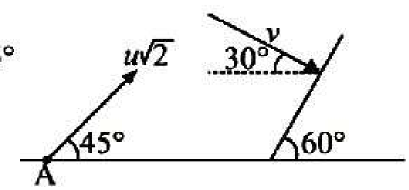


(a) $\frac{\sqrt{3}u}{2}$ *(b)* $\frac{u}{2}$ *(c)* $\frac{2u}{\sqrt{3}}$ *(d)* u

Short-cut solution :

If v is the required velocity, then its horizontal component when it hits the plane is $v \cos 30°$, so

$$v \cos 30° = u\sqrt{2} \cos 45°$$

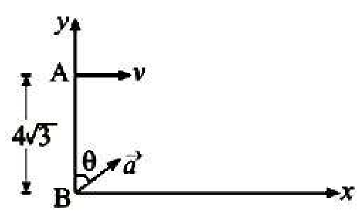


or $$v\frac{\sqrt{3}}{2} = u\sqrt{2} \times \frac{1}{\sqrt{2}}$$

or $$v = \frac{2u}{\sqrt{3}}$$

Ans. (c)

Illustration 14



Particle A moves along the line $y = 4\sqrt{3}$ with constant velocity $\vec{v}$ of magnitude 2.0 m/s and directed parallel to the positive x-axis. Particle B starts at the origin with zero speed and constant acceleration a (of magnitude 4.0 m/s^2) at the same instant that the particle A passes the y-axis. The angle θ between $\bar{a}$ and y-axis that would result in a collision between these two particles should have a value equal to :

(a) 30° *(b)* 45° *(c)* 50° *(d)* 80°

Short-cut solution :

If t is the time of collision, then for particle A

$$x = vt$$

and for particle B

$$x = 0+\frac{1}{2}(a\sin\theta)t^2$$

and $$y = 0+\frac{1}{2}(a\cos\theta)t^2$$

Therefore, $$vt = \frac{1}{2}a\sin\theta t^2 \quad \text{...(i)}$$

and $$4\sqrt{3} = \frac{1}{2}(a\cos\theta)t^2 \quad \text{...(ii)}$$

On substituting $a = 4\text{m/s}^2$, $v = 2$ m/s and simplifying we get

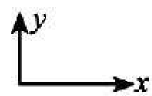
$$\cos\theta = \frac{\sqrt{3}}{2}$$

or $$\theta = 30°.$$ ***Ans. (a)***

Illustration 15

Rain is falling vertically with a speed of 20 m/s. A person is running in the rain with a velocity of 5 m/s and a wind is also blowing with a speed at 15 m/s (both from west). Find the angle with the vertical at which the person should hold his umbrella so that he may not get drenched.

Short-cut solution :

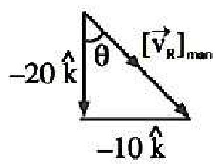


$$\vec{v}_{rain} = -20\hat{j} \text{ m/s}$$

$$\vec{v}_{man} = 5\hat{i} \text{ m/s}, \text{ and } \vec{v}_{wind} = 15\hat{i} \text{ m/s}.$$

The resultant velocity of rain and wind

$$\vec{v}_R = -20\hat{j}+15\hat{i}$$



Now velocity of rain relative to the man is

$$[\vec{v}_R]_{man} = -20\hat{j}+15\hat{i}-5\hat{i} = 10\hat{i}-20\hat{j}$$

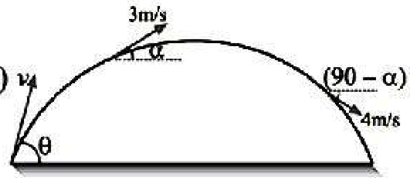
Therefore $$\tan\theta = \frac{10}{20} = \frac{1}{2} \text{ or } \theta = \tan^{-1}\left(\frac{1}{2}\right)$$ ***Ans.***

Illustration 16

A projectile of mass m has velocities 3 m/s and 4 m/s at two points during its flight in the uniform gravitational field of the earth. If these two velocities are perpendicular to each other, then find the minimum K.E. of the particle during its flight.

Short-cut solution :

According to given condition



$$3 \cos \alpha = 4 \cos (90 - \alpha)$$

or $$3 \cos \alpha = 4 \sin \alpha$$

or $$\tan \alpha = \frac{3}{4}$$

As horizontal component of velocity at highest point is, $v \cos \theta = 3 \cos \alpha$.

Therefore,

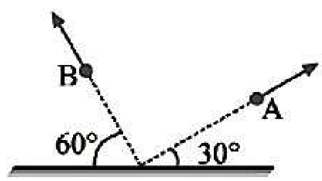
$$K_{min} = \frac{1}{2}m(3\cos\alpha)^2 = \frac{1}{2}m \times 3^2 \times \left(\frac{4}{5}\right)^2 = \frac{72}{25}m \quad \textit{Ans.}$$

Illustration 17

Two particles A and B are projected in air. A is thrown with a speed of 3 m/s and B with a speed of 4 m/s as shown in figure. What is the separation between them after 1 second ?

Short-cut solution :

Both the particles move under gravity, so



$$|\vec{a}_{BA}| = g - g = 0$$

and $$|\vec{v}_{BA}| = \sqrt{4^2 + 3^2} = 5 \text{ m/s}$$

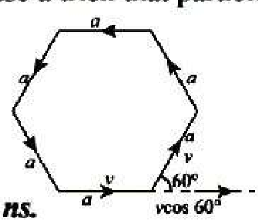
∴ $$s = v_{BA}t = 5 \times 1 = 5 \text{ m} \quad \textit{Ans.}$$

Illustration 18

Six particles situated at the corners of a regular hexagon of side 'a' move at constant speed v. Each particle maintain a direction towards the particle at the next corner. Find the time the particles will take to meet each other.

Short-cut solution :

Particles will meet at the centre of the hexagon. But we can use a trick that particle will approach to other particle with a relative velocity,



$$v = v - v \cos 60°$$
$$= v - v/2 = v/2.$$

The distance to be moved $= a$

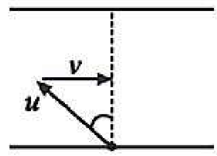
∴ Required time, $$t = \frac{a}{v/2} = \frac{2a}{v}. \quad \textit{Ans.}$$

Illustration 19

A man crosses a 320 m wide river perpendicular to the current in 4 minute. If in still water he can swim with a speed $\frac{5}{3}$ times that of the current, then the speed of the current, in m/min is:

(a) 30 *(b)* 40 *(c)* 50 *(d)* 60

Short-cut solution :



$$\sqrt{u^2 - v^2} = \frac{320}{4} = 80$$

$$\sqrt{\left(\frac{5}{3}v\right)^2 - v^2} = 80$$

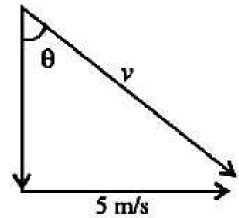
$\therefore \quad v = 60$ m/min *Ans. (d)*

Illustration 20

Raindrops are falling vertically when no wind is blowing. Now when a wind is blowing horizontally at the speed of 5 m/s, raindrops are observed to be striking the ground at an angle θ with the vertical. The speed of the raindrops is

(a) $5 \sin\theta$ *(b)* $\frac{5}{\sin\theta}$ *(c)* $5 \cos\theta$ *(d)* $5 \tan\theta$

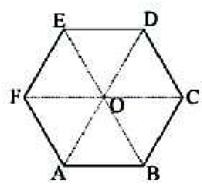
Short-cut solution :



$\frac{5}{v} = \sin\theta$, or $v = \frac{5}{\sin\theta}$. *Ans. (b)*

Concept Booster Exercise

1. In a regular hexagon *ABCDEF*,
prove that $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF} = 6\vec{AO}$

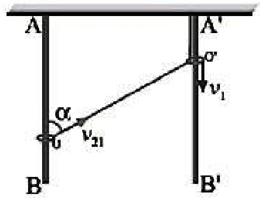


2. The resultant vector $\vec{P}$ and $\vec{Q}$ is $\vec{R}$. On reversing the direction of $\vec{Q}$, the resultant vector becomes S. Show that : $R^2 + S^2 = 2(P^2 + Q^2)$.

3. The angle of inclination between two vectors $\vec{P}$ and $\vec{Q}$ is θ. If $\vec{P}$ and $\vec{Q}$ are interchanged in position, show that the resultant will be turn through an angle φ, where $\tan\frac{\phi}{2} = \left[\frac{P-Q}{P+Q}\right]\tan\frac{\theta}{2}$.

4. Two forces (P + Q) and (P – Q) make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that P tan θ = Q tan α.

5. Two rings O and O' are put on two vertical stationary rods AB and $A'B'$ respectively. An inextensible thread is fixed at point A' and on ring O and is passed through ring O' (see figure). Assuming that ring O' moves downwards at constant velocity v_1, determine the velocity v_2 of ring O if $\angle AOO' = \alpha$.



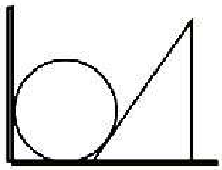
6. Magnitude of resultant of two vectors $\vec{P}$ and $\vec{Q}$ is equal to magnitude of $\vec{P}$. Find the angle between $\vec{Q}$ and resultant of $2\vec{P}$ and $\vec{Q}$. *[JEE Main 2020]*

Numeric/Integer

7. Position of two particles A and B as a function of time are given by $X_A = -3t^2 + 8t + c$ and $Y_B = 10 - 8t^3$. The velocity of B with respect to A at $t = 1$ is $\sqrt{v}$. Find v. *[JEE Main 2020]*

Numeric/Integer

8. A sphere of radius 10 cm is in contact with a wedge as shown. The point of contact of the sphere is at height of 2 cm above the ground. If the wedge is moving leftwards with 20 cm/s, the sphere is moving with :

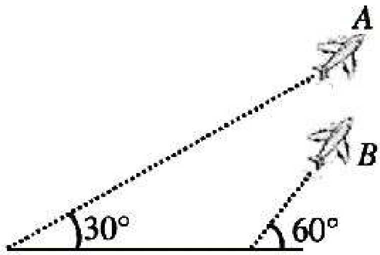


Numeric/Integer

(a) 12 cm/s (b) 15 cm/s
(c) 18 cm/s (d) 21 cm/s

9. Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in figure. The speed of A is $100\sqrt{3}$ m/s. At time $t = 0$ s, an observer in A finds B at a distance of 500 m. The observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at $t = t_0$, A just escapes being hit by B, t_0 in seconds is *[JEE Adv. 2014]*

Numeric/Integer



10. A boat which has a speed of 5 km/h, in still water crosses a river of width 1 km along the shortest possible path in 15 minute. The velocity of the river water in km/h is

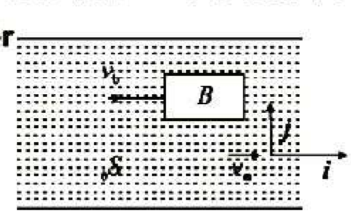
Numeric/Integer

(a) 1 (b) 3 (c) 4 (d) $\sqrt{41}$

11. An aeroplane is to along straight line from A to B, and back again. The relative speed with respect to wind is V. The wind blows perpendicular to line AB with speed v. The distance between A and B is ℓ. The total time for the round trip is:

(a) $\frac{2\ell}{\sqrt{V^2 - v^2}}$ (b) $\frac{2v\ell}{V^2 - v^2}$ (c) $\frac{2V\ell}{V^2 - v^2}$ (d) $\frac{2V\ell}{\sqrt{V^2 + v^2}}$

12. A boat B is moving upstream with velocity 3 m/s with respect to ground. An observer standing on boat observes that a swimmer S is crossing the river perpendicular to the direction of motion of boat. If river flow velocity is 4 m/s and swimmer crosses the river of width 100 m in 50 sec, then : **Numeric/Integer**



(a) velocity of swimmer w.r.t. ground is $\sqrt{13}$ m/s
(b) drift of swimmer along river is zero
(c) drift of swimmer along river will be 50 m
(d) velocity of swimmer w.r.t. ground is 2 m/s

13. From the top of a tower, two balls are thrown horizontally with velocities u_1 and u_2 in opposite directions. If their velocities are perpendicular to each other just before they strike the ground, find the height of the tower.

(a) $\frac{u_1^2}{2g}$ (b) $\frac{u_1u_2}{2g}$ (c) $\frac{u_2^2}{2g}$ (d) $\frac{u_1u_2}{g}$

14. A particle is projected with velocity $10\sqrt{2}$ m/s at an angle of 45° with the horizontal. The interval between the moments when speed is $\sqrt{125}$ m/s ($g = 10$ m/s^2)

Numeric/Integer

(a) 1 s (b) 2 s (c) 3 s (d) 4 s

Solutions

1. $\vec{R} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF}$

$= \overrightarrow{AB} + (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}) + (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}) + (\overrightarrow{CD})$

As $\overrightarrow{DE} = -\overrightarrow{AB}$

$\therefore$ $\vec{R} = 3(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD})$

$= 3 \times \overrightarrow{AD} = 3 \times 2\overrightarrow{AO}$

$= 6\,\overrightarrow{AO}$

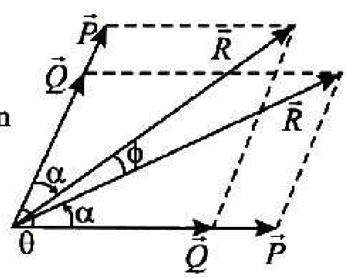
2. We have $R^2 = P^2 + Q^2 + 2PQ\cos\theta$ ...(i)

and $S^2 = P^2 + Q^2 - 2PQ\cos\theta$ ...(ii)

Adding equations (i) and (ii), we get

$R^2 + S^2 = 2(P^2 + Q^2)$.

3. If α is the angle which resultant $\vec{R}$ makes with $\vec{P}$, then



$2\alpha + \phi = \theta$

$\therefore$ $\phi = (\theta - 2\alpha)$

or $$\frac{\phi}{2} = \left(\frac{\theta}{2}-\alpha\right)$$

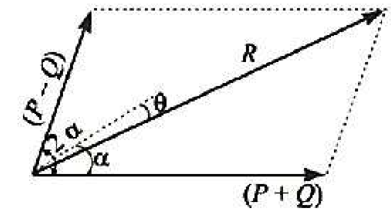
or $$\tan\frac{\phi}{2} = \tan\left(\frac{\theta}{2}-\alpha\right) \quad ...(i)$$

where $$\tan\alpha = \frac{Q\sin\theta}{P+Q\sin\theta} \quad ...(ii)$$

After solving equations (i) and (ii), we get

$$\tan\frac{\phi}{2} = \left[\frac{P-Q}{P+Q}\right]\tan\frac{\theta}{2}.$$

4. The angle which the resultant makes with $P+Q$ will be $(\alpha-\theta)$.



Thus $$\tan(\alpha-\theta) = \frac{(P-Q)\sin 2\alpha}{(P+Q)+(P-Q)\cos 2\alpha}$$

or $$\frac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)} = \frac{(P-Q)\sin 2\alpha}{(P+Q)+(P-Q)\cos 2\alpha}$$

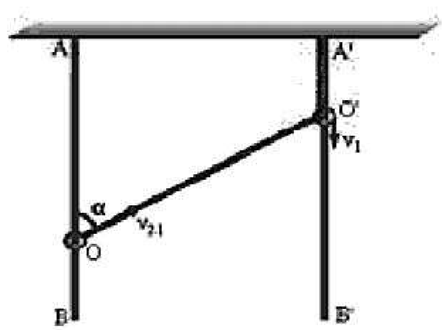
or $$(P+Q)\sin(\alpha-\theta) = (P-Q)\sin(\alpha+\theta)$$

or $$P[\sin(\alpha+\theta)-\sin(\alpha-\theta)] = Q[\sin(\alpha+\theta)+\sin(\alpha-\theta)]$$

or $$P\times 2\cos\alpha\sin\theta = Q\times 2\sin\alpha\cos\theta$$

$$\therefore \quad P\tan\theta = Q\tan\alpha$$

5. Let us go through the reference frame fixed to ring O'. As the ring O' move down, the string is pulled at a constant velocity v_{21} relative to O'. Thus the velocity of ring O relative to O'.



$$v_{21} = \frac{v}{\cos\alpha}, \text{ directed upwards.}$$

Therefore, the velocity of ring O relative to the straight line AA' (which is stationary w.r.t. ground) is;

$$\vec{v}_{21} = \vec{v}_2 - \vec{v}_1$$

or $$\vec{v}_2 = \vec{v}_{21} + \vec{v}_1$$

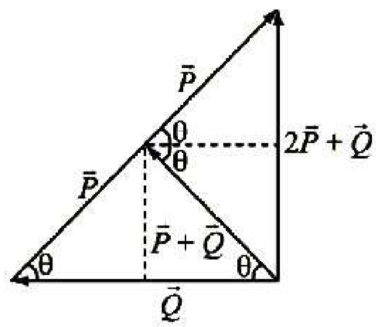
or $$v_2 = v_{21} - v_1$$

$$= \frac{v_1}{\cos\alpha} - v_1 = v_1\left(\frac{1}{\cos\alpha} - 1\right)$$

$$= v_1\left(\frac{1-\cos\alpha}{\cos\alpha}\right) = v_1 \frac{2\sin^2\alpha/2}{\cos\alpha}$$

6. **(90°)**

Alternate solution

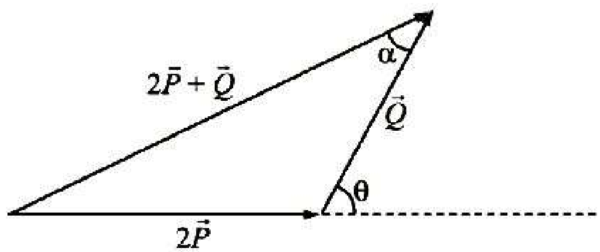


$$|\vec{P}+\vec{Q}| = |\vec{P}|$$

$$P^2 + Q^2 + 2PQ\cos\theta = P^2$$

$\Rightarrow$ $$Q + 2P\cos\theta = 0$$

$\Rightarrow$ $$\cos\theta = -\frac{Q}{2P}$$



$$\tan\alpha = \frac{2P\sin\theta}{2P\cos\theta + Q} = \infty \quad \because [2P\cos\theta + Q = 0]$$

$$\alpha = 90°.$$

7. **(580)**

$$X_A = -3t^2 + 8t + c$$

$$\vec{v}_A = (-6t+8)\hat{i}$$

$$= 2\hat{i}$$

$$Y_B = 10 - 8t^3$$

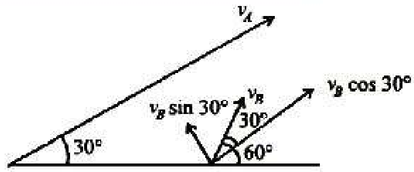
$$\vec{v}_B = -24t^2\hat{j}$$

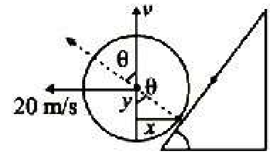
$$\sqrt{v} = |\vec{v}_B - \vec{v}_A| = |-24\hat{j} - 2\hat{i}|$$

$$\sqrt{v} = \sqrt{24^2 + 2^2}$$

$$v = 580.$$

8. **(b)** We have, $x^2 + y^2 = R^2$





or $$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

or $$x\,v_x + y\,v_y = 0$$

$$\therefore \quad v_y = v_x\left(\frac{x}{y}\right)$$

$$= 20 \times \frac{3}{4} = 15 \text{ m/s.}$$

9. **5** Here

$$v_A = v_B \cos 30°$$

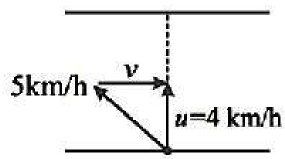
$$\therefore \quad 100\sqrt{3} = v_B \times \frac{\sqrt{3}}{2}$$

$$\therefore \quad v_B = 200 \text{ ms}^{-1}$$

$$\text{Time} = \frac{\text{displacement}}{\text{velocity}}$$

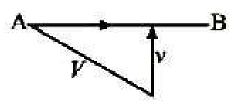
$$\therefore \quad t_0 = \frac{500}{v_B \sin 30°} = \frac{500}{200 \times \sin 30°} = 5 \text{ sec}$$

10. **(b)** $$u = \frac{1}{1/4} = 4 \text{ km/h}$$



$$\therefore \quad v = \sqrt{5^2 - 4^2} = 3 \text{ km/h.}$$

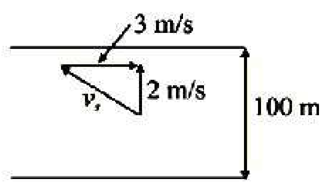
11. **(a)** Velocity along the line in both side will be same,



$$v' = \sqrt{V^2 - v^2}$$

so, $$t = \frac{2\ell}{\sqrt{V^2 - v^2}}$$

12. **(a)** $v = \dfrac{100}{50} = 2 \text{ m/s}$



$v_s = \sqrt{3^2 + 2^2} = \sqrt{13}$ m/s.

13. **(b)** $\vec{v}_1 = u_1\hat{i} - gt\hat{j}$ and $\vec{v}_2 = -u_2\hat{i} - gt\hat{j}$

Given, $\vec{v}_1 . \vec{v}_2 = 0$

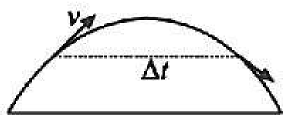
or $(u_1\hat{i} - gt\hat{j}) \cdot (-u_2\hat{i} - gt\hat{j}) = 0$

$$\therefore \quad t = \frac{\sqrt{u_1 u_2}}{g}.$$

Now, $$h = \frac{1}{2}gt^2 = \frac{1}{2}g \times \left(\frac{\sqrt{u_1u_2}}{g}\right)^2$$

$$= \frac{u_1u_2}{2g}.$$

14. **(a)** Using, $$v^2 = {v_x}^2 + {v_y}^2$$



or $$(\sqrt{125})^2 = (10\sqrt{2}\cos 45°)^2 + {v_y}^2$$

$\therefore$ $$v_y = 5 \text{ m/s}.$$

The required time interval

$$\Delta t = \frac{2v_y}{g} = \frac{2 \times 5}{10} = 1\text{s}.$$

Laws of Motion and Circular Motion 4

***TOPIC 4.1**: Newton's Laws of Motion, Momentum, Impulse-Momentum Theorem, Motion of Connected Bodies, Lift and Pulley Problems.*

Review of Formulae

1. Momentum, $\vec{P} = m\vec{v}$.
2. Impulse of force, $\vec{J} = F\Delta t$.
3. Newton's second law of motion, $\vec{F}_{ext} = \frac{d\vec{P}}{dt} = m\vec{a}$.
4. Impulse – Momentum theorem, $\vec{J} = \vec{P}_f - \vec{P}_i$.
5. Newton's third law of motion, $\vec{F}_{AB} = -\vec{F}_{BA}$.
6. Action and Reaction act simultaneously and two different objects.
7. Motion in a lift : The apparent weight of an observer in lift, $W' = m(g \pm a)$
8. Hooke's law, $\vec{F}_{ext} = k\vec{x}$.

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. Newton's second law in inertial frame,

   $$\left[\vec{F}_{net}\right]_{real} = m\vec{a}.$$

   Here $\vec{a}$ is the acceleration of the particle in inertial frame.

2. Newton's second law in non-initial frame,

   $$\vec{F}_{real} + \vec{F}_{pseudo} = m\vec{a}.$$

   Here $\vec{a}$ is the acceleration of the particle is non-inertial frame.

3. Force on a perfectly reflecting surface, if particles each of mass m strikes normally with velocity v

   $F = 2mvn$, here n is the number of particles per second.

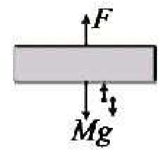
Illustration 1

A disc of mass M is kept floating by firing bullets below it. Each bullet is of mass m and strikes normally with a velocity v. If each bullet rebounds with the same speed then number of bullets fired per second are:

(a) $\frac{Mg}{mv}$ (b) $\frac{2Mg}{mv}$ (c) Mg (d) $\frac{Mg}{2mv}$

Short-cut solution :

For the floating disc,

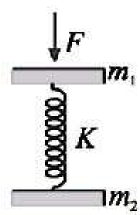


$$F = Mg$$

or $$2mvn = Mg$$

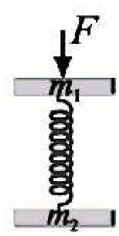
$\therefore$ $$n = \frac{Mg}{2mv}$$ ***Ans. (d)***

Illustration 2



Two disks of masses m_1 and m_2 are connected by a spring of force constant k. The lower disk of mass m_2 lies on a table and the upper disk is vertically above it (see figure). What vertical force F should be applied to the upper disk so that when the force is withdrawn, the lower disk is lifted off the table ?

Short-cut solution :



When force F is applied to the upper disk and withdrawn the upward force generated in the spring will also be F. This force is equal to lift the weight of the system, so

$$F \geq (m_1 + m_2)g.$$ ***Ans.***

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

Step I : For a system of blocks placed in contact or connecting by light strings, assume them a single body of total mass M.

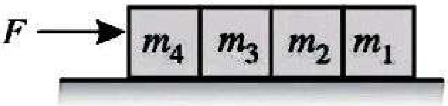
Step II : Then find net force or unbalanced force in the direction of motion, say F.

Step III : Now using Newton's second law to find acceleration,

$$a = \frac{\text{unbalanced force}}{\text{total mass}} = \frac{F}{M}.$$

Illustration 3

Blocks of masses m_1, m_2, m_3 and m_4 are placed in contact on a smooth horizontal surface. A horizontal force F is pushed them as shown. Find force of interaction between the blocks.



Short-cut solution :

The blocks are in contact and so their common acceleration

$$a = \left[\frac{F}{m_1 + m_2 + m_3 + m_4}\right]$$

Force between the blocks m_1 and m_2

$$F_1 = m_1 a$$

Force between blocks of m_2 and m_3

$$F_2 = (\text{total mass beyond } m_3) \times a$$
$$= (m_1 + m_2)a$$

Force between blocks of masses m_3 and m_4

$$F_3 = (\text{total mass beyond } m_4) \times a$$
$$= (m_1 + m_2 + m_3)\, a \qquad \textit{Ans.}$$

Illustration 4

Four blocks are connected by light strings and pulled upwards by a constant force F. Find the tensions in the strings.

Short-cut solution :

Acceleration of the blocks,

$$a = \left[\frac{F - (m_1 + m_2 + m_3 + m_4)g}{m_1 + m_2 + m_3 + m_4}\right]$$

Tensions : $T_1 = (\text{ mass below } m_2)\,(g + a)$

$= m_1(g + a)$

$T_2 = (\text{total mass below } m_3)\,(g + a)$

$= (m_1 + m_2)\,(g + a)$

$T_3 = (\text{total mass below } m_4)\,(g + a)$

$= (m_1 + m_2 + m_3)\,(g + a)$ *Ans.*

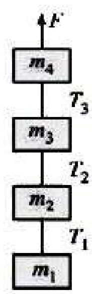
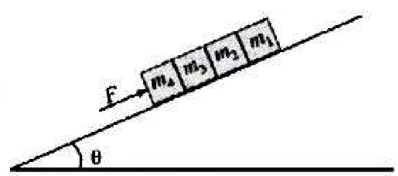


Illustration 5

Four blocks are in contact on a smooth inclined placed are pushed up by a constant force F. Find force of interaction between the blocks.



Short-cut solution :

Common acceleration of the blocks

$$a = \left[\frac{F-(m_1+m_2+m_3+m_4)g\sin\theta}{m_1+m_2+m_3+m_4}\right]$$

Force between m_1 and m_2

$$F_1 = (\text{mass beyond } m_2)(g\sin\theta + a)$$
$$= m_1(g\sin\theta + a)$$

Force between m_2 and m_3

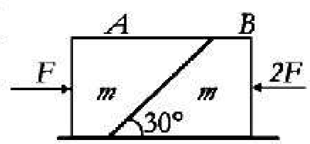
$$F_2 = (\text{total mass beyond } m_3)(g\sin\theta + a)$$
$$= (m_1 + m_2)(g\sin\theta + a)$$

Force between m_3 and m_4

$$F_3 = (\text{total mass beyond } m_4)(g\sin\theta + a)$$
$$= (m_1 + m_2 + m_3)(g\sin\theta + a)$$ ***Ans.***

Illustration 6

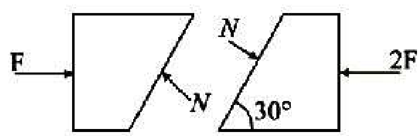
Two blocks 'A' and 'B' each of mass 'm' are placed on a smooth horizontal surface. Two horizontal forces F and 2F are applied on the two blocks 'A' and 'B' respectively as shown in figure. The block A does not slide on block B. Then the normal reaction acting between the two blocks is



(a) F *(b)* $\frac{F}{2}$ *(c)* $\frac{F}{\sqrt{3}}$ *(d)* $3F$

Short-cut solution :

$$a = \frac{2F - F}{2m} = \frac{F}{2m}$$



Now $2F - N\cos 60° = ma$

or $2F - \frac{N}{2} = m \times \frac{F}{2m}$

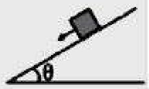
$\therefore$ $N = 3F$. ***Ans. (d)***

Tips and Tricks for Simple Pulley Devices Shortcut

Generally in pulley devices, we have asked acceleration of blocks and tension in the string. The following short-cut method can be used for simple pulley devices (blocks connected have same acceleration):

Acceleration, $a = \dfrac{\text{unbalanced load}}{\text{total mass of the system}}$

Load effect : If block moves vertically it will be mg.
If block moves horizontally it will be zero.
If block moves on inclined it will be $mg \sin\theta$.



For tension in the string : Take total mass hanging from the string, if they move up, then

$$T = m_{up}(g + a).$$

And if they move downwards, then

$$T = m_{down}(g - a).$$

Illustration 7

Consider the system shown in figure. The system is released from rest, find the tension in the cord connected between 1 kg and 2 kg blocks. ($g = 10\ m/s^2$)

Short-cut solution :

$$a = \frac{\text{unbalanced load}}{\text{total mass}} = \frac{[(2+1)-2]g}{(1+2+2)} = 2m/s^2$$

$$T_1 = m_{down}(g - a) = 1\ (10 - 2) = 8\text{N}$$

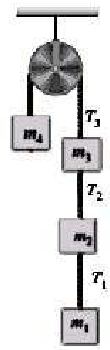
$$T_2 = m_{up}\ (g + a) = 2\ (10 + 2) = 24\ \text{N}.$$ ***Ans.***

Illustration 8

Four blocks of given masses are connected by massless strings and pass over massless pulley as shown in figure. Find tensions in the strings.

Short-cut solution :

Acceleration magnitude of the blocks together,



$$a = \frac{\text{unbalanced load}}{\text{total mass}}$$

$$= \left[\frac{(m_1 + m_2 + m_3) - m_4}{m_1 + m_2 + m_3 + m_4}\right]g$$

Tensions :

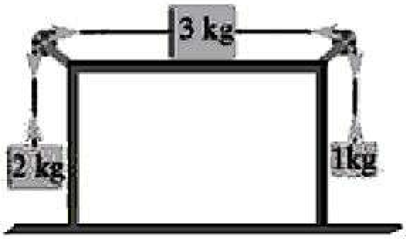
$$T_1 = m_1(g - a)$$

$$T_2 = (m_1 + m_2)\ (g - a)$$

$$T_3 = (m_1 + m_2 + m_3)\ (g - a)$$ ***Ans.***

Illustration 9

The system shown in figure is released from rest. Calculate the tension in the strings and force exerted by the strings on the pulleys. Assuming pulleys and strings are massless.



Short-cut solution :

$$a = \frac{\text{unbalanced load}}{\text{total mass}}$$

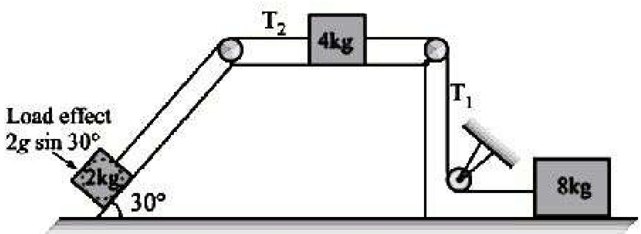
$$= \frac{(2-1)g}{(1+3+2)} = \frac{g}{6} m/s^2.$$

$$T_1 = m_{up}(g+a) = 1\left(g+\frac{g}{6}\right) = \frac{7g}{6}\text{N}.$$

$$T_2 = m_{down}(g-a) = 2\left(g-\frac{g}{6}\right) = \frac{5g}{3}\text{N}.$$

Illustration 10

In the device shown, find acceleration of the blocks and tensions in the strings. All the contact surfaces are smooth. Take g = 10 m/s².



Short-cut solution :

The common acceleration,

$$a = \frac{\text{unbalanced load}}{\text{total mass}}$$

$$= \frac{2g\sin 30°}{2+4+8} = \frac{g}{14} = \frac{5}{7}\text{ m/s}^2$$

Tensions : $T_1 = 8a$

$$= 8\times\frac{5}{7} = \frac{40}{7}\text{ N}$$

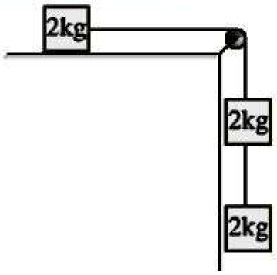
And $T_2 - T_1 = 4a$

or $T_2 = T_1 + 4a$

$$= \frac{40}{7} + 4\times\frac{5}{7}$$

$$= \frac{60}{7}\text{N}.$$ ***Ans.***

Illustration 11



Three blocks each of mass 2kg are connected with the help of light strings which passes over a massless pulley. One block is placed on smooth table and other two are hanging. Find acceleration of the blocks and tension in the strings.

Short-cut solution :

Acceleration of the blocks,

$$a = \frac{\text{unbalanced load}}{\text{total mass}}$$

$$= \frac{(2+2)g}{2+2+2} = \frac{g}{3} \text{ m/s}^2$$

Tension: $T_1 = 2(g - a)$; as lower 2 kg coming down

$$= 2\left(g - \frac{2g}{3}\right)$$

$$= \frac{2g}{3} N$$

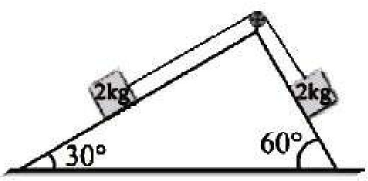
and $T_2 = 4(g - a)$; as (2 +2)kg coming down

$$= 2 \times \frac{2g}{3}$$

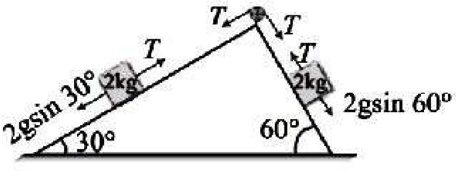
$$= \frac{4g}{3} N \qquad \textit{Ans.}$$

Illustration 12

Two blocks, each of mass 2 kg are connected by a light string which passes of a massless pulley. Blocks move on smooth inclined as shown in figure. Find acceleration of the blocks and tension in the string. Take $g = 10$ m/s^2.



Solution :



Using Newton's second law

we have, $2g \sin 60° - T = 2a$...(i)

and $T - 2g\sin 30° = 2a$...(ii)

On simplifying above equations, we get

$$a = \left(\frac{\sqrt{3}-1}{4}\right)g$$

$$= 1.83 \text{ m/s}^2$$

and $T = 13.66$ N. ***Ans.***

Short-cut solution :

Acceleration, $a = \dfrac{\text{unbalanced load}}{\text{total mass}}$

$$= \frac{2g\sin 60° - 2g\sin 30°}{(2+2)}$$

$$= \frac{(\sqrt{3}-1)}{4}g \text{ m/s}^2$$

$$= 1.83 \text{ m/s}^2$$

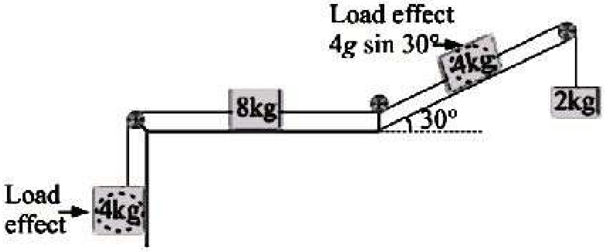
Tension: $T = m(g\sin 30° + a)$; as left side block moves up the plane

$$= 2(10\times\frac{1}{2}+1.83)$$

$$= 13.66 \text{ N}$$ ***Ans***

Illustration 13

In the device shown all the surfaces are smooth and pulleys are light and smooth. Find acceleration of the blocks.



Short-cut solution :

Acceleration, $a = \dfrac{\text{unbalanced load}}{\text{total mass}}$

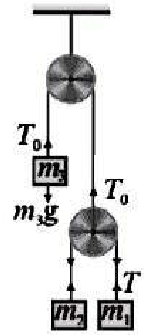
$$= \frac{(4g + 4g\sin 30°) - 2g}{4+8+4+2}$$

$$= \frac{4g}{18} = \frac{2g}{9} \text{ m/s}^2.$$ ***Ans.***

Illustration 14

In the device shown pulleys are light and smooth. Find the mass of block m_3 in terms of m_1 and m_2 so that it remains at rest.

Short-cut solution :



The tension, $$T = \left[\frac{2m_1m_2g}{m_1+m_2}\right]$$

For m_3 to be at rest,

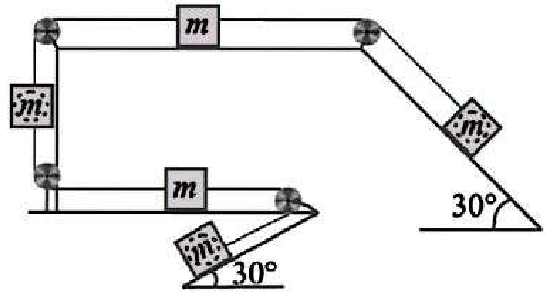
$$T_0 = 2T = m_3g$$

or $$2\left[\frac{2m_1m_2g}{m_1+m_2}\right] = m_3g$$

or $$m_3 = \left(\frac{4m_1m_2}{m_1+m_2}\right) \qquad \textit{Ans.}$$

Illustration 15

In the device shown all the surfaces are smooth, and pulleys are massless, find acceleration of the blocks. (Take g = 10 m/s²)



(a) $1\ m/s^2$ *(b)* $2\ m/s^2$ *(c)* $5\ m/s^2$ *(d)* $4\ m/s^2$

Short-cut solution :

$$a = \frac{mg + mg\sin 30° - mg\sin 30°}{5m}$$

$$= \frac{g}{5}\ \text{m/s}^2. \qquad \textit{Ans. (b)}$$

TIPS & TRICKS Tips and Tricks for Movable Pulley Shortcut

For constraint relations, we may use following short-cut approach :

(i) $\Sigma Tv\cos\theta = 0$; Here θ is the angle between $\vec{T}$ and $\vec{v}$.

(ii) $\Sigma Ta\cos\theta = 0$; Here θ is the angle between $\vec{T}$ and $\vec{a}$.

(iii) In a very specific problem in which one end of the string is fixed and a block is connected to movable pulley and other block at the free end of the string, then acceleration of the block at free end

= twice the acceleration of the block connected to movable pulley

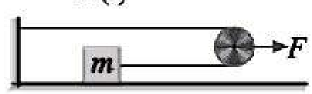
Illustration 16

A block of mass 'm' is tied to a fixed point on a horizontal table through a string passing round a massless smooth pulley as shown in the figure. A force F is applied on the pulley. Find the acceleration of the pulley.

Short-cut solution :

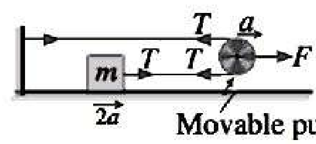
If acceleration of movable pulley is a, then acceleration of the block will be $2a$

For pulley, $F - 2T = 0 \times a$ ...(i)



or $T = \dfrac{F}{2}$

For block, $T = m(2a)$

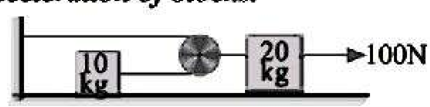


or $\dfrac{F}{2} = 2ma$

$\therefore$ $a = \dfrac{F}{4m}$. *Ans.*

Illustration 17

In the device shown pulley is massless and smooth. Two blocks of masses 10 kg and 20 kg are connected by strings and the block of mass 20 kg is acted by 100 N force as shown. Find acceleration of blocks.

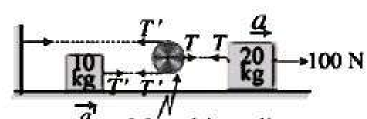


Short-cut solution :

The acceleration of the block 20 kg mass connected to movable pulley has acceleration 'a', then acceleration of 10 kg will be '$2a$'.

Constraint relations can also be obtained by using

$\Sigma Ta \cos 0 = 0$



or $\underbrace{Ta\cos 180^\circ}_{\text{for 20 kg}} + \underbrace{T'a'\cos 0^\circ}_{\text{for 10 kg}} = 0$

Also $T = 2T'$ ...(i)

$\therefore$ $2T'a(-1) + T'a' = 0$

or $a' = 2a$

Now by Newton's second law

$$100 - T = 20\,a \qquad \text{...(ii)}$$

and $$T' = 10(2a) \qquad \text{...(iii)}$$

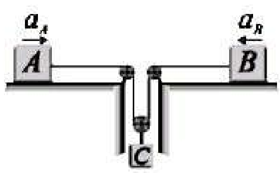
After simplifying above equations, we get

$$a = \frac{5}{3}\ \text{m/s}^2$$

$$\therefore \qquad a' = 2a = \frac{10}{3}\ \text{m/s}^2 \qquad \textit{Ans.}$$

Illustration 18

In the arrangement of three blocks, the string is in extensible and blocks A and B are given accelerations as indicated in the figure. Find acceleration of the block C.

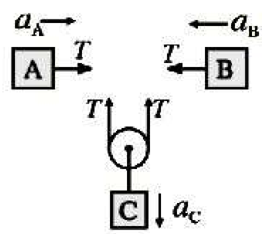


(i)

(ii)

Short-cut solution :

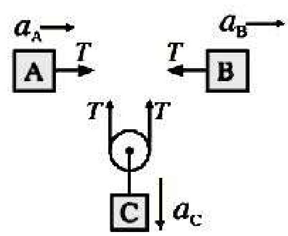
(i) If a_C is the acceleration of block C, then



$$Ta_A \cos 0° + Ta_B \cos 0° + 2Ta_C \cos 180° = 0$$

or $$a_C = \left[\frac{a_A + a_B}{2}\right] \qquad \textit{Ans.}$$

(ii) Using $\Sigma Ta \cos\theta = 0$.



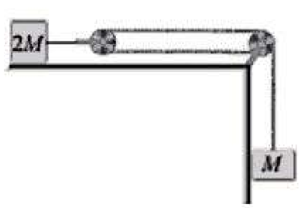
or $Ta_A \cos 0° + Ta_B \cos 180° + 2Ta_C \cos 180° = 0$

$$\therefore \qquad a_C = \left[\frac{a_A - a_B}{2}\right] \qquad \textit{Ans.}$$

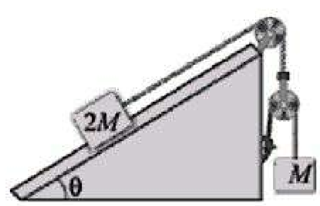
Illustration 19

Find the acceleration of the blocks in the following devices from the data shown in figure. Pulleys are massless and frictionless.

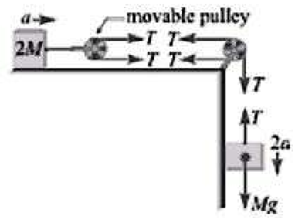
(i)



(ii)



Short-cut solution :



(i) For block $2M$;

$$2T = 2M(a) \quad \dots \text{(i)}$$

For block M;

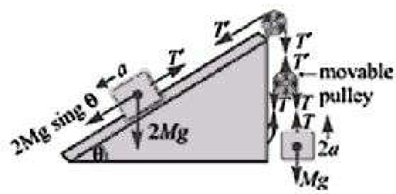
$$Mg - T = M(2a) \quad \dots \text{(ii)}$$

After solving above equations, we get

$$a = g/3 \text{ m/s}^2.$$

Force on clamp which holds the pulley

$$F = \sqrt{(2T)^2 + T^2} = \sqrt{5}T$$



(ii) For block $2M$;

$$2Mg\sin\theta - T' = 2M(a) \quad \dots \text{(i)}$$

For movable pulley ;

$$T' - 2T = 0 \times a \quad \dots \text{(ii)}$$

For block M ;

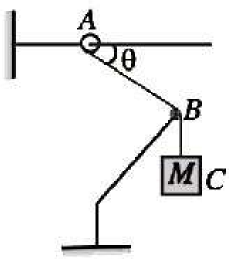
$$T - Mg = M(2a) \dots \text{(iii)}$$

After solving above equations, we get

$$a = -g/3\,(1 - \sin\theta). \qquad \textit{Ans.}$$

Illustration 20

A smooth ring A of mass m can slide on a fixed horizontal rod. A string tied to the ring passes over a fixed pulley B and carries a block C of mass M = 2m as shown in figure. At an instant the string between the ring and the pulley makes an angle θ with th rod. Find the acceleration of the block and the ring just after released from, θ = 30°.



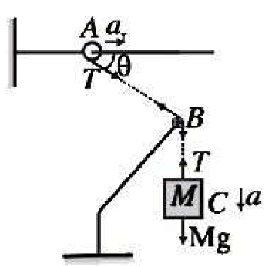
Short-cut solution :

If a is the acceleration of block and a_r is the acceleration of the ring, then

$$Ta_r\cos\theta + Ta\cos 180° = 0$$

or
$$a = a_r\cos\theta$$

or
$$a_r = a/\cos\theta \quad \dots \text{(i)}$$



Now using Newton's second law, we have

$$Mg - T = Ma \quad \dots \text{(ii)}$$

and
$$T\cos\theta = ma_r \quad \dots \text{(iii)}$$

On solving above equations, we get

$$a = \frac{Mg\cos^2\theta}{m + M\cos^2\theta}$$

and $$a_r = \frac{a}{\cos\theta} = \frac{Mg\cos\theta}{m + M\cos^2\theta}$$

Putting $M = 2m$ and $\theta = 30°$, $g = 9.8\ \text{m/s}^2$; $a = 6.78\ \text{m/s}^2$ ***Ans.***

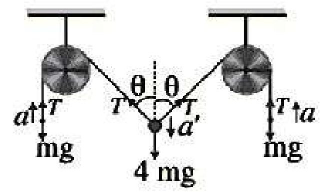
Illustration 21

In the figure shown, the pulleys and strings are massless. Find acceleration of the block of mass 4 m just after the system is released from rest. ($\theta = \sin^{-1}\frac{3}{5}$)

Short-cut solution :

If 'a' is the acceleration of block of mass m, then

$$\underbrace{Ta\cos 0° + Ta\cos 0°}_{\text{for both blocks of mass m}} + \underbrace{2T\cos\theta \times a'\cos 180°}_{\text{for block of 4m}} = 0$$



or $$a' = \frac{a}{\cos\theta} \downarrow$$

Using Newton's second law,

$$T - mg = ma \quad \text{...(i)}$$

and $$4\,mg - 2T\cos\theta = 4m\left(\frac{a}{\cos\theta}\right)$$

or $$4mg - 2T \times \frac{4}{5} = 4m\frac{a}{(4/5)} \quad \text{...(ii)}$$

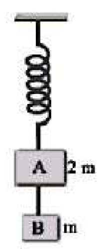
On simplifying above equations, we get

$$a = \frac{12g}{33}$$

and $$\frac{a}{\cos\theta} = \frac{5g}{11} \quad \textbf{\textit{Ans.}}$$

Illustration 22

Two blocks A and B of masses 2 m and m, respectively, are connected by a massless and inextensible string. The whole system is suspended by a massless spring as shown in the figure. The magnitude of acceleration of A and B, immediately after the string is cut, are respectively : ***[JEE Adv. 2006]***



(a) $g, \frac{g}{2}$ *(b)* $\frac{g}{2}, g$ *(c)* g, g *(d)* $\frac{g}{2}, \frac{g}{2}$

Short-cut solution :

The tension in the string connected between A and B will be, $T = mg$. when this string is cut, the unbalanced force for both the blocks becomes equal to mg. Therefore,

$$a_A = \frac{\text{unbalanced force}}{\text{mass}} = \frac{mg}{2m} = \frac{g}{2}$$

and $$a_B = \frac{mg}{m} = g. \qquad \textit{Ans. (b)}$$

Short-cut approach for friction :

Acceleration, $$a = \frac{\text{unbalanced load} - \text{friction}}{\text{total mass}}$$

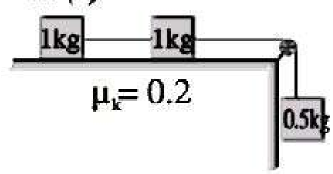
$$= \left[\frac{\text{unbalanced load} - f}{M}\right]$$

Tension, $T = m_{up}(g + a)$

or $T = m_{down}(g - a)$

For inclined plane $T = m(g \sin\theta \pm a)$. *Ans. (b)*

Illustration 23

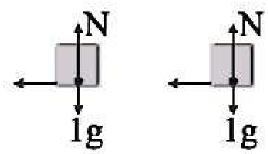


In the device shown, find

(i) acceleration of the blocks.

(ii) tension in the string attached to 0.50 kg. Take $g = 10\ m/s^2$.

Short-cut solution :



Total frictional force, $f = \mu(N + N)$
$= 0.2(1g + 1g)$
$= 0.4g = 4\ N$.

Acceleration, $$a = \frac{\text{unbalanced load} - f}{M}$$

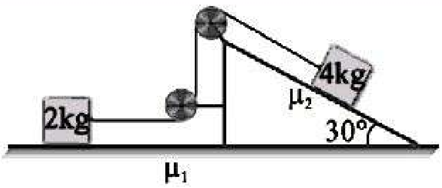
$$= \frac{0.5g - 4}{1 + 1 + 0.5}$$

$$= \frac{5 - 4}{2.5} = 0.4\ m/s^2$$

Tension, $T = m(g - a)$
$= 0.5\,(10 - 0.4) = 4.8\ N$ *Ans.*

Illustration 24

In the device shown, the acceleration of each block is $0.5\ m/s^2$ and tension in the string is 16 N. Find coefficient of friction (COF) at two contact surfaces with the blocks. (Take $g = 10\ m/s^2$)



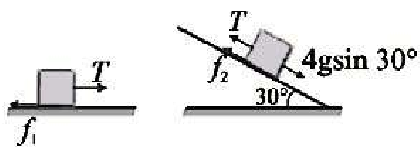
Short-cut solution :

For 2kg block, $a = \dfrac{T - f_1}{m}$

or $0.5 = \dfrac{16 - \mu_1 \times 2g}{2}$

$\therefore$ $\mu_1 = \dfrac{15}{2g} = \dfrac{15}{20} = 0.75$. ***Ans.***

For 4 kg block,



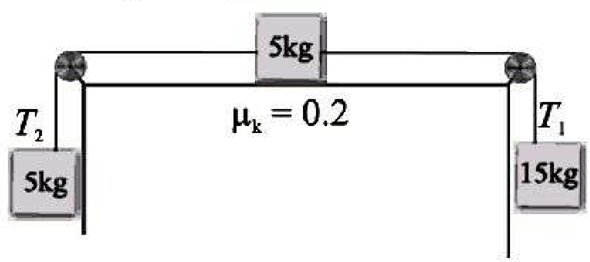
$$a = \frac{(4g\sin 30° - T - f_2)}{4}$$

or $0.5 = \dfrac{4 \times 10 \times \frac{1}{2} - 16 - \mu_2 \times 4g\cos 30°}{4}$

$\therefore$ $\mu_2 = 0.06$. ***Ans.***

Illustration 25

In the device shown, the friction between the table and the block is 0.2. Find tensions in the two strings. Take g = 10 m/s^2.



Short-cut solution :

Acceleration, $a = \dfrac{\text{unbalanced load} - f}{M}$

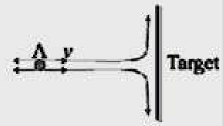
$= \dfrac{(15-5)g - 0.2 \times 5g}{5+5+15}$

$= 3.6 \text{ m/s}^2$

Tension, $T_1 = 15(g - a) = 15(10 - 3.6) = 96$

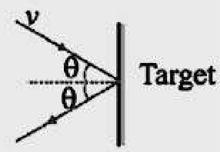
$T_2 = 5(g + a) = 5(10 + 3.6) = 68$ N. ***Ans.***

Tips and Tricks for Force Exerted by Jet Shortcut Solutions



1. Rate of flow, $Q = Av$
2. Force exerted, $F = \rho Q v$.

   Here v is the velocity of jet with respect to target.

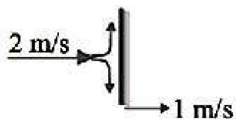


3. If jet strikes at an angle θ and rebounds with same angle, then force exerted

$$F = 2\rho Q v \cos\theta$$

Illustration 26

A jet of water area 10 cm² strikes a wall normally with velocity 2 m/s. If wall is moving away from the jet with a velocity 1 m/s. Find force exerted by jet on the wall. Assuming water splashes parallel to wall after strike.



Short-cut solution :

Force exerted,
$$F = \rho Q v$$
$$= \rho(Av)v = \rho A v^2$$
$$= 1000 \times 10 \times 10^{-4} \times (2-1)^2$$
$$= 1\text{ N} \qquad \textit{Ans.}$$

TOPIC 4.2: *Circular Motion and Banking of Road.*

Review of Formulae

1. UCM : There is only centripetal acceleration $a_c = \dfrac{v^2}{r} = \omega^2 r$

Centripetal force $F_c = \dfrac{mv^2}{r}$

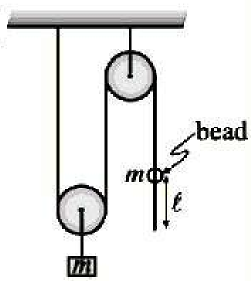
2. Non-UCM : $a_c = \dfrac{v^2}{r}$ and $a_t = \alpha r$.

Resultant acceleration, $a = \sqrt{a_c^2 + a_t^2}$

3. Banking of road: $\dfrac{\tan\theta \pm \mu}{1 \mp \mu_s \tan\theta} = \dfrac{v^2}{rg}$.

Video Solution

Q. In the figure shown, friction force between the bead and the light string is mg/4, the time in which the bead looses contact with string after the system is released from rest



(a) $\sqrt{\frac{7\ell}{8g}}$ (b) $\sqrt{\frac{8\ell}{7g}}$

(c) $\sqrt{\frac{4\ell}{7g}}$ (d) $\sqrt{\frac{2\ell}{7g}}$

To see the video solution, scan the QR code:

OR Visit https: //www.youtube.com/watch?v=glB2EYXpvLw

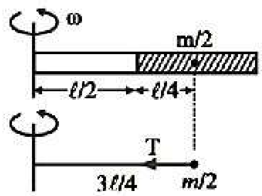
Illustration 27

A rod of length ℓ and mass m is rotated in a horizontal plane about its one end with constant angular velocity ω. The tension at the middle section of the rod is :

(a) $m\omega^2\ell$ (b) $\frac{m\omega^2\ell}{2}$ (c) $\frac{3}{4}m\omega^2\ell$ (d) $\frac{3}{8}m\omega^2\ell$

Short-cut solution :

The tension at the middle of the rod is due to only $m/2$ mass of the rod which assumed to rotate about its CM. So



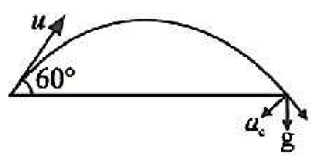
$$T = \left(\frac{m}{2}\right)\omega^2 r = \frac{m}{2}\omega^2\left(\frac{3\ell}{4}\right) = \frac{3}{8}m\omega^2\ell.$$ *Ans. (d)*

Illustration 28

A particle is projected with velocity 10 m/s at an angle 60° with the horizontal. The radius of curvature of the path of the particle, when it hits the ground is: (Take g = 10 m/s²)

(a) 10 m (b) 20 m (c) 25 m (d) 40 m

Short-cut solution :



$$a_C = g\cos 60°$$

or $$\frac{v^2}{R} = g\cos 60°$$

$\therefore$ $$R = \frac{u^2}{g\cos 60°} = \frac{10^2}{10\times\frac{1}{2}} = 20\,\text{m}.$$ *Ans. (b)*

Video Solution

Q. A particle of mass m slides from the top of the surface of a sphere of radius R. It looses contact and strikes the ground. At what depth below the top the particle will loose contact with the surface?

To see the video solution, scan the QR code:
OR Visit https://www.youtube.com/watch?v=GPWJF4QB2dM

Illustration 29

Two cars S_1 and S_2 are moving in coplanar concentric circular tracks in the opposite sense with the periods of revolution 3 min and 24 min, respectively. At time t = 0, the cars are farthest apart. Then, the two cars will be : *[KVPY-2017]*

(a) closest to each other at t = 12 min and farthest at t = 18 min.

(b) closest to each other at t = 3 min and farthest at t = 24 min.

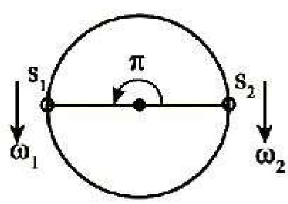
(c) closest to each other at t = 6 min and farthest at t = 12 min.

(d) closest to each other at t = 12 min and farthest at t = 24 min.

Short-cut solution :

$\omega_1 = \frac{2\pi}{3}$ and $\omega_2 = \frac{2\pi}{24}$,

$\therefore \quad \omega_1 + \omega_2 = \frac{3\pi}{4}$



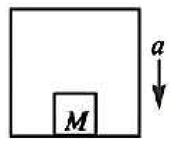
Time for meet (closest) = $\frac{\pi(2n+1)}{\omega_1+\omega_2} \Rightarrow \frac{4}{3}(2n+1)$

$t \Rightarrow \frac{4}{3}$ sec, 4, $\frac{20}{3}$, $\frac{28}{3}$, 12

Time of farthest $\Rightarrow \frac{2\pi n}{\omega_1+\omega_2} = \frac{2\pi n}{3\pi/4} = \frac{8n}{3}; \frac{8}{3}, 8, 24$ ***Ans. (d)***

Concept Booster Exercise

1. The acceleration should the box of figure descend so that the block of mass M exerts force $\frac{Mg}{4}$ on the floor of the box is :



(a) $\frac{g}{4}$ (b) $\frac{3g}{4}$ (c) $\frac{g}{2}$ (d) $\frac{3g}{2}$

2. A small block A is placed on another block B of mass 10 kg and length 2 m. Initially the block A is near right end of block B. A constant force of 10 N is applied to the block B. All surfaces are assumed frictionless. The time elapsed before the block A separates from B is : **Numeric/Integer**

(a) 1 s (b) 2 s (c) 0.5 s (d) 4 s

3. Two small blocks of masses 1 kg and 2 kg are connected by light string which passes over massless smooth pulley. After 1 second of motion the larger block is stopped for a moment. The time elapsed before the string is tight again is: **Numeric/Integer**

(a) 1s (b) 2 s (c) $\frac{1}{3}$ s (d) $\frac{2}{3}$ s

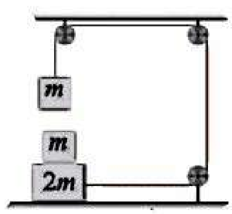
4. A block is kept on a floor of a lift at rest. The lift starts descending with an acceleration of 11 m/s^2. The displacement of the block during the first 1 s after the start is : **Numeric/Integer**

(a) 5.5 m (b) 4.5 m (c) 5 m (d) 10 m

5. A man of mass 60 kg sitting on a scooter which is moving on a straight road with constant acceleration 10 m/s^2. The force exerted by man on the seat of scooter is : **Numeric/Integer**

(a) 600 N (b) 1200 N (c) 860 N (d) 848 N

6. In the arrangement shown, the coefficient of friction (COF) between 'm' and '$2m$' so that they move together is : **Numeric/Integer**



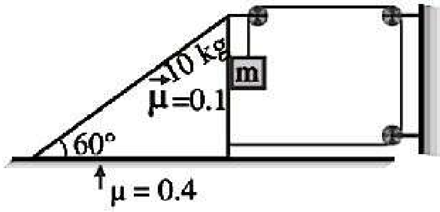
(a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

7. In the figure the block of mass 1 kg is struck by jet releasing water at a rate of 1 kg/s and at a speed of 3 m/s. The force exerted by the block of mass 1 kg on 2 kg is : **Numeric/Integer**

(a) 1 N (b) 2 N (c) 3 N (d) 4 N

8. The maximum value m so that the arrangement shown in the figure is in equilibrium is given by :

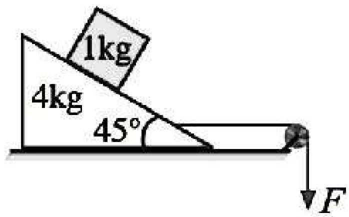
Numeric/Integer



(a) 1 kg (b) 2 kg (c) 4 kg (d) $\frac{5}{2}$ kg

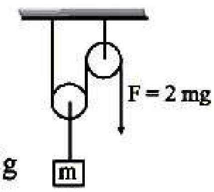
9. In the device shown, find the value of 'F' so that both the blocks move together. The inclination and surface are smooth. (Take g = 10 m/s^2)

Numeric/Integer



(a) 10 N (b) 50 N (c) 40 N (d) 100 N

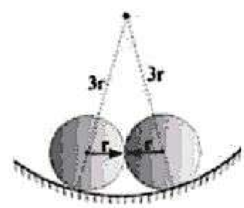
10. In the shown mass pulley system. Pulleys and string are massless. The one end of the string is pulled by the force $F = 2mg$. The acceleration of the block will be



(a) g/2 (b) 0 (c) g (d) 3g

11. Two equal heavy spheres, each of radius r, are in equilibrium within a smooth cup of radius $3r$. The ratio of reaction between the cup and one sphere and that between the two sphere is

Numeric/Integer



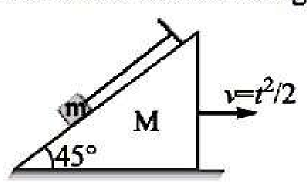
(a) 1 (b) 2
(c) 3 (d) none

12. A block of mass 10kg is suspended from string of length 4m. When pulled by a force F along horizontal from midpoint. Upper half of string makes 45° with vertical, value of F is *[JEE Main 2020]*

(a) 100 N (b) 90 N (c) 75 N (d) 70 N

13. In the given figure a block of mass m is tied on a wedge by an ideal string as shown in figure. String is parallel to inclined plane. All the surface involved are smooth. Wedge is spring moved towards right with a time varying velocity $v = t^2/2$ (m/s). At what time block will just leave the contact with the wedge (take $g = 10$ m/s^2)

Numeric/Integer



(a) 2 s (b) 4 s (c) 5 s (d) 10 s

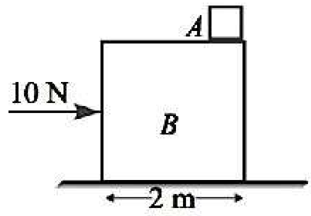
Solutions

1. **(b)** $N = m(g-a)$

or $\frac{mg}{4} = mg - ma \Rightarrow a = \frac{3g}{4}$

2. **(b)** Acceleration of block B

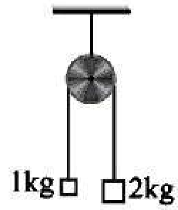
$$a = \frac{10}{10} = 1 \text{ m/s}^2$$



Now, $2 = 0 + \frac{1}{2} \times 1 \times t^2$

or $t = 2$ s

3. **(c)** Acceleration, $a = \frac{(2-1)g}{2+1} = \frac{g}{3}$ m/s^2



Using, $v = u + at$, $v = 0 + a \times 1 = \frac{g}{3} \times 1 = \frac{g}{3}$ m/s^2

When larger block stopped for moment, the smaller block move under gravity, so again,

$$0 = v - gt$$

or $t = \frac{v}{g} = \frac{g/3}{g} = \frac{1}{3}$s. ***Ans.***

4. **(c)** The acceleration of lift is greater than 10 m/s^2 and so block no remain in contact with the lift, it falls under gravity. So

$$h = 0 + \frac{1}{2} \times 10 \times 1^2 = 5 \text{ m}$$

5. **(d)** $f = ma = 60 \times 10$

$= 600$ N,

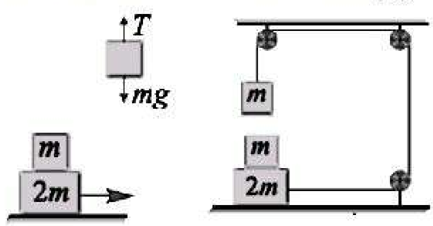
$$F = \sqrt{600^2 + 600^2} = 600\sqrt{2}\ N$$

6. **(a)** If μ the COF between the blocks then acceleration of these blocks can be μg.

So $mg - T = m(\mu g)$

and $T = 3m(\mu g)$

$\Rightarrow$ $\mu = \frac{1}{4}$



7. **(b)** Force exerted by jet $= \left(\frac{dm}{dt}\right)v = 1 \times 3 = 3$ N

Acceleration of blocks $= \frac{3}{3} = 1$ m/s^2

Thus required force, $F = 2 \times 1 = 2$ N.

8. **(d)** Here $N' = 0$, $\therefore f' = 0$

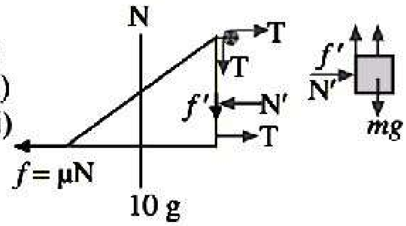
$\therefore \quad T = mg \quad ...(i)$

For large block, $10\,g + T = N \quad ...(ii)$

and $2T = f = \mu N \quad ...(iii)$

On solving, we get

$$m = \frac{5}{2}\text{ kg}.$$



9. **(b)**

$$F = ma = m(g \tan 45°)$$
$$= (4 + 1) \times 10 = 50\,N.$$

10. **(d)**

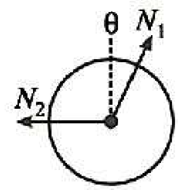
$$a = \frac{2T - mg}{m} = \frac{2 \times 2\,mg - mg}{m} = 3g.$$

11. **(b)**

$$\sin\theta = \frac{1}{2}$$

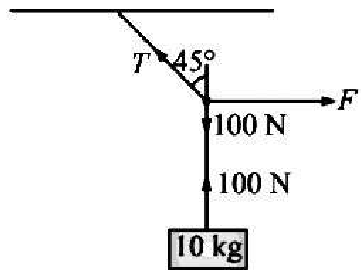
Thus, $N_1 \sin\theta = N_2$

$$\therefore \quad \frac{N_1}{N_2} = \frac{1}{\sin\theta} = 2.$$



12. **(a)**

$$\frac{T}{\sqrt{2}} = 100$$
$$\frac{T}{\sqrt{2}} = F$$
$$F = 100\text{ N}$$



Ans.

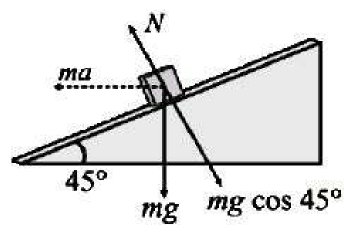
13. **(d)** $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{t^2}{2}\right) = i$

For $N = 0$,

$$Ma \sin 45° = mg \cos 45°$$

or

$$t = g \cot 45°$$
$$= 10\text{ s}$$



5 Work, Energy and Power

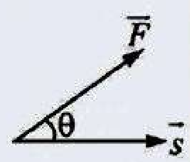
TOPIC: *Work done, Kinetic, Potential Energy and Power.*

Review of Formulae

1. Work done, $W = Fs \cos \theta$
2. $K = \frac{1}{2}mv^2$, $U = mgh$
3. Potential energy of the spring, $U = \frac{1}{2}k\,(\pm x)^2$
4. Power $= \frac{dW}{dt} = \vec{F} \cdot \vec{v}$
5. Efficiency, $\eta = \frac{\text{Output}}{\text{Input}}$
6. The power of a machine gun firing 'n' bullets per second each of mass m with a speed v will be $P = n\left(\frac{1}{2}mv^2\right)$.
7. Power of pump needed to lift water, $P = \left(\frac{dm}{dt}\right)gh$.

Tips and Tricks for Shortcut Solutions

1. Work done by constant force
   = force × displacement in the direction of force.
   $W = F(s \cos \theta) = (F \cos \theta)s$



2. Work done by variable force, $W = \int_{s_1}^{s_2} \vec{F} \cdot d\vec{s}$
3. Kinetic energy, $K = \frac{p^2}{2m}$.
4. Kinetic energy is associated with all kind of forces, but potential energy is associated only with conservative forces.

5. Work-energy theorem, $W_{all} = \Delta K$.

6. Conservative force, $W_c = -\Delta U$ or $F_c = -\frac{\Delta U}{\Delta s}$.

7. In case of variable force, work-energy theorem is short and easy in use.

8. Under conservative forces, $[K + U]$ = constant.

9. Work done and kinetic energy depends on reference frame. In two inertial frames with same relative speed, work done and K.E. will be different but acceleration and force will be same.

10. Work done by static friction may be zero, positive or negative.

11. Work done by kinetic friction may be zero or negative.

12. When spring loaded gently, its extension or compression $x_0 = \frac{Mg}{k}$, because at the extreme position it kept in equilibrium by the load Mg.

13. When spring loaded suddenly (or maximum extension or compression is asked) the block will not be in equilibrium by the load Mg. So energy equation is to be used to get maximum extension or compression. It generally $x = 2x_0$.

Illustration 1

Under the action of force, 2 kg body moves such that its position x as a function of time t is given by $x = \frac{t^3}{3}$, x is in metre and t in second. Calculate the work done by the force in the first 2 second.

Short-cut solution :

By work-energy theorem

Given $x = t^3/3$

$\therefore$ Velocity $v = \frac{dx}{dt} = t^2$

At $t = 0$, $v_i = 0^2 = 0$

At $t = 2$, $v_f = 2^2 = 4$ m/s

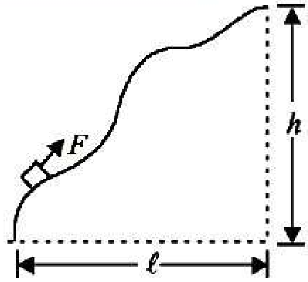
Work done $W = \frac{1}{2}m(v_f^2 - v_i^2)$

$= \frac{1}{2} \times 2 \times (4^2 - 0)$

$= 16$ J. ***Ans.***

Illustration 2

A body of mass 'm' was slowly hauled up the hill as shown in figure, by a force F which at each point was directed along a tangent to the trajectory. Find work performed by this force, if the height of the hill is h, the length of its base is ℓ, and coefficient of friction μ.



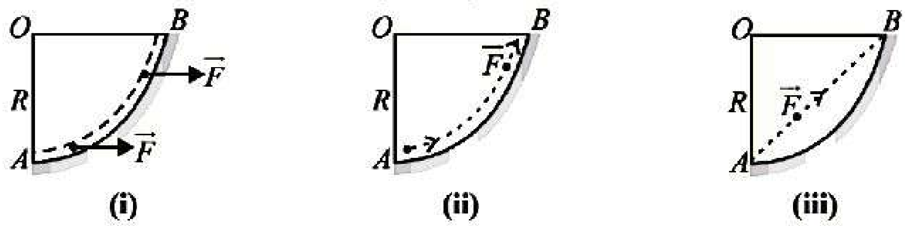
Short-cut solution :

Here gravitational force acting vertically, so frictional force will be horizontally. Therefore

$$W_F = \text{w.d. against friction} + \text{w.d. against gravity}$$
$$= f_k s + (mg)h$$
$$= \mu mgl + mgh. \qquad \textbf{\textit{Ans.}}$$

Illustration 3

Figure shows a smooth circular track of radius 'R' in the vertical plane which subtends an angle $\frac{\pi}{2}$ at the centre O. A particle of mass m is taken from A to B under the action of force F in the following situations. Find the work done by the force F in each of these cases separately.



Short-cut solution :

(i) The displacement of the particle in the direction of force is, $s = R$. Therefore, work done $= Fs = FR$. ***Ans.***

(ii) The displacement of the particle in the direction of force, $s = \frac{\pi R}{2}$. Therefore work done

$$W = Fs = \frac{F\pi R}{2} = \frac{\pi FR}{2} \qquad \textbf{\textit{Ans.}}$$

(iii) The displacement of the particle in the direction of force, $s = \sqrt{2}R$. Therefore work done

$$W = Fs = F(\sqrt{2}R) = \sqrt{}FR. \qquad \textbf{\textit{Ans.}}$$

Illustration 4

A block is connected to a massless spring of force constant K. The other end of spring is fixed to the wall. The spring is stretched by x_0 from its normal length. Calculate work done by the spring on the block in following situations:

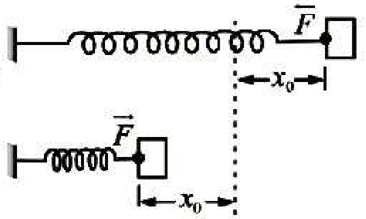
(i) When spring attained its normal length

(ii) When spring gets compressed by x_0.

Short-cut solution :

(i) In this case force exerted by the spring F and displacement of the block (point B) are in the same direction, so

$$W = \int_0^{x_0} F dx = \int_0^{x_0} kx dx$$



$$= \frac{1}{2} k x_0^2$$

In this case work done is positive.

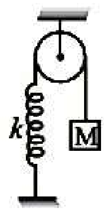
(ii) In this case force exerted by spring on the block and its displacement are opposite, so

$$W = \int_0^{x_0} (-kx) dx = -\frac{1}{2} k x_0^2 .$$

In this case work done by spring is negative. ***Ans.***

Illustration 5

A block of mass M is connected to a massless string-spring of constant k and passes over smooth and massless pulley as shown. Initially spring is unstreched when the block is released. Find tension in the string-spring device if



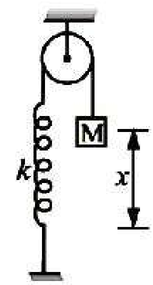
(i) block is allowed to move slowly, so that it will be in equilibrium at lowest position.

(ii) block is left suddenly.

Short-cut solution :

(i) The block is in equilibrium at its lowest position so, tension in the string-spring device

$$T = Mg$$



(ii) In this case extension of the spring will be twice that in case (i). So tension in the string will be 2 mg.

By conservation of ME energy, we have

$$Mgx = \frac{1}{2} kx^2$$

or $$kx = 2\,Mg$$

or $$T = 2\,Mg$$ ***Ans.***

Illustration 6

The potential energy for a conservative force is given by $U = k(x + y)$. The work done by the conservative force in moving a particle from the point $A(1, 1)$ to point $B(2, 3)$ is given by:

(a) $3k$ ***(b)*** $-3k$ ***(c)*** $6k$ ***(d)*** $7k$

Short-cut solution :

We know that

$$W_c = -\Delta U$$
$$= -(U_f - U_i)$$
$$= (U_i - U_f)$$
$$U_i = k(x + y) = k(1 + 1) = 2k$$

and

$$U_f = k(2 + 3) = 5k$$

$\therefore$

$$Wc = 2k - 5k = -3k$$

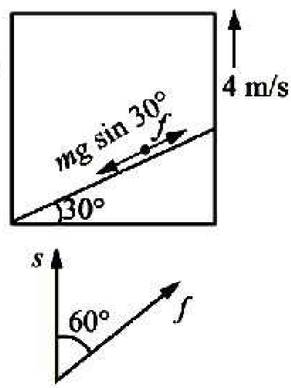
Ans. (b)

Illustration 7

A small block of mass m is kept on a rough inclined surface of inclination θ = 30° fixed in a lift. The lift goes up with a uniform velocity v = 4 m/s and the block does not slide on the wedge. The work done by friction on the block in 1 second will be: (Take g = 10 m/s²)

(a) ***5J*** *(b)* ***10J*** *(c)* ***20J*** *(d)* ***zero***

Short-cut solution :



As the block does not slide on the inclined, so friction

$$f = mg \sin 30°$$
$$= 1 \times 10 \times \frac{1}{2}$$
$$= 5 \text{ N}.$$

The displacement in 1 second

$$s = vt = 4 \times 1 = 4\text{m}$$

Now work done by friction

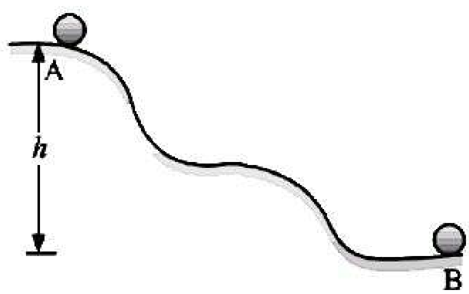
$$W = fs \cos 60° = 5 \times 4 \times \frac{1}{2} = 10 \text{ J}.$$

Ans. (b)

Illustration 8

An object of mass m is sliding down on rough zigzag path and stops after travelling a certain horizontal distance because of friction. The COF is same throughout and independent on velocity, find work done by a force to return the object to its initial position along the same path :

Short-cut solution :



Journey from A to B:

$$W_{gravity} + W_{friction} = \Delta K$$
$$mgh + W_{friction} = 0$$

$\therefore$

$$W_{friction} = -mgh.$$

Journey from B to A:

$$W_F + W_{\text{gravity}} + W_{\text{friction}} = \Delta K$$

or $$W_F - mgh - mgh = 0$$

$\therefore$ $$W_F = 2mgh.$$ *Ans.*

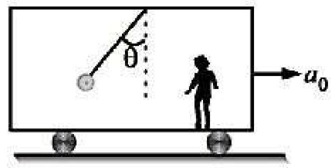
Illustration 9

A pendulum is suspended inside a trolley. Find the angle deflected by pendulum from vertical, if

(i) trolley attain acceleration a_0 very slowly.

(ii) trolley suddenly attains acceleration a_0.

Short-cut solution :

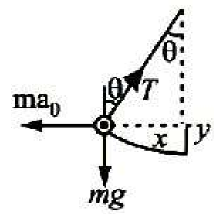


(i) In this case bob will be in equilibrium with respect to trolley, at its extreme position

$$T \sin \theta = ma_0$$

and $$T \cos \theta = mg$$

$\therefore$ $$\tan \theta = \frac{a_0}{g}$$



or $$\theta = \tan^{-1}\left(\frac{a_0}{g}\right)$$ *Ans.*

(ii) In this case bob will not be in equilibrium with respect to trolley at its extreme position, so

$$W_{\text{pseudo}} + W_{\text{gravity}} + W_{\text{tension}} = \Delta K$$

$$ma_0 \times x - mgy + 0 = 0$$

or $$ma_0(\ell \sin \theta) - mg\ell(1 - \cos \theta) = 0$$

or $$a_0 \times 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} - g \times 2\sin^2 \frac{\theta}{2} = 0$$

$\therefore$ $$\tan \frac{\theta}{2} = \frac{a_0}{g}$$

or $$\theta = 2\tan^{-1}\left(\frac{a_0}{g}\right)$$ *Ans.*

Illustration 10

The potential energy of 1 kg particle free to move along the x-axis is given by $U(x) = \left(\frac{x^4}{4} - \frac{x^2}{2}\right)$J. The total mechanical energy of the particle is 2 J then, the maximum speed in (m/s) is : ***[JEE Main 2006]***

(a) 2 *(b)* $\frac{3}{\sqrt{2}}$ *(c)* $\sqrt{2}$ *(d)* $\frac{1}{\sqrt{2}}$

Solution :

$U + K = 2$

For maximum K, U should be minimum, and so

$$\frac{dU}{dx} = 0 \text{, or } \frac{d}{dx}\left(\frac{x^4}{4} - \frac{x^2}{2}\right) = 0$$

or $$x = 0, \pm 1$$

$$U_{min} = \frac{(1)^4}{4} - \frac{(1)^2}{2} = -\frac{1}{4} J$$

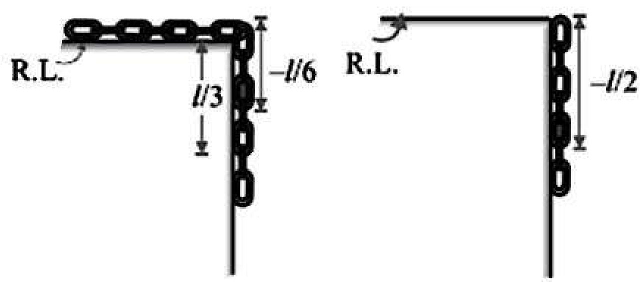
$$\therefore \quad -\frac{1}{4} + \frac{1}{2} \times 1 \times v_{max}^2 = 2$$

or $$v_{min} = \frac{3}{\sqrt{2}} \text{ m/s.}$$ *Ans. (b)*

Illustration 11

A uniform chain of length ℓ and mass m overhangs a smooth table with its two third part lying on the table. Find the kinetic energy of the chain as it completely slip of the table.

Solution :



With respect to the top of the table, the initial potential energy of the chain U_i = P.E. of the chain lying on the table +*P.E.* of the over hanging part of the chain

$$= \frac{2m}{3} g \times 0 + (m/3) g \times (-\ell/6) = -\frac{mg\ell}{18}.$$

P.E. of chain at the instant of slip

$$U_f = 0 + mg(-\ell/2) = -\frac{mg\ell}{2}.$$

Since only gravity is acting on the chain, therefore we have

$$-\Delta U = \Delta K$$

or $$\Delta K = -(U_f - U_i)$$

$$= -\left[\frac{-mg\ell}{2} - \left(\frac{-mg\ell}{18}\right)\right] = \frac{4}{9} mg\ell$$

Since $K_i = 0$

$\therefore$ $K_f = \frac{4}{9}mg\ell$. ***Ans.***

Illustration 12

A particle is delivered constant power, its displacement is proportional to :

(a) $t^{\frac{1}{2}}$ *(b)* $t^{\frac{3}{2}}$ *(c)* $t^{\frac{-3}{2}}$ *(d)* $t^{\frac{-1}{2}}$

Short-cut solution :

Dimensions of power

$$P \rightarrow ML^2T^{-3}$$

$\therefore$ $L \propto T^{\frac{3}{2}}$ ***Ans. (b)***

Illustration 13

The blades of a wind mill sweep out a circle of area A. If wind flows at a velocity v perpendicular to the circle, the power produced by wind mill :

(a) $P \propto v$ *(b)* $P \propto v^2$ *(c)* $P \propto v^3$ *(d)* $P \propto v^4$

Short-cut solution :

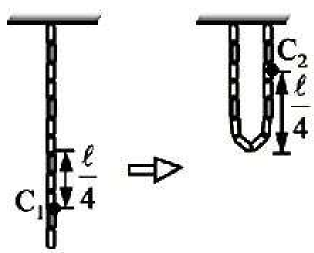
Power,

$$P \propto Fv$$
$$\propto (\rho Qv)v \quad [Q = Av]$$
$$\propto \rho(Av)v^2$$
$$\propto v^3$$

Ans. (c)

Illustration 14

A uniform chain of mass 'M' and of length 'ℓ' is suspended vertically. The lower end of the chain is lifted upto point of suspension. Calculate work done in the process.



Short-cut solution :

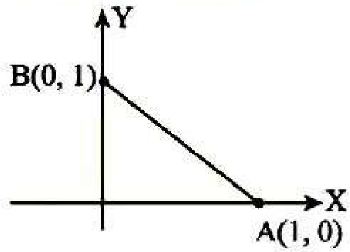
The rise in CG from C_1 to C_2,

$$h = \frac{\ell}{4} + \frac{\ell}{4} = \frac{\ell}{2}$$

$$\therefore \quad W = \left(\frac{m}{2}\right)gh = \frac{m}{2}g \times \frac{\ell}{2} = \frac{mg\ell}{4}$$

Illustration 15

Particle moves from point A to point B along the line shown in figure under the action of force. $\vec{F} = -x\hat{i} + y\hat{j}$. Determine the work done on the particle by $\vec{F}$ in moving the particle from point A to point B. *[JEE Main 2020]*



(a) 1 J *(b)* $\frac{1}{2}$ J *(c)* 2 J *(d)* 3 J

Short-cut solution :

$$W = \int \vec{F}.d\vec{s} = (-x\hat{i} + y\hat{j}).(dx\hat{i} + dy\hat{j})$$

$$= \int_1^0 -xdx + \int_0^1 ydy$$

$$= -\frac{x^2}{2}\bigg|_1^0 + \frac{y^2}{2}\bigg|_0^1 = \left(0+\frac{1}{2}\right)+\left(\frac{1}{2}\right) = 1\text{J}$$

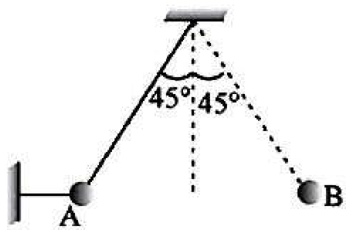
Ans. (a)

Concept Booster Exercise

1. A particle in equilibrium displaces from (1, 1, 2)m to (2, 0, 3)m. If one of the forces acting on the particle is $(\hat{i} - \hat{j} + 3\hat{k})$ N, total work done by the remaining forces acting on the particle is : **Numeric/Integer**

   (a) 5 J (b) –5 J
   (c) 10 J (d) 0

2. A ball is held at rest in position A in figure by two light cords. The horizontal string is cut and ball starts swinging as a pendulum. The ratio of the tension in the supporting cord in position B to that in position A is : **Numeric/Integer**



   (a) 1 (b) 1/2
   (c) 2 (d) 3/2

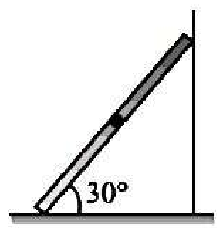
3. A body of mass 0.3 kg is taken up an inclined plane of inclination 30° with the horizontal and then allowed to slide down to bottom again. Work done by frictional force over the round trip, if $\mu = 0.15$: **Numeric/Integer**

   (a) –5 J (b) –5.8 J
   (c) –7.6 J (d) 0

4. Potential energy of a particle is given by $U = (2x + 3y + 3z)J$. The force on the particle: **Numeric/Integer**

   (a) 8 N (b) $\sqrt{29}N$
   (c) 36 N (d) 45 N

5. A uniform ladder of mass 'm' and length ℓ resting horizontally on the floor is lifted and held against a vertical wall at an angle of 30° with the floor. Calculate work done by gravity.

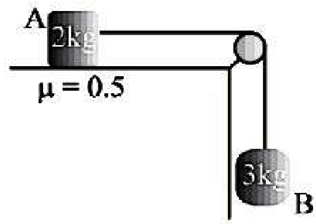


(a) $mg\ell$ (b) $-mg\ell$

(c) $-mg\frac{\ell}{2}$ (d) $-mg\frac{\ell}{4}$

6. The system shown is released at rest. Speed of block A after block B has descended by 2 cm is :

(Take $g = 10\ m/s^2$) **Numeric/Integer**



(a) 0.4 m/s (b) 0.5 m/s

(c) 0.6 m/s (d) 0.8 m/s

7. A particle of mass 'm' is projected at an angle θ to the horizontal with an initial velocity u. The work done by gravity during the time it reaches the highest point is :

(a) $\frac{-1}{2}mu^2 \sin^2\theta$ (b) $\frac{1}{2}mu^2 \cos^2\theta$

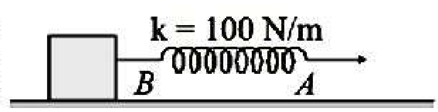
(c) $mu^2 \sin^2\theta$ (d) Zero

8. A man places a chain of mass 'm' and length ℓ on a table slowly. Initially the lower end of the chain just touches the table. The man drops the chain when half of the chain in vertical position. Then work done by man in this process is :

(a) $-mg\frac{\ell}{2}$ (b) $-\frac{mg\ell}{4}$

(c) $-\frac{3mg\ell}{8}$ (d) $-\frac{mg\ell}{8}$

9. A block lying on a smooth surface with spring connected to it is pulled by an external force as shown. Initially the velocity of ends A and B of the spring are 4 m/s and 2 m/s respectively. If the energy of the spring is increasing at the rate of 20 J/s, then the stretch in the spring is :



Numeric/Integer

(a) 1.0 cm (b) 2.0 cm

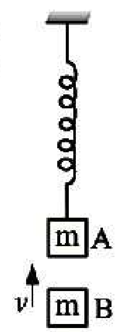
(c) 10 cm (d) 20 cm

10. A body of mass m accelerates uniformly from rest to velocity v_1 in time t_1. The instantaneous power delivered to the body is :

(a) $mv_1^2\, t_1/t^2$ (b) $mv_1^2 t_1^2/t$

(c) $mv_1^2\, t/t_1^2$ (d) $mv_1^2 t^2/t_1^3$

11. Block 'A' is hanging from a vertical spring and is at rest. Block 'B' strikes the block 'A' with velocity 'v' and sticks to it. Then the value of v for which the spring just attains natural length is :



(a) $\sqrt{\frac{12mg^2}{k}}$ (b) $\sqrt{\frac{6mg^2}{k}}$

(c) $\sqrt{\frac{5mg^2}{k}}$ (d) none of these

Solutions

1. **(b)** Supposing, $\vec{F}_1 = (\hat{i} - \hat{j} + 3\hat{k})N$

Given, $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ... + \vec{F}_n = 0$

$\therefore \quad \vec{F}_2 + \vec{F}_3 + ... \vec{F}_n = -\vec{F}_1$

work done by remaining force = work done by $(-\vec{F}_1)$.

$$= -\vec{F}_1 \cdot (\vec{r}_2 - \vec{r}_1)$$

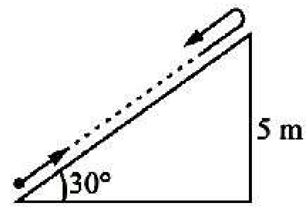
$$= -(\hat{i} - \hat{j} + 2\hat{k}) \cdot \{(2\hat{i} + 3\hat{k}) - (\hat{i} + \hat{j} + 2\hat{k})\}$$

$$= -5J.$$

2. **(b)** $\frac{T_B}{T_A} = \frac{mg\cos 45°}{mg/\cos 45°} = \cos^2 45° = \frac{1}{2}$.

3. **(c)** $W = -2f\ell = -2\ \mu mg \cos\theta \times 10$

$= -7.6\ J.$



4. **(b)** $\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$.

5. **(d)** $W = -mgh = -mg\frac{\ell}{2}\sin 30°$.

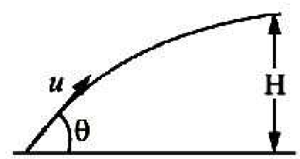
6. **(a)** $W_g + W_f = \Delta K$

or $3g \times 0.02 - 0.5 \times 2g \times 0.02 = \frac{1}{2}(2+3)v^2$

or $v = 0.4$ m/s.

7. **(a)** $W = -mg \times H$

$$= -mg\left(\frac{u^2\sin^2\theta}{2g}\right) = -\frac{1}{2}mg^2\sin^2\theta$$



8. **(c)** $W_g = (U_i - U_f)$

$$= \left(mg\frac{\ell}{2} - \frac{m}{2}g\frac{\ell}{4}\right)$$

$$= \frac{3}{8}mg\ell$$

$\therefore \quad W_{man} = -W_g = \frac{-3}{8}mg\ell$.

9. **(c)** If F is the force in the spring, then power delivered

$$F \times v_A - F \times v_B = 20$$

or $$F \times 4 - F \times 2 = 20$$

or $$F = 10 \text{ N}$$

Now by Hooke's law,

$$x = \frac{F}{k} = \frac{10}{100} = 0.1 \text{ m}$$

10. **(c)** $a = \frac{v_1}{t_1}$, $\therefore$ $F = ma = \frac{mv_1}{t_1}$; $s = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{v_1}{t_1}\right)t^2$

$$W = Fs = \frac{mv_1}{t_1} \times \frac{1}{2}\left(\frac{v_1}{t_1}\right)t^2 = \frac{mv_1^2t^2}{2t_1^2}$$

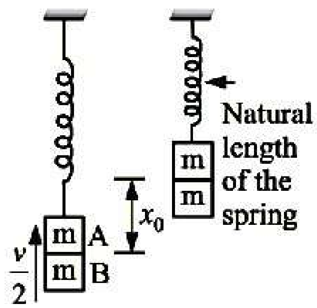
$$P = \frac{dW}{dt} = \frac{mv_1^2t}{t_1^2}$$

11. The initial extension on in spring

$$x_0 = \frac{mg}{k}$$

Just after collision, the speed of combined mass becomes $\frac{v}{2}$.

Now using conservation of ME, we have



$$\frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 + \frac{1}{2}k\left(\frac{mg}{k}\right)^2 = (2m)\, g\left(\frac{mg}{k}\right)$$

$$\Rightarrow \quad v = \sqrt{\frac{6mg^2}{k}}.$$

Collisions and Centre of Mass 6

TOPIC 6.1: *Perfectly Elastic and Inelastic Collision in 1D and 2D.*

Review of Formulae

1. For one-dimensional elastic collision

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{.......... (i)}$$

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \quad \text{.......... (ii)}$$

2. During collision, energy stored in the colliding bodies

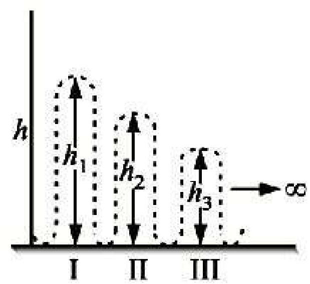
$$= \left[\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right] - \left[\frac{1}{2}(m_1 + m_2)v^2\right]$$

Here v is the velocity of bodies during collision.

3. When a steady stream of particles, each of mass m and speed v collide with a fixed body, the average force on fixed body

$$F = n\frac{\Delta P}{\Delta t} = nm\left(\frac{\Delta v}{\Delta t}\right).$$

4. A ball falls from height 'h' and rebounds again and again: If COR between ball and the floor is 'e', then



velocity, $v_n = e^n\sqrt{2gh}$

Height attained, $h_n = e^{2n}h$

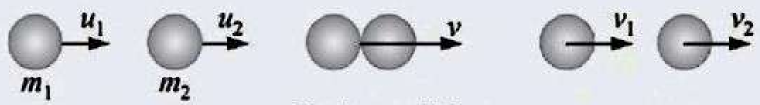
Total distance till last collision, $D = \left(\frac{1+e^2}{1-e^2}\right)h$

Total time of motion, $T = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. Collision is the phenomenon of mutual interaction, so momentum of the system remains constant in every collision. For two bodies $\Delta\vec{P}_1 + \Delta\vec{P}_2 = 0$ or $\Delta\vec{P}_1 = -\Delta\vec{P}_2$.

2. During collision, colliding bodies will have same velocity, and some part of their KE will be stored in elastic PE, so



So, $$U_e = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_1u_2^2\right) - \frac{1}{2}(m_1 + m_2)v^2$$

3. In 1-D, elastic collision, $u_1 - u_2 = -(v_1 - v_2)$.

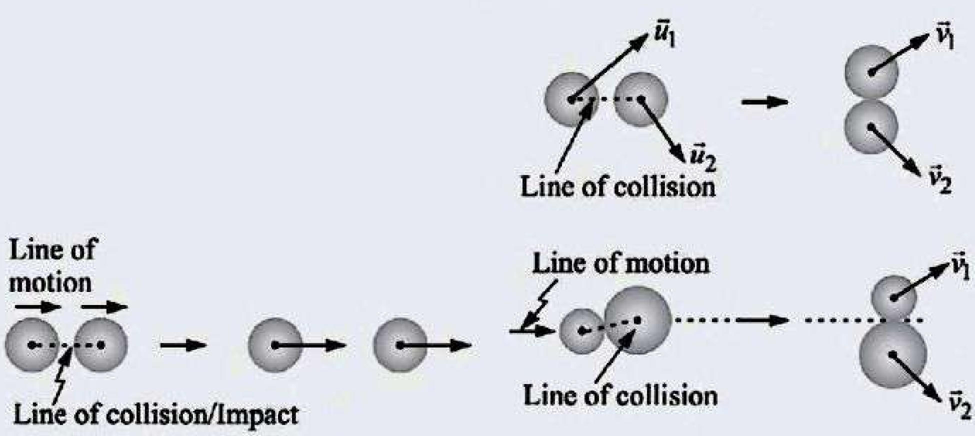
4. In perfectly elastic collision of two identical bodies, their velocities get exchanged.

5. KE transferred from projectile (m_1, u_1) to the target (m_2, u_2),

$$\frac{\Delta K}{K} = 1 - \left(\frac{v_1}{u_1}\right)^2$$

6. When line of motion of body/bodies coincides with line of *impact/line of collision*, the bodies will move in the same line after collision, and so called 1-D or head-on collision, otherwise 2-D or oblique collision.

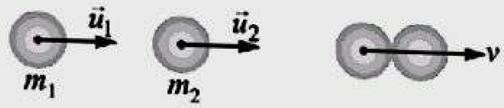
   Line of collision is the line joining geometric centres of the bodies.



1-D collision **2-D collision**

7. Perfectly *inelastic* collision in 1-D:

In this collision the loss of KE is maximum and given by,



$$\text{Loss} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(u_1 - u_2)^2.$$

General Analysis of 1-D Collision:

Coefficient of restitution (COR):

$$e = \left[\frac{v_2 - v_1}{u_1 - u_2}\right] = -\left[\frac{v_1 - v_2}{u_1 - u_2}\right] \quad \text{...(i)}$$

The value of e: $0 \le e \le 1.$

and $m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$ ...(ii)

On solving above equations, we get

$$v_1 = \left(\frac{m_1 - e m_2}{m_1 + m_2}\right)u_1 + \left[\frac{(1+e)m_2}{m_1 + m_2}\right]u_2 \quad \text{...(iii)}$$

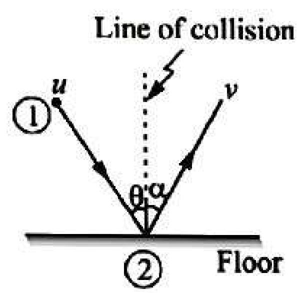
and

$$v_2 = \left(\frac{m_2 - e m_1}{m_1 + m_2}\right)u_2 + \left[\frac{(1+e)m_1}{m_1 + m_2}\right]u_1 \quad \text{...(iv)}$$

If $m_1 = m_2$ and $u_2 = 0$, then $\frac{v_1}{v_2} = \left(\frac{1-e}{1+e}\right)$.

More about COR (e):

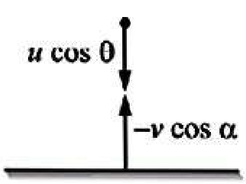
Definition of COR can be applied along line of collision



(i) In the collision of ball with the floor or wall, the line of collision is normal to the floor/wall (see figure)

In case when ball strikes the floor ($u_2 = v_2 = 0$), then

$$u_1 = u\cos\theta \text{ and } v_1 = -v\cos\alpha$$



Therefore, COR, $e = -\left[\frac{-v\cos\alpha - 0}{u\cos\theta - 0}\right]$ ...(i)

Parallel to the wall/floor

$$u\sin\theta = v\sin\alpha \quad \text{...(ii)}$$

From above equations, we get

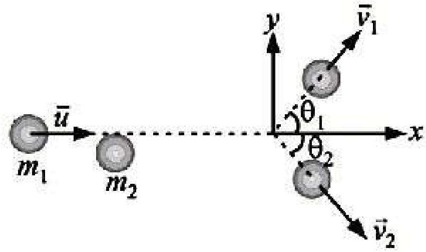
$$e = \frac{\tan\theta}{\tan\alpha}. \qquad [\text{as } e < 1, \text{ so } \theta < \alpha, u < v]$$

More About Line of Collision and Conservation of Linear Momentum

In oblique collision students generally face a difficulty in choosing the direction in which one can use conservation of momentum and the line of collision along which the definition of coefficient of restitution can be applied.

➢ The direction along which the momentum of the system is conserved is obviously the direction along which $\vec{F}_{ext} = 0$.

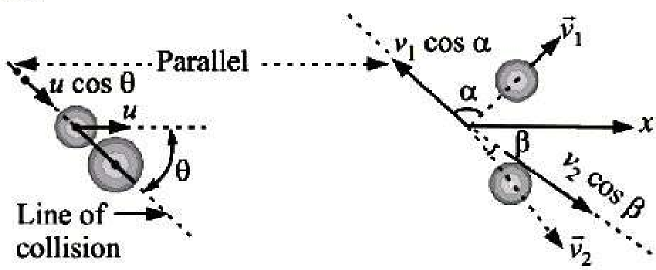
Analysis -1 : Let us take the example of oblique collision between the balls.



➢ In this collision conservation of momentum of the whole system can be applied along any direction in the plane of the bodies. Also momentum of each ball remains constant perpendicular to line of collision.

➢ Using conservation of momentum:
along x-axis; $m_1u + 0 = mv_1 \cos\theta_1 + m_2v_2 \cos\theta_2$
along y-axis; $0 + 0 = m_1v_1 \sin\theta_1 - m_2v_2 \sin\theta_2$

➢ In using the equation of coefficient of restitution (COR) e, first find line of collision, and then find velocities components of both the balls along this line.



➢ If α and β are the angles made by $\vec{v}_1$ and $\vec{v}_2$ from line of collision, (see figure), then

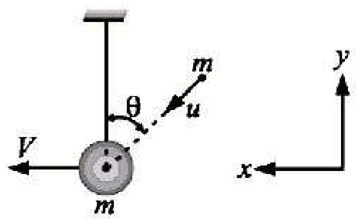
$$e = -\left[\frac{v_1 - v_2}{u_1 - u_2}\right]_{\text{along line of collision}}$$

$$= -\left[\frac{-v_1\cos\alpha - v_2\cos\beta}{u\cos\theta - 0}\right].$$

- In this type of problem, some of the angles or condition are given after collision.

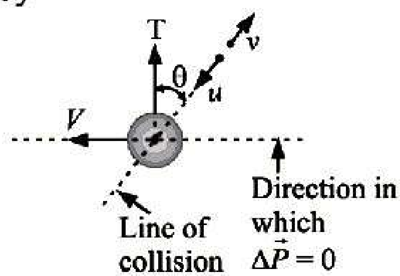
Analysis -2 : Collision of particle with the hanging ball (1 - D collision)

In this collision the direction along which momentum is constant is only x-direction, because in other directions, the tension in the string (external force) will change the momentum of the system. Also, this is 1-D collision, so striking ball will rebound in the line of collision. Therefore along horizontal direction by conservation momentum, we have



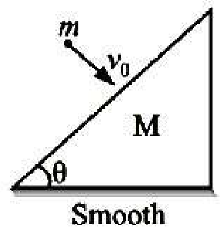
$$mu\sin\theta + 0 = MV - mv\sin\theta$$

and $$e = -\left[\frac{-v - V\sin\theta}{u - 0}\right].$$



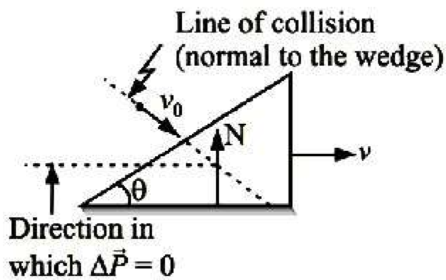
Analysis -3 : Collision between ball and wedge: Ball hits the wedge normally (1-D collision)

In this case the direction in which momentum is conserved is the horizontal direction only. In other directions the normal force from the ground will change the momentum of the system. Also the collision is 1-D, so ball will move in this direction after collision. If v_1 is the velocity of ball after collision, then we can write



$$mv_0\sin\theta + 0 = mv_1\sin\theta + Mv$$

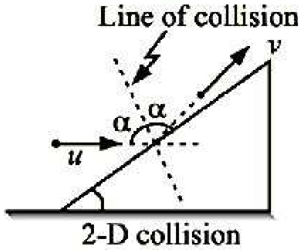
and $$e = -\left[\frac{v_1 - v\sin\theta}{v_0 - 0}\right].$$



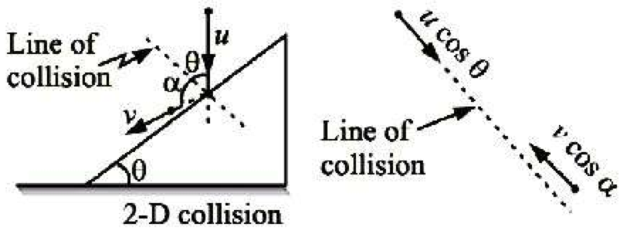
Analysis -4 : Collision of ball with the wedge (2-D collision)

Case 1: When wedge is fixed, and collision is perfectly elastic. In this case ball will rebound as the angle of incidence.

- Perpendicular to the line of collision the momentum of ball remains conserved. So $u = v$.



Case 2: When wedge is fixed, and collision is inelastic. In this case ball will rebound at an angle greater than angle of incidence.



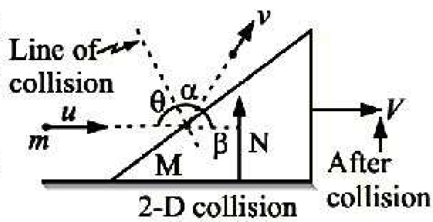
➢ Perpendicular to line of collision, momentum of ball remains conserved.

$\therefore \qquad u \sin\theta = v \sin\alpha \quad \Rightarrow \quad v < u$ and $\alpha > \theta$.

Also,
$$e = -\left[\frac{-v\cos\alpha - 0}{u\cos\theta - 0}\right].$$

On solving above equations, we get $e = \tan\theta / \tan\alpha$.

Case 3: When wedge is free to move and ball hits the wedge at some angle with the normal of the wedge.

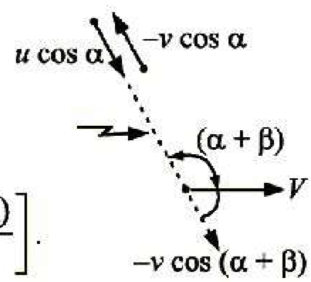


We can use conservation of momentum for ball perpendicular to line of collision, so

$$mu \sin 0 = mv \sin\alpha$$

And conservation of momentum of the system along horizontal direction only

$$mu + 0 = mv \sin\beta + MV.$$

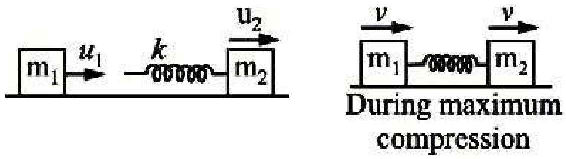


Also
$$e = -\left[\frac{-v\cos\alpha - (-V\cos(\alpha+\beta))}{u\cos\theta - 0}\right].$$

Maximum Compression of the Spring:

In such a problem in which spring is connected to either of the colliding blocks, the spring compresses till one block approaches the other block. The maximum compression will occur when the velocity of both the blocks becomes equal.

The energy stored in the spring will come back to the KE of the blocks. Take two blocks of masses m_1 and m_2 with a spring of force constant k. If their velocities are u_1 and u_2 $(u_1 > u_2)$, then

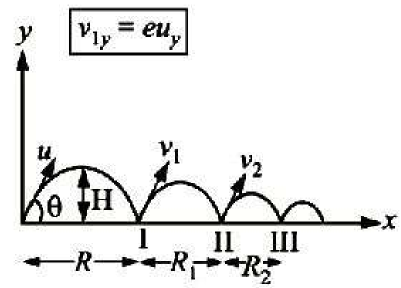


$$m_1u_1 + m_2u_2 = (m_1 + m_2)v \quad ...(i)$$

and $$\frac{1}{2}m_1u_1^2 + \frac{1}{2}mu_2^2 = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}kx_{max}^2 \quad ...(ii)$$

Above equations can give the value of x_{max}.

A ball projected at an angle θ with COR '*e*'



(i) Time for I collision, $T = \frac{2u\sin\theta}{g}$

(ii) Height attained, $H = \frac{u^2\sin^2\theta}{2g}$

(iii) Horizontal range, $R = \frac{u^2\sin 2\theta}{g}$

(iv) Total time of motion till last collision, $T_0 = \frac{T}{(1-e)}$.

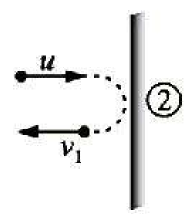
(v) Total horizontal distance, $R_0 = \frac{R}{(1-e)}$.

(vi) Sum of total maximum heights, $H_0 = \frac{H}{(1-e^2)}$.

Illustration 1

A ball hits the wall normally with velocity 'u'. If COR between ball and wall is 'e', find rebound velocity of the ball.

Short-cut solution :



$$e = \left[\frac{v_2 - v_1}{u_1 - u_2}\right]$$

As $u_2 = v_2 = 0, u_1 = u$

∴ $v_1 = -eu$. ***Ans.***

Illustration 2

Two identical balls one of them initially at rest make head-on collision. The velocity of second ball becomes two times that of first ball after collision. Find COR.

Short-cut solution :

Using, $\frac{v_1}{v_2} = \frac{1-e}{1+e}$

or $\frac{1}{2} = \frac{1-e}{1+e}$

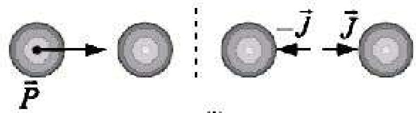
$\therefore$ $e = \frac{1}{3}$. ***Ans.***

Illustration 3

A body 'A' with a momentum 'P' collides with another identical stationary body 'B' one dimensionally. During the collision, 'B' gives an impulse 'J' to the body 'A'. Then find COR.

Short-cut solution :

$$P + 0 = P_1 + P_2$$



or $P - P_1 = P_2$

For ball B: $P_2 - 0 = J$

$\therefore$ $P - P_1 = J$ ...(i)

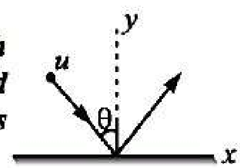
Now $e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{mv_2 - mv_1}{mu - 0}$

$= \frac{P_2 - P_1}{P}$ ...(ii)

From above equations, we get $e = \frac{2J}{P} - 1$. ***Ans.***

Illustration 4

A ball of mass m collides with the ground at an angle θ with the normal of the ground. If collision lasts for time t, then find average force exerted by the ground on the ball. The COR is 'e'.

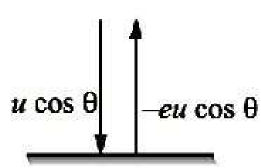


Short-cut solution :

The velocity of ball along normal before collision

$= u \cos \theta$

So after collision, $v = -e(u \cos \theta)$



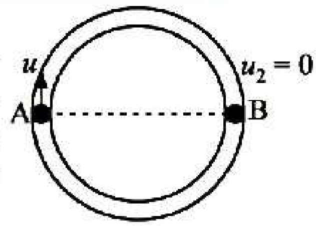
Force on the ball, $F_y = \frac{\Delta P}{\Delta t} = -\frac{m(eu\cos\theta - u\cos\theta)}{t}$

$= -\frac{(1+e)}{t} mu\cos\theta$ ***Ans.***

No change in momentum of the ball parallel to ground, so $F_x = 0$.

Illustration 5

Two equal spheres A and B lie on a smooth horizontal circular groove at opposite ends of a diameter. At $t = 0$, sphere 'A' is projected along the groove and it first impinges on B at time $t = T$, and again at time $t = T_2$. If COR is 'e' then find $\frac{T_2}{T_1}$.



Short-cut solution :

$$T_1 = \frac{s}{v} = \frac{\pi R}{u_1} \qquad \text{...(i)}$$

Using, $$e = \frac{v_2 - v_1}{u_1 - 0} \quad \Rightarrow \quad v_2 - v_1 = eu_1$$

Time taken by A to collide with B again,

$$T_2 - T_1 = \frac{2\pi R}{v_2 - v_1} = \frac{2\pi R}{eu_1} \qquad \text{...(ii)}$$

From above equations, we have $\frac{T_2}{T_1} = \frac{2+e}{e}$ ***Ans.***

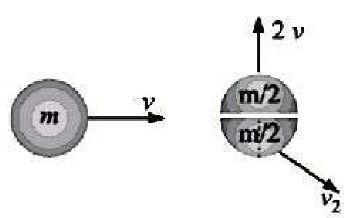
Illustration 6

A bomb explodes in air when it has a horizontal speed of v. It breaks into two identical pieces of equal mass. If one goes vertically up at a speed of 4v, find the velocity of other immediately after the explosion.

Solution :

Momentum of bomb before explosion

$$\vec{\mathbf{P}} = mv\hat{\mathbf{i}}.$$



Using conservation of momentum

$$\vec{\mathbf{P}} = \vec{\mathbf{P}}_1 + \vec{\mathbf{P}}_2$$

$$\therefore \quad \vec{\mathbf{P}}_2 = \vec{\mathbf{P}} - \vec{\mathbf{P}}_1$$

where $$\vec{\mathbf{P}}_1 = \frac{m}{2} \times 4v\hat{\mathbf{j}}$$

$$\therefore \quad \vec{\mathbf{P}}_2 = mv\hat{\mathbf{i}} - 2mv\hat{\mathbf{j}}$$

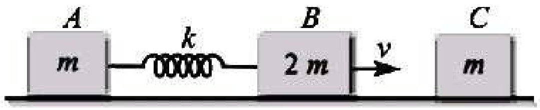
or $$\frac{m}{2}\vec{\mathbf{v}}_2 = mv\hat{\mathbf{i}} \quad mv\hat{\mathbf{j}}$$

or $$\vec{\mathbf{v}}_2 = 2v\hat{\mathbf{i}} - 4v\hat{\mathbf{j}}.$$ ***Ans.***

Illustration 7

Two blocks A and B of mass m and 2m respectively are connected by a spring of force constant k. The masses are moving to the right with uniform velocity v each, the heavier mass, leading the lighter one. The spring is of natural length in the motion. Block B collides head on with a third block C of mass m, at rest, the collision being completely inelastic. Determine the velocity of blocks at the instant of maximum compression of the spring.

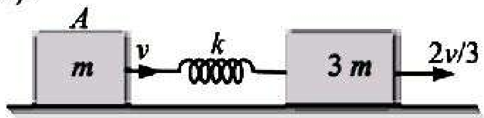
Solution :



Collision between blocks B and C

$$2mv = (2m+m)v'$$

$$\Rightarrow \quad v' = \frac{2v}{3}.$$



After the collision the blocks move as shown in *figure*.

When both the blocks will get equal velocity the spring will have maximum compression. Let velocity be v_0.

Using principle of conservation of momentum, we have

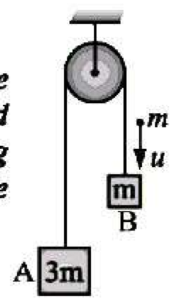
$$mv + 3m \times \frac{2v}{3} = mv_0 + 3mv_0$$

After solving, we get

$$v_0 = \frac{3v}{4}. \qquad \textit{Ans.}$$

Illustration 8

A system of two blocks A and B are connected by an inextensible massless string as shown in figure. The pulley is massless and frictionless. Initially the system is at rest, a bullet of mass 'm' moving with a velocity 'u' hits the block B and gets embedded into it. The impulse imparted by tension force to the block of mass '3]m' is:

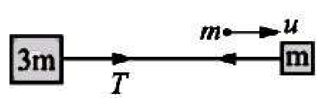


(a) $\frac{5}{4}mu$ (b) $\frac{4}{5}mu$ (c) $\frac{2}{5}mu$ (d) $\frac{3}{5}mu$

Short-cut solution :

We can assume the system as shown,

Using conservation of linear momentum, we have



$$4m \times 0 + mu = (4m+m)v$$

or $$v = \frac{u}{5}.$$

Impulse of 'A' : $J = M(v_f - v_i) = 3m\left(\frac{u}{5} - 0\right)$

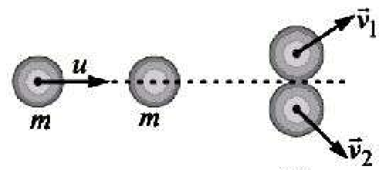
$= \frac{3}{5}mu$. **Ans. (d)**

Illustration 9

Two identical bodies, one of them initially at rest, make elastic oblique collision, show that after collision they will move mutually perpendicular.

Short-cut solution :

Using conservation of momentum, we have



$$m\vec{u} = m\vec{v}_1 + m\vec{v}_2$$

or $$\vec{u} = \vec{v}_1 + \vec{v}_2 \quad ...(i)$$

Also $$\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

or $$v_1^2 + v_2^2 = u^2 \quad ...(ii)$$

From (i), $$\vec{u}\cdot\vec{u} = (\vec{v}_1 + \vec{v}_2)\cdot(\vec{v}_1 + \vec{v}_2)$$

or $$u^2 = v_1^2 + v_2^2 + 2\vec{v}_1\cdot\vec{v}_2 \quad ...(iii)$$

From (ii) & (iii), we have $\vec{v}_1 \cdot \vec{v}_2 = 0$ or $\theta = 90°$ **Ans.**

TOPIC 6.2: Centre of Mass

Review of Formulae

1. The centre of mass of a system of discrete particles in cartesian coordinates is given by

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i ,$$

$$y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

$$z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

2. In polar coordinates system

$$\vec{r}_{cm} = \frac{1}{M}\sum m_i \vec{r}_i$$

3. Centre of mass of a rigid body,

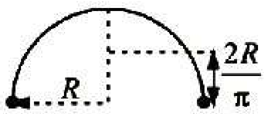
$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,(dm)$$

4. Shift in position of CM is given by

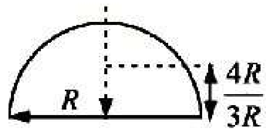
$$\Delta\vec{r}_{cm} = \left[\frac{m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_2}{m_1 + m_2}\right]$$

5. If not net external force acts on the system $\Delta\vec{r}_{cm} = 0$

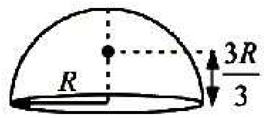
$$\therefore \qquad m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_2 = 0.$$



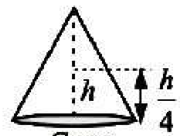
Semi-circular wire



Semi-circular disc



Hemisphere



Cone

Illustration 10

(n – 1) equal point masses each of mass m are placed at the vertices of a regular n-polygon. The vacant vertex has a position vector a with respect to the centre of the polygon. The position vector of centre of mass is :

(a) $-\frac{1}{(n-1)}a$ *(b)* $\frac{a}{(n+1)}$ *(c)* $\frac{a}{n}$ *(d)* $\frac{n}{a+1}$

Short-cut solution :

Centre of mass of whole system lies at the centre so,

$$r_{cm} = \frac{nm \times 0 - ma}{nm - m} = -\frac{1}{(n-1)}a.$$ *Ans. (a)*

Illustration 11

A uniform thin rod AB of length L has linear mass density $\mu(x) = a + \frac{bx}{L}$***, where x is measured from A. If the CM of the rod lies at a distance of*** $\left(\frac{7}{12}\right)$ ***L from A, then a and b are related as :***

(a) $a = 2b$ *(b)* $2a = b$ *(c)* $a = b$ *(d)* $3a = 2b$

Short-cut solution :

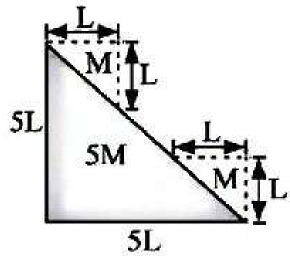
$$x_{cm} = \frac{\int_0^L (dm)x}{\int_0^L dm} = \frac{\int_0^L \left(a + \frac{bx}{L}\right)dx.x}{\int_0^L \left(a + \frac{bx}{L}\right)dx}$$

$$\text{or}\quad \frac{7L}{12}=\frac{\left|\frac{ax^2}{2}+\frac{bx^3}{3L}\right|_0^L}{\left|ax^2+\frac{bx^2}{2L}\right|_0^L}$$

$\therefore\quad 2a = b.$ **Ans. (b)**

Illustration 12

In the adjoining diagram, the small prism of mass M slides down on the bigger prism of mass 5 M from position shown at the top of the bigger prism to the position at the bottom of the bigger prism as shown in figure. By what distance does the combination move to the left if the bigger prism initially rests on a frictionless floor;



(a) $\frac{L}{5}$ *(b)* $\frac{4L}{5}$ *(c)* $\frac{2L}{3}$ *(d)* $\frac{L}{6}$

Short-cut solution :

$$5M(-\Delta x_1) + M(4L - \Delta x_1) = 0$$

$$\therefore\quad \Delta x_1 = \frac{2L}{3}.$$ **Ans. (c)**

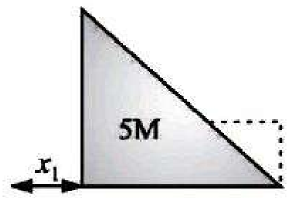
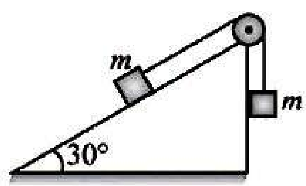


Illustration 13

Two identical blocks are connected by a massless string which passes over a light smooth pulley as shown in figure. If there is no friction between the block and the inclined, then find acceleration of CM.

Short-cut solution :

Acceleration of the blocks, $a = \left(\frac{mg - mg\sin 30°}{m+m}\right) = \frac{g}{4}$ m/s^2.

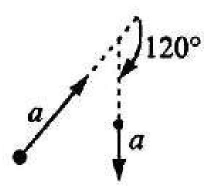


Acceleration of CM

$$a_{cm} = \frac{F_{net}}{2m}$$

$$= \frac{m\sqrt{a^2 + a^2 + 2aa\cos 120°}}{2m} = \frac{a}{2}$$

$$= \frac{\frac{g}{4}}{2} = \frac{g}{8}\ \text{m/s}^2.$$ **Ans.**



Concept Booster Exercise

1. An object of mass 3m splits into three equal fragments. Two fragments have velocities $v\hat{j}$ and $v\hat{i}$. The velocity of the third fragment is:

(a) $(v\hat{j} - v\hat{i})$ (b) $(v\hat{i} - v\hat{j})$ (c) $-(v\hat{i} + v\hat{j})$ (d) $\frac{v(\hat{i}+\hat{j})}{\sqrt{2}}$

2. Two particles having position vectors $\vec{r}_1 = (3\hat{i} + 5\hat{j})$ meter and $\vec{r}_2 = (-5\vec{i} - 3\vec{j})$ meter are moving with velocities $\vec{v}_1 = (4\vec{i} + 3\vec{j})$ m/s and $\vec{v}_2 = (a\vec{i} + 7\vec{j})$ m/s. If they collide after 2 second, the value of a ; **Numeric/Integer**

(a) 2 (b) 4 (c) 6 (d) 8

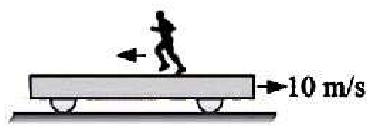
3. Two particles each of mass 0.10 kg are moving with velocities 3m/s along x-axis and 5m/s along y-axis respectively. After an elastic collision one of the mass moves with a velocity $4\hat{i} + 4\hat{j}$. The energy of other particle after collision is $\frac{x}{10}$, then x is.

[JEE Main 2020] **Numeric/Integer**

4. A child is sitting at one end of a long trolley moving with a uniform speed v on a smooth horizontal track. If the child starts running towards the other end of the trolley with a speed u, the speed of the centre of mass of the system will;

(a) $u + v$ (b) $v - u$ (c) v (d) none

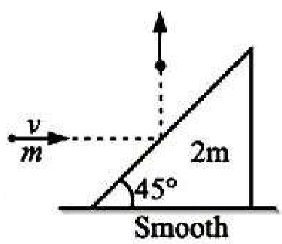
5. A trolley of mass 200 kg moves with a uniform speed of 10 m/s on a frictionless track. A child of mass 20 kg runs on the trolley from one end to the other with a speed of 4 m/s relative to the trolley in a direction opposite to its motion, and jumps out of the trolley. The final speed of the trolley is



Numeric/Integer

(a) 10.4 m/s (b) 12.4 m/s
(c) 16 m/s (d) 0

6. A bullet of mass m hits the triangular wedge of inclination 45° and mass 2 m with a velocity $v = \sqrt{2}$ m/s. After elastic collision the bullet moves perpendicular to its initial direction. If there is no friction between wedge and the surface, then velocity of bullet after collision is : **Numeric/Integer**

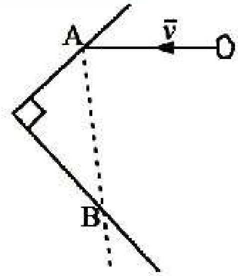


(a) 1 m/s (b) $\sqrt{2}$ m/s (c) 2 m/s (d) $2\sqrt{2}$ m/s

7. An object is dropped from a height h from the ground. Every time it hits the ground it looses 50 % of its kinetic energy. The total distance covered as $t \to \infty$ is :

(a) $3h$ (b) ∞ (c) $\frac{5}{3}h$ (d) $\frac{8}{3}h$

8. AB is an L-shaped obstacle fixed on a horizontal smooth table. A ball strikes it at A, gets deflected and restrikes it at B. If the velocity vector before collision is $\vec{v}$ and COR of each collision is 'e' then the velocity of ball after its second collision at B is:



(a) $e^2\vec{v}$ (b) $-e^2\vec{v}$

(c) $-e\vec{v}$ (d) none of these

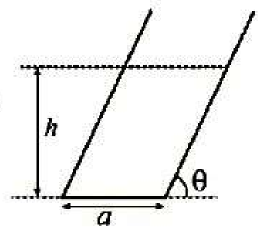
9. A pendulum consists of a wooden bob of mass 'm' and length 'ℓ'. A bullet of mass m_1 is fired towards the pendulum with a speed v_1 and emerges out the bob with a speed of $\frac{v_1}{3}$. The initial speed of the bullet if the bob just completes the vertical circle is:

(a) $\frac{m}{m_1}\frac{3\sqrt{5gl}}{2}$ (b) $\frac{m_1}{m}\frac{\sqrt{3gl}}{2}$ (c) $\frac{m}{m_1}\sqrt{5gl}$ (d) none of these

10. A hollow tilted cylindrical vessel of negligible mass rest on a horizontal plane as shown. The diameter of the base is a and the side of the cylinder makes an angle θ with the horizontal. Water is then slowly poured into the cylinder. The cylinder topples over when the water reaches a certain height h, given by : *[KVPY - 2015]*

(a) $h = 2a\tan\theta$ (b) $h = a\tan^2\theta$

(c) $h = a\tan\theta$ (d) $h = \frac{a}{2}\tan\theta$



Solutions

1. (c)
$$0 = m\vec{v}_1 + m\vec{v}_2 + m\vec{v}_3$$
or
$$0 = m(v\hat{i} + v\hat{j}) + m\vec{v}_3$$
$\therefore$
$$\vec{v}_3 = -v(\hat{i} + \hat{j}).$$

2. (d) For collision,
$$\vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$
or
$$(3\hat{i} + 5\hat{j}) + (4\hat{i} + 3\hat{j}) \times 2 = (-5\hat{i} - 3\hat{j}) + (a\hat{i} + 7\hat{j}) \times 2$$
$\therefore$
$$a = 8$$

3. (1) For elastic collision
$$KE_i = KE_f$$
$$\frac{1}{2}m \times 25 + \frac{1}{2} \times m \times 9 = \frac{1}{2}m \times 32 + \frac{1}{2}mv$$
$$34 = 32 + v^2$$
$$KE = \frac{1}{2} \times 0.1 \times 2 = 0.1\ \text{J} = \frac{1}{10}$$
$$x = 1.$$

4. (c) Child is the internal part of the system, so velocity of centre of mass will not change due to his movement.

5. (a) The speed of the child = – 4 + v

Now $220 \times 10 = 20 \times (-4 + v) + 200\,v$

$\therefore \quad v = 10.4$ m/s.

6. (a) $mv + 0 = (2m)v2$

$$\Rightarrow \quad v_2 = \frac{v}{2}$$

Also $$\frac{1}{2}mv^2 + 0 = \frac{1}{2}mv_1^2 + \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2$$

or $$v_1 = \frac{v}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1 \text{ m/s.}$$

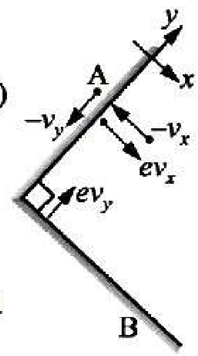
7. (a) $$e = \frac{v}{u}$$

$$= \frac{\sqrt{K_f}}{\sqrt{K_i}} = \sqrt{\frac{1}{2}}$$

Now $$H = \left(\frac{1+e^2}{1-e^2}\right)h = \frac{\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}h = 3h.$$

8. (c) At A: $\vec{v} = -(v_x\hat{i} + v_y\hat{j})$

After collision at A the velocity after collision becomes $+ ev_x$. $(-v_y)$ component strikes the side B and becomes $+ ev_y$ after collision



Therefore, $$\vec{v} = e(v_x\hat{i} + v_y\hat{j})$$

$$= -e\vec{v}$$

9. (a) $$mv_1 + 0 = m\frac{v_1}{3} + mv \quad \text{or} \quad v = \frac{m_1}{m} \times \frac{2v_1}{3}$$

To describe the circle, $$v = \sqrt{5g\ell}$$

$$\therefore \quad \sqrt{5g\ell} = \frac{m_1}{m} \times \frac{2v_1}{3}$$

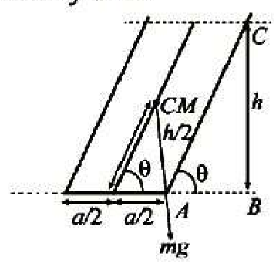
$$\Rightarrow \quad v_1 = \frac{m}{m_1}\frac{3\sqrt{5g\ell}}{2}.$$

10. (c) Line of action of weight must be passed through base of the cylinder

From geometry,

$$\tan\theta = \frac{h/2}{a/2}$$

$\therefore \quad h = a\tan\theta$



Rotational Mechanics 7

***TOPIC 7.1:** Axis of Rotation; Angular-Displacement, Velocity, Acceleration and Equations of Rotational Motion.*

Review of Formulae

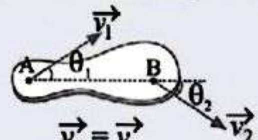
$\omega = \omega_0 + \alpha t$ $\qquad\qquad$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha\theta$ $\qquad\qquad$ $\omega = 2\pi t = \frac{2\pi}{T}$ and $\theta = 2\pi n$.

Tips and Tricks for Shortcut Solutions

1. Rigid body is one whose each point has same angular velocity in rotation about fixed axis and each point has same linear velocity in translation.

$\vec{\omega}_A = \vec{\omega}_B$



$\vec{v}_A = \vec{v}_B$

$v_1 \cos\theta_1 = v_2 \cos\theta_2$

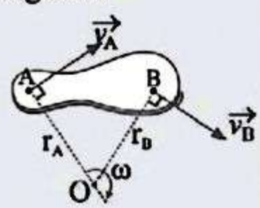
2. **Instantaneous Axis of Rotation (IAOR)**

   It is the axis of rotation of a body at some instant, about which body has pure rotation. To determine the IAOR, we should know the velocities of two points of the body.

   Case 1 : When velocities of any two points of a body are given.

   The intersection point O, of the perpendiculars down to $\vec{v}_A$ and $\vec{v}_B$ will be IAOR.

   Here $\omega = \frac{v_A}{r_A} = \frac{v_B}{r_B}$.



   Case 2: When two antiparallel velocities are given.
   In this case join the tails of velocity vectors and join the heads of velocity vectors, their intersection point, O will be IAOR,

   Here $\omega = \frac{v_A}{r_A} = \frac{v_B}{r_B}$.

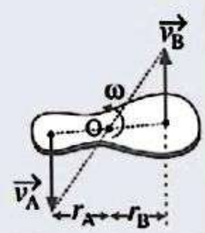
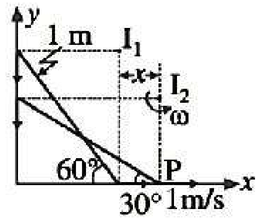


Illustration 1

A metre stick slides against a smooth vertical wall. Find horizontal distance between two IAOR when stick makes 60° and 30° from the horizontal.
Also find its angular velocity, when it makes 30° with the horizontal. The velocity of its lower end is 1 m/s at this instant.

Short-cut solution :

The IAOR at two positions is shown in figure. The distance



$$x = 1\cos 30° - 1\cos 60°$$

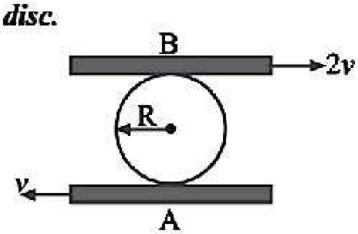
$$= \frac{\sqrt{3}-1}{2} \text{ m} \qquad \textit{Ans.}$$

The angular velocity,
$$\omega = \frac{v_P}{I_2P} = \frac{1}{1\sin 30°}$$

$$= \frac{1}{1/2} = 2 \text{ rad/s.} \qquad \textit{Ans.}$$

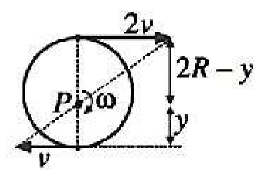
Illustration 2

A disc is rolling without sliding between two horizontal planks. If the velocities of the planks A and B are v and 2v respectively, find (i) position of IAOR and (ii) angular velocity of the disc.



Short-cut solution :

If P is the IAOR, then



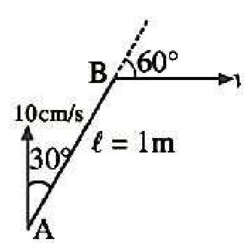
$$\omega = \frac{v}{y} = \frac{2v}{2R-y}$$

$$\therefore \qquad y = \frac{2R}{3}.$$

Now
$$\omega = \frac{v}{y} = \frac{v}{2R/3} = \frac{3v}{2R}. \qquad \textit{Ans.}$$

Illustration 3

The velocities of two ends of rod of length ℓ = 1m are given as 10 cm/s and v, as shown in figure. Find angular velocity of the rod.



Short-cut solution :

Rod on being a rigid body,

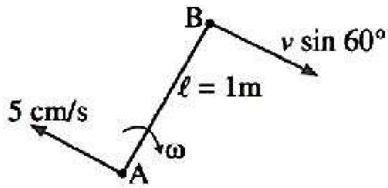
$$v \cos 60^\circ = 10 \cos 30^\circ$$

or
$$v \times \frac{1}{2} = 10 \times \frac{\sqrt{3}}{2}$$

or
$$v = 10\sqrt{3} \text{ cm/s}$$

The angular velocity of the rod about 'A'

$$\omega = \frac{v_{AB}}{r}$$



$$= \frac{(10\sqrt{3})\sin 60^\circ + 10 \sin 30^\circ}{100}$$

$$= \frac{10\sqrt{3} \times \frac{\sqrt{3}}{2} + \frac{10}{2}}{100} = 0.20 \text{ rad/s.}$$ ***Ans.***

Illustration 4

The step pulley shown in figure starts from rest and accelerates at 2 rad/s^2. What time is required for block A to move 20m ? Find also the velocity of A and B at that time.

Solution :

When A moves 20 m, its angular displacement θ is given by

$$s = r\theta$$

or
$$\theta = \frac{s}{r} = \frac{20}{1} = 20 \text{ rad}$$

Given $\alpha = 2$ rad/s^2 and $\omega_0 = 0$.

By second equation of motion, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$20 = 0 + \frac{1}{2} \times 2 \times t^2$$

or
$$t = 4.47 \text{ s}$$ ***Ans.***

Angular velocity of pulley at this time

$$\omega = \omega_0 + \alpha t$$

$$= 0 + 2 \times 4.47 = 8.94 \text{ rad/s}$$ ***Ans.***

Now velocity of A,
$$v_A = \omega r_A = 8.94 \times 1$$

$$= 8.94 \text{ m/s}$$

and
$$v_B = \omega\, r_B = 8.94 \times 0.75 = 6.70 \text{ m/s.}$$ ***Ans.***

TOPIC 7.2: *Moment of Force or Torque, Moment of Inertia, Radius of Gyration, Theorem of Parallel & Perpendicular Axis, Angular Momentum and K.E. of Rotation.*

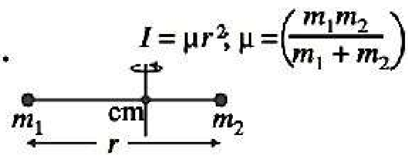
Review of Formulae

Moment of Force or Torque

$$\vec{\tau} = \vec{r} \times \vec{F} \text{ and moment arm} = \frac{|\vec{r} \times \vec{F}|}{|\vec{F}|}.$$

Moment of Inertia

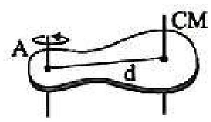
$$I = mr^2.$$



Radius of Gyration (ROG)

$$k = \sqrt{\frac{I}{M}}.$$

Parallel-Axis Theorem



$$I_A = I_{cm} + Md^2.$$

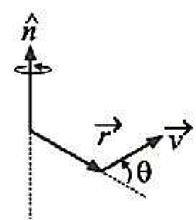
Parallel-axis may be inside or outside the body.

Perpendicular-Axis Theorem

$$I_z = I_x + I_y$$

Intersection of axis need not be the centre of mass of the body.

Angular Momentum



$$\vec{L} = m(\vec{r} \times \vec{v}) = mvr\hat{n}$$

Angular Momentum Due to Pure Rotation

$$\vec{L} = I\vec{\omega}.$$

Angular Momentum Due to Translation and Rotation Both

$$\vec{L} = m(\vec{r} \times \vec{v}) + I\vec{\omega}.$$

Newton's Second Law for Rotating Body

$$\vec{\tau}_{ext} = \frac{d\vec{L}}{dt} \quad \text{or} \quad \vec{\tau}_{ext} = I\vec{\alpha}.$$

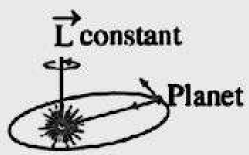
KE of Rotation : $K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}.$

Tips and Tricks for Shortcut Solutions

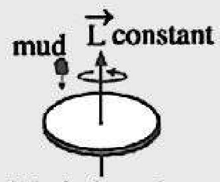
For $\vec{\tau}_{ext} = 0$ $\vec{L}$ = constant.

- Angular momentum remains constant along the axis, in which τ_{ext} is zero.
- Angular momentum remains conserved in collision, expansion of system and contraction of system etc.

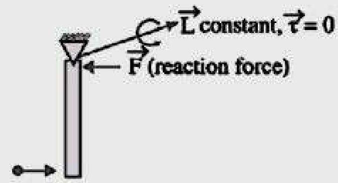
Following are few examples in which Angular momentum about Certain-Axis remains constant.



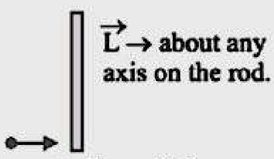
Motion of planet about sun



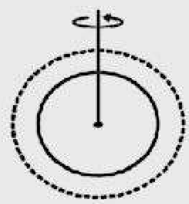
Mud piece drops onto disc



Bullet collides with the rod



Bullet collides with the rod placed on smooth table


Body expands /contracts

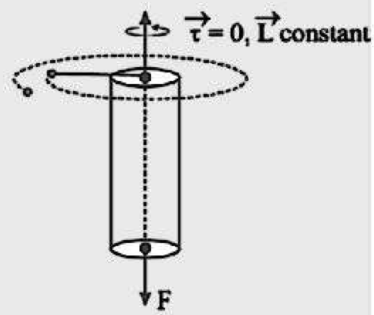
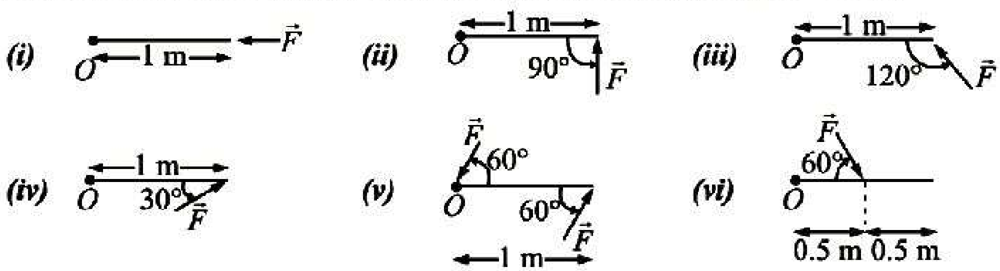


Illustration 5

Calculate the torque (magnitude and direction) about point 'O' due to force $\vec{F}$ in the following cases. In each case, the force $\vec{F}$ and the rod both lie in the plane of the page. The rod has length 1m and the force has magnitude F = 10 N.



Solution :

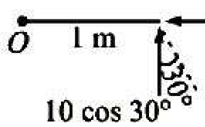
Key concept: Torque, $\tau = Fr \sin \theta$.

Using, $\tau = Fr \sin \theta$

(i) In this case, r = 1m, $\theta = 180°$; $\therefore$ $\tau_o = Fr \sin \theta = 10 \times 1 \times \sin 180° = 0$.

(ii) In this case, r = 1m, $\theta = 90°$, $\therefore$ $\tau_o = 10 \times 1 \times \sin 90° = 10$ N-m, anticlockwise.

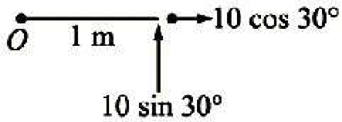
(iii)



$\tau_o = 10 \cos 30° \times 1 + 10 \sin 30° \times 0$

$= 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ N-m, anticlockwise.

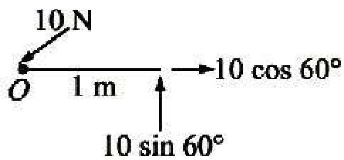
(iv)



$\tau_o = 10 \sin 30° \times 1 + 10 \cos 30° \times 0$

$= 10 \times \frac{1}{2} = 5$ N-m, anticlockwise.

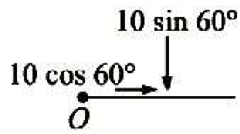
(v)



$\tau_o = 10 \sin 60° \times 1 + 10 \times \cos 60° \times 0 + 10 \times 0$

$= 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ N-m, anticlockwise.

(vi)

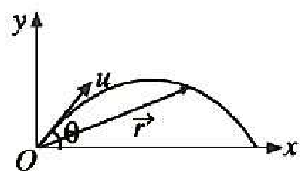


$\tau_o = 10 \sin 60° \times 0.5 + 10 \cos 60° \times 0$

$= 5 \times \frac{\sqrt{3}}{2} = 2.5\sqrt{3}$ N-m, clockwise. *Ans.*

Illustration 6

A particle is projected from a point 'O' with a speed 'u' at an angle 'θ' to the horizontal. Find torque of gravitational force on projectile about 'O' at any time t. (x, y plane is vertical plane)



Short-cut solution :

Torque, $\tau_0 = mgx$

$= mg(u \cos \theta t)$ *Ans.*

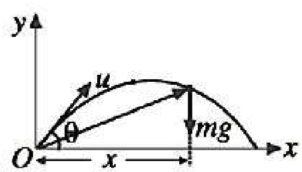
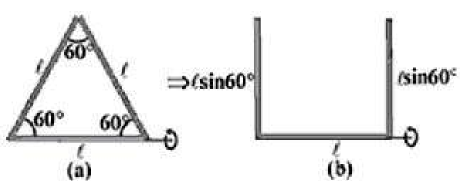


Illustration 7

(a) ***Three identical thin rods, each of mass 'm' and length ℓ are joint to form an equilateral triangle. Find moment of inertia of the triangle about one of its sides.***



(b) ***Also find moment of inertia of this system about an axis passing through centroid of the triangle and perpendicular to the plane of the rods.***

Short-cut solution :

(a) The given system of rods for its moment of inertia is equivalent to the system shown in figure (b). Thus the moment of inertia about the axis given

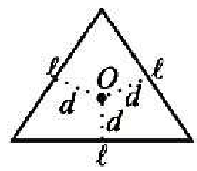
$$I = 0 + \frac{m(\ell \sin 60°)^2}{3} + \frac{m(\ell \sin 60°)^2}{3}$$

$$= 2\frac{m\ell^2}{3} \times \sin^2 60°$$

$$= \frac{2}{3}m\ell^2 \times \frac{3}{4} = \frac{m\ell^2}{2}$$ ***Ans.***

(b) If ℓ is the side of triangle (length of rod), then $d = \dfrac{\ell}{2\sqrt{3}}$.

Moment of inertia of one rod about 'O'.



$$I = I_{cm} + md^2$$

$$= \frac{m\ell^2}{12} + m\left(\frac{\ell}{2\sqrt{3}}\right)^2$$

$$= \frac{m\ell^2}{6}$$ ***Ans.***

Now moment of inertia of whole system

$$I_0 = 3I$$

$$= 3 \times \frac{m\ell^2}{6} = \frac{m\ell^2}{2}.$$ ***Ans.***

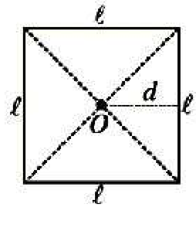
Illustration 8

(a) ***Four identical rods each mass 'm' and length 'ℓ' are joint to form a square. Find moment of inertia of the system about an axis passing through the centre of mass of the system and perpendicular to the plane of the rods.***

(b) ***Also find moment of inertia about the diagonal of the square.***

Short-cut solution :

(a) Moment of inertia of the system



$$I_0 = 4I$$

$$= 4[I_{cm} + md^2]$$

$$= 4\left[\frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2\right]$$

$$= \frac{4}{3}m\ell^2$$ ***Ans.***

(b) Using perpendicular axis theorem, we have

$$I + I = I_0$$

$$\therefore \quad I = \frac{I_0}{2}$$

$$= \frac{\frac{4}{3}m\ell^2}{2}$$

$$= \frac{2}{3}m\ell^2. \qquad \textit{Ans.}$$

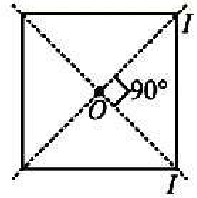
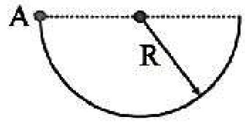


Illustration 9

A uniform rod of mass m is bent into the form of a semicircle of radius R. The moment of inertia of the rod about an axis passing through 'A' and perpendicular to the plane of the semicircle is :



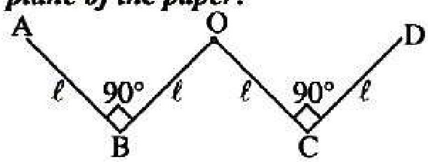
(a) $\frac{2}{3}mR^2$ *(b)* mR^2 *(c)* $2mR^2$ *(d)* $\frac{5}{\pi}mR^2$

Short-cut solution :

$$I_A = \text{MI of the ring about tangent parallel to the axis}$$
$$= I_{cm} + md^2$$
$$= mR^2 + mR^2 = 2mR^2. \textit{Ans. (c)}$$

Illustration 10

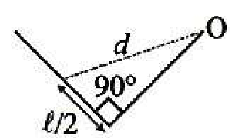
A uniform rod of length 4ℓ, mass 4 m is bent in the shape as shown in figure, what is the moment of inertia of the rod about the axis passing through point O and perpendicular to the plane of the paper?



Short-cut solution :

MI of rod AB :

$$d = \sqrt{\ell^2 + \left(\frac{\ell}{2}\right)^2} = \frac{\sqrt{5}\ell}{2}$$



$$I = I_{cm} + md^2$$

$$= \frac{m\ell^2}{12} + m\left(\frac{\sqrt{5}\ell}{2}\right)^2$$

$$= \frac{4}{3}m\ell^2$$

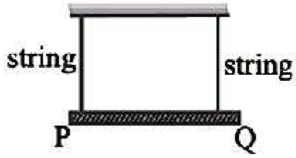
MI of rod OB, $I = \frac{m\ell^2}{3}$.

The rod is symmetrical about 'O', so

$$I_O = 2\left[\frac{4}{3}m\ell^2 + \frac{m\ell^2}{3}\right] = \frac{10}{3}m\ell^2.$$ ***Ans.***

Illustration 11

A rod PQ of mass m is kept suspended in horizontal position as shown. Linear mass density of rod varies with distance x from its left end P as λ = kx (k is constant). Left string is now cut. Torque about point 'Q' just after the string is cut is :

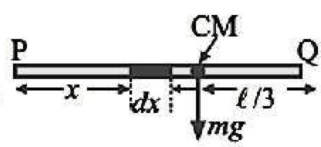


(a) $\frac{mg\ell}{2}$ *(b)* $\frac{mg}{\;}$ *(c)* $\frac{mg\ell}{4}$ *(d)* $\frac{mg\ell}{6}$

Short-cut solution :

The CM of the rod, from P

$$x_{cm} = \frac{\int_0^\ell (dm)x}{\int_0^\ell dm} = \frac{\int_0^\ell (\lambda dx)x}{\int_0^\ell \lambda dx}$$

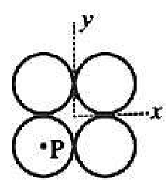


$$= \frac{\int_0^\ell (kxdx)x}{\int_0^\ell kxdx} = \frac{2\ell}{3}.$$

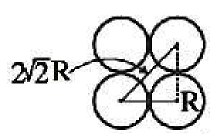
Now torque, $\tau_Q = mg \times \frac{\ell}{3} = \frac{mg\ell}{3}$. ***Ans. (b)***

Illustration 12

Four identical discs each of mass m and radius R are placed in contact as shown in figure. Find MI of the system about an axis passing through P and perpendicular to the plane of the figure.



Short-cut solution :



MI of the system about CM axis and perpendicular to it

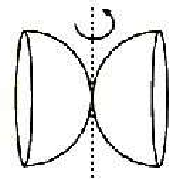
$$I_{cm} = 4\left[\frac{mR^2}{2} + m(\sqrt{2}R)^2\right]$$

$$= 10\,mR^2.$$

Now $$I_P = I_{cm} + Md^2 = 10\,mR^2 + 4m\,(\sqrt{2}R)^2$$

$$= 18mR^2. \qquad \textit{Ans.}$$

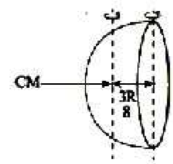
Illustration 13



Moment of inertia of a sphere about its diameter is I. The sphere is cut in two equal parts and placed as shown in figure. The moment of inertia about the axis shown is :

(a) I *(b)* $\frac{5}{7}I$ *(c)* $\frac{2}{5}I$ *(d)* $\frac{13}{8}I$

Short-cut solution :



M.I. about diameter $$I = \frac{2}{5}mR^2$$

M.I. about cm, $$I_{cm} = I_D - m\left(\frac{3R}{8}\right)^2 = \frac{2}{5}mR^2 - \frac{9}{64}mR^2$$

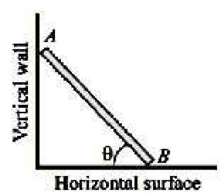
Now M.I. about tangent $$I' = I_{cm} + m\left(\frac{5R}{8}\right)^2$$

$$= \frac{2}{5}mR^2 - \frac{9}{64}mR^2 + \frac{25}{64}mR^2$$

$$= \left(\frac{2}{5} + \frac{1}{4}\right)mR^2 = \frac{13}{20}mR^2$$

$$= \frac{13I}{8}. \qquad \textit{Ans. (d)}$$

Illustration 14



A thin rod AB (of mass m and length l) is sliding keeping in contact with a vertical wall and a smooth surface as shown. Moment of inertia of the rod about instantaneous axis of rotation at the instant shown is :

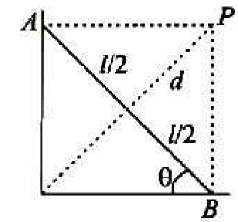
(a) $\frac{ml^2}{6}\sin\theta$ *(b)* $\frac{ml^2}{6}\cos\theta$ *(c)* $\frac{ml^2}{3}$ *(d)* $\frac{ml^2}{6}$

Short-cut solution :

From geometry,

$$d = \frac{l}{2}$$

So $I_p = I_{cm} + md^2.$



$$= \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2$$

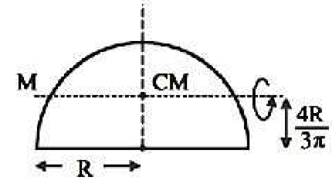
$$= \frac{ml^2}{3}.$$

Ans. (c)

Illustration 15

Given a semi-circular disc of mass M and radius R. Find its MI about an axis passing through CM and parallel to its diameter.

Short-cut solution :



Using, $I_D = I_{CM} + md^2$

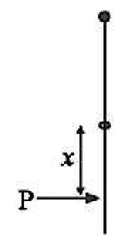
or $I_{CM} = I_D - md^2$

$$= \frac{mR^2}{4} - m\left(\frac{4R}{3\pi}\right)^2 = mR^2\left[\frac{1}{4} - \frac{16}{9\pi^2}\right].$$

Ans.

Illustration 16

A uniform rod of length ℓ is given an impulse P at a distance x from its CM as shown in figure. The position of the IAOR from the centre of mass is

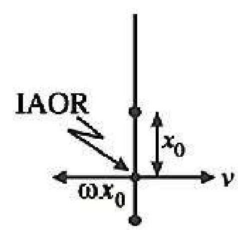


(a) $\frac{\ell^2}{x}$ *(b)* $\frac{\ell^2}{12x}$ *(c)* $\frac{x^2}{12\ell}$ *(d)* $\frac{\ell^2}{3x}$

Short-cut solution :

Using conservation of angular momentum about CM of the rod. Assuming after impulse velocity of CM becomes v. Then

$$mvx = I_{CM}\,\omega$$



or $$\omega = \frac{mvx}{I_{CM}}$$

$$= \frac{mvx}{\frac{m\ell^2}{12}} = \frac{12vx}{\ell^2}$$

If x_0 is the distance of IAOR, then

$$0 = v - \omega x_0$$

or $$0 = v - \left(\frac{12vx}{\ell^2}\right)\omega$$

$\therefore$ $$x_0 = \frac{\ell^2}{12x}.$$

Ans. (b)

Illustration 17

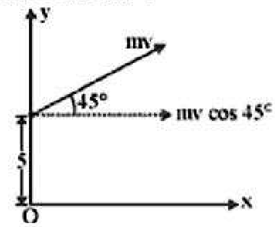
A particle of mass m is moving with velocity $\vec{v}$ along a line $y = x + 5$. Find the angular momentum of the particle about a perpendicular axis passing through origin O.

Short-cut solution :

The linear momentum of the particle $p = mv$. Compare given equation with $y = mx + c$, we have $m = 1$ or $\theta = 45°$ and $c = 5$ unit. The angular momentum about O

$$L_O = mv \cos 45° \times 5$$

$$= \frac{5}{\sqrt{2}} mv \text{ unit}$$

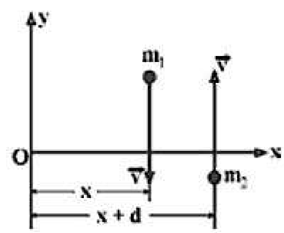


Its direction is along negative z-axis. Thus

$$\vec{L}_O = -\frac{5}{\sqrt{2}} mv\hat{k} \text{ unit.} \quad \textit{Ans.}$$

Illustration 18

Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d, show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken.

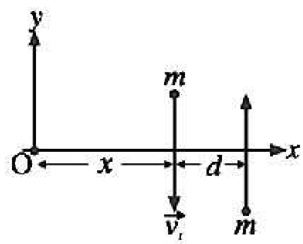


Short-cut solution :

Angular momentum about 'O'

$$\vec{L}_0 = mv(d + x) - mvx$$

$$= mvd\hat{k}$$



The result is independent of x.

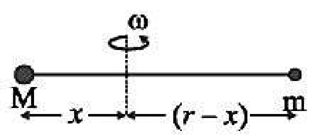
Illustration 19

Two particles of masses m and M (m << M) are separated a distance r. The system is to be rotated to get an angular velocity ω, about an axis perpendicular to the line joining the particles. Find the position of the axis, if work done in the process is minimum.

Short-cut solution :

MI of the system about axis shown

$$I = Mx^2 + m(r - x)^2.$$



Work done, $$W = K = \frac{1}{2} I\omega^2$$

$$= \frac{1}{2}\left[Mx^2 + m(r - x)^2\right]\omega^2$$

For minimum work, $$\frac{dW}{dx} = 0.$$

$$\text{or}\quad \frac{d}{dx}\left[\frac{1}{2}\{Mx^2 + m(r-x)^2\}\omega^2\right] = 0$$

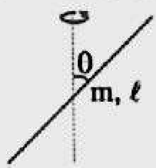
$$\text{or}\qquad M \times 2x - 2m(r-x) = 0$$

$$\therefore\qquad x = \left[\frac{mr}{m+M}\right]. \qquad \textbf{Ans.}$$

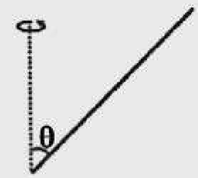
As $m << M$, so $x \to 0$. So system is to be rotated about M.

Tips and Tricks for Shortcut Solutions

1. MI of thin rod about oblique axis :



$I = \frac{m\ell^2}{12}\sin^2\theta$



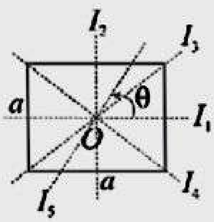
$I = \frac{m\ell^2}{3}\sin^2\theta$

2. For square plate (m, a)

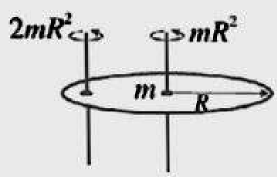
$I_1 = I_2 = I_3 = I_4 = I_5 = \frac{ma^2}{12}$

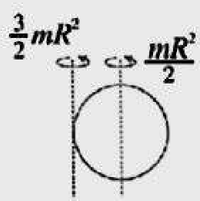
and $I_0 = I_1 + I_2 = I_3 + I_4 = I_1 + I_5$

$= \frac{ma^2}{6}$.



3. MI of ring or hoop :

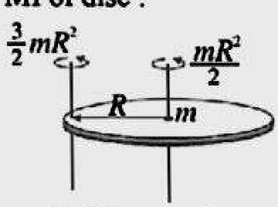


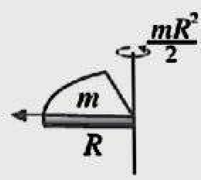


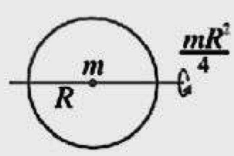
MI about tangent parallel to diameter

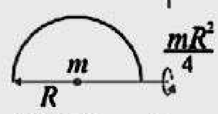
The minimum MI of ring about any axis = $\frac{mR^2}{2}$.

4. MI of disc :





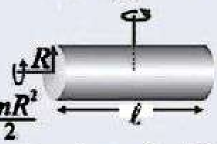




Half disc of mass m

5. MI of solid cylinder and thin pipe:

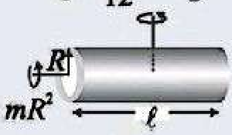
$I_2 = \frac{m\ell^2}{12} + \frac{mR^2}{4}$



$I_1 = \frac{mR^2}{2}$

For $I_1 = I_2 \Rightarrow \ell = \sqrt{3}R$

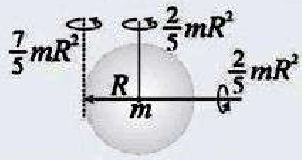
$I_2 = \frac{m\ell^2}{12} + \frac{mR^2}{2}$

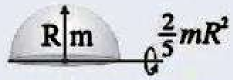


$I_1 = mR^2$

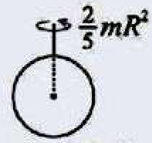
MI of thin pipe

6. MI of sphere and thin shell :





MI of hemisphere about its diameter



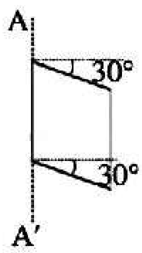
MI of shell

7. Work done in rotating any system for constant angular velocity will be minimum, when system is rotated about its CM.

Illustration 20

Four identical rods, each of mass m and length ℓ are joint to form a rhombus as shown. Find MI of this system about an axis A – A′.

Short-cut solution :



$$I_{AA'} = 0 + 2\left[\frac{m}{3}(\ell\cos 30°)^2\right] + m(\ell\cos 30°)^2$$

$$= 2\times\frac{m\ell^2}{3}\times\left(\frac{\sqrt{3}}{2}\right)^2 + m\ell^2\times\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{5}{4}m\ell^2.$$ ***Ans.***

Illustration 21

If earth contracts to half the present radius, then what will be the new duration of the day?

Short-cut solution :

Key Concept : Conservation of angular momentum

Using, $I_i\omega_i = I_f\omega_f$

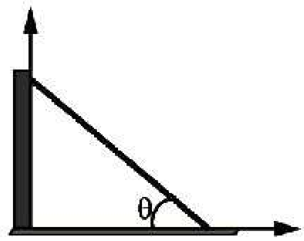
or $\left(\frac{2}{5}MR^2\right)\left(\frac{2\pi}{T}\right) = \frac{2}{5}M\left(\frac{R}{2}\right)^2\left(\frac{2\pi}{T_f}\right)$

∴ $T_f = \frac{T}{4}$

$= \frac{24}{4} = 6\text{h}.$ ***Ans.***

Illustration 22

A uniform bar of length ℓ and mass m stands vertically touching a vertical wall (y-axis). When slightly displaced, its lower end begins to slide along the floor (x-axis). Obtain an expression for the angular velocity (ω) of the bar as a function of θ. Neglect friction everywhere.

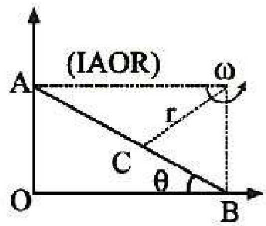


Short-cut solution :

The position of instantaneous axis of rotation (IAOR) is shown in figure.

$$C = \left(\frac{\ell}{2}\cos\theta, \frac{\ell}{2}\sin\theta\right)$$

$$r = \frac{\ell}{2} = \text{half of the diagonal}$$



All surfaces are smooth. Therefore, mechanical energy will remain conserved.

∴ Decrease in gravitational potential energy of bar = increase in rotational kinetic energy of bar about IAOR

$$\therefore \quad mg\frac{\ell}{2}(1-\sin\theta) = \frac{1}{2}I\omega^2 \quad \text{...(1)}$$

Here, $$I = \frac{m\ell^2}{12} + mr^2 \text{ (about IAOR)}$$

or $$I = \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$$

Substituting in Eq. (1), we have

$$mg\frac{\ell}{2}(1-\sin\theta) = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$$

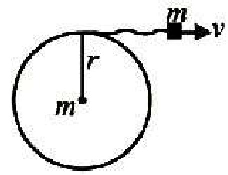
or $$\omega = \sqrt{\frac{3g(1-\sin\theta)}{\ell}} \quad \textit{Ans.}$$

Illustration 23

A block of mass m is attached to the pulley disc of equal mass m and radius r by means of slack string as shown. The pulley is hinged about its centre on a horizontal table and the block is projected with an initial velocity of v. Find its velocity when the string becomes taut.

Short-cut solution :

Using conservation of angular momentum about the hinge. If v' is the required velocity then



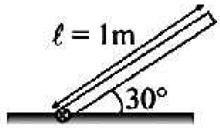
$$mvr = mv'r + I_c\omega$$

$$= mv'r + \frac{mr^2}{2} \times \frac{v'}{r}$$

or $$v' = \frac{2v}{3}. \qquad \textit{Ans.}$$

Illustration 24

A rod of length 1 m is released from rest as shown in the figure below.



If ω of rod is $\sqrt{n}$ at the moment it hits the ground, then find n. [JEE Main 2020]

Short-cut solution :

$$mg\frac{\ell}{2}\sin 30° = \frac{1}{2}\frac{m\ell^2}{3}\omega^2$$

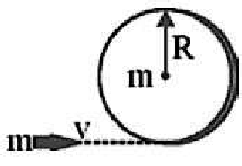
Solving

$$\omega^2 = 15$$

$$\omega = \sqrt{15} \qquad \textit{Ans.}$$

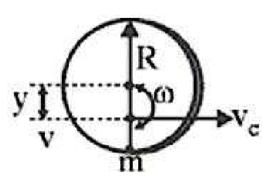
Illustration 25

A circular wooden hoop of mass m and radius R rests flat on a frictionless surface. A bullet, also of mass m, and moving with a velocity v strikes the hoop and gets embedded in it. The thickness of hoop is much smaller than R. Find the angular velocity with which the system rotates after the bullet strikes the hoop



Short-cut solution :

Let the velocity of the C.M. of the system after strike is v_c. By conservation of linear momentum



$$mv = 2m \times v_c$$

$\Rightarrow$ $$v_C = \frac{v}{2}$$

Position of C.M. $y = \dfrac{m\times 0 + m\times R}{m+m} = \dfrac{R}{2}$

Using conservation of angular momentum about C.M. of the system (hoop + bullet)

$$mv\times\frac{R}{2} = I_C\omega$$

or $$\frac{mvR}{} = (I_{\text{bullet}} + I_{\text{hoop}})_c\omega$$

or $$\frac{mvR}{2} = \left[m\left(\frac{R}{2}\right)^2 + \left\{mR^2 + m\left(\frac{R}{2}\right)^2\right\}\omega\right]$$

After solving, we get

$$\omega = \frac{v}{3R}. \qquad \textit{Ans.}$$

Illustration 26

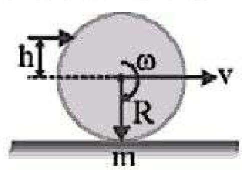
A thin spherical shell of radius R lying on a rough horizontal surface is hit sharply and horizontally by a cue. Where should it be hit so that the shell does not slip on the surface?

Short-cut solution :

Due to the impact of the cue, friction starts acting on the shell at the point of contact, which constitutes a torque about centre of the shell. But the process of impact is of very short duration ($\Delta t \to 0$), and so there is negligible change in angular momentum due this torque. So applying conservation of angular momentum about the centre of the shell. If v is the velocity of the centre of mass of the shell, then we can write,

$$mvh = I_{cm}\omega$$

$$= \left(\frac{2}{3}mR^2\right)\times\frac{v}{R}$$



or $$h = \frac{2R}{3}. \qquad \textit{Ans.}$$

Illustration 27

A circular platform is mounted on a frictionless vertical axle. Its radius R = 2m and its moment of inertia about the axle is 200 kgm². It is initially at rest. A 50kg man stands on the edge of the platform and begins to walk along the edge at the speed of 1 m/s relative to the ground. Time taken by the man to complete one revolution is:

(a) π s *(b)* $\dfrac{3\pi}{2}$ s *(c)* 2π s *(d)* $\dfrac{\pi}{2}$ s

Short-cut solution :

According to the conservation of momentum

$$L_i = L_f$$

$$\Rightarrow \quad mvR - I\omega = 0$$

$$mvR = I\omega$$

or $$50 \times 1 \times 2 = 200 \times \omega$$

or $$\omega = \frac{1}{2}.$$

Now $$(v + \omega R)\, t = 2\pi R$$

$$t\left(1 + \frac{1}{2} \times 2\right) = 2\pi \times 2$$

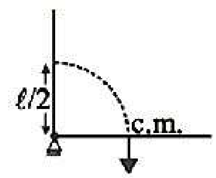
$$t = 2\pi \text{ s.}$$ ***Ans. (c)***

Illustration 28

A rod of mass m and length ℓ is hinged at its lower end and allowed to move from vertical position. Find reaction forces at the hinge when rod becomes horizontal. Assuming hinge is smooth.

Solution :

Using conservation of mechanical energy,



$$mg\frac{\ell}{2} = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$$

$$\therefore \quad \omega = \sqrt{\frac{3g}{\ell}}.$$

In the moving condition CM of rod has two accelerations ; Centripetal and tangential acceleration. If α is the angular acceleration, then using,

$$\tau = I\alpha, \text{ we have}$$

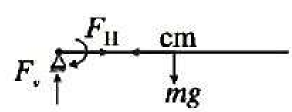
$$mg\frac{\ell}{2} = \left(\frac{m\ell^2}{3}\right)\alpha$$

or $$\alpha = \frac{3g}{2\ell}$$

and $$a_{cm} = \alpha\frac{\ell}{2} = \frac{3g}{2\ell} \times \frac{\ell}{2} = \frac{3g}{4}.$$

Now horizontal reaction, $$F_H = m\omega^2 r$$

$$= m\left(\sqrt{\frac{3g}{\ell}}\right)^2 \times \frac{\ell}{2}$$



$$= \frac{3mg}{2}.$$ ***Ans.***

And vertical reaction ;

$$mg - F_v = ma_{cm}$$

$$= m \times \frac{3g}{4}$$

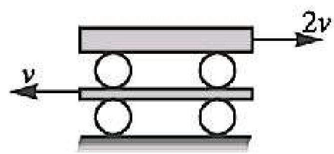
or $$F_v = \frac{mg}{4}.$$ *Ans.*

Illustration 29

A system of uniform cylinders and plates is shown in figure. All the cylinders are identical and there is no slipping at any contact. Velocity of lower and upper plate is v and 2v respectively as shown in figure. Then the ratio of angular speed of the upper cylinders to lower cylinders is :

***(a)* 1 *(b)* 2 *(c)* 3 *(d)* 4**

Short-cut solution :



Angular speed of lower cylinder

$$\omega_1 = \frac{v-0}{2R}.$$

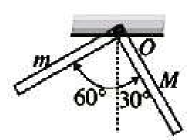
Angular speed of upper cylinder

$$\omega_2 = \frac{(2v+v)}{2R} = \frac{3v}{2R}.$$

$\therefore$ $$\frac{\omega_2}{\omega_1} = \frac{3v/2R}{v/2R} = 3.$$ *Ans. (c)*

Illustration 30

Two uniform rods of equal length but different masses are rigidly joined to form an L-shaped body, which is then pivoted about O as shown in the figure. If in equilibrium the body is in the shown configuration, ratio $\frac{M}{m}$ will be :



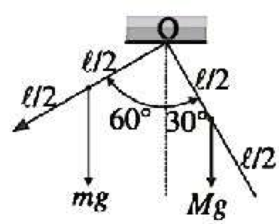
***(a)* 2 *(b)* 3 *(c)* $\sqrt{2}$ *(d)* $\sqrt{3}$**

Short-cut solution :

For equilibrium

$$\Sigma\vec{\tau}_O = 0$$

or $$Mg \times \frac{\ell}{2}\sin 30° - mg \times \frac{\ell}{2}\sin 60° = 0$$



or $$\frac{M}{m} = \frac{\sin 60°}{\sin 30°}$$

$$= \frac{\sqrt{3}/2}{1/2}$$

$$= \sqrt{3}. \qquad \textit{Ans. (d)}$$

Illustration 31

A uniform rod of mass 'm' and length 'ℓ' placed on a smooth horizontal floor is hit by a particle also of mass 'm' moving on the floor, at a distance ℓ/4 from one end elastically. The distance travelled by the centre of the rod after the collision when it has completed three revolutions will be:

(a) $\pi\ell$ *(b)* $2\pi\ell$ *(c)* $4\pi\ell$ *(d)* *none of these*

Short-cut solution :

Let v and v' are velocities of particle before and after collision. If V is the velocity of CM of the rod, then

$$mv = mv' + mV$$

or $$v = v' + V \qquad \text{...(i)}$$

Applying conservation of angular momentum about point of collision, we have

$$mv \times 0 = mV\left(\frac{\ell}{4}\right) - \left(\frac{m\ell^2}{12}\right)\omega$$

or $$\omega\ell = 3V \qquad \text{...(ii)}$$

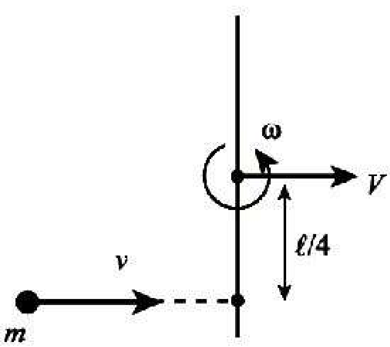
Now using, $$e = -\left[\frac{v_1 - v_2}{u_1 - u_2}\right]$$

or $$1 = -\left[\frac{v' - V}{v - 0}\right]$$

or $$V - v' = v \qquad \text{...(iii)}$$

After solving above equations, we get

$$\omega = \frac{3v}{\ell}$$



or $$\frac{v}{\omega} = \frac{\ell}{3}$$

or $$\frac{vt}{\omega t} = \frac{\ell}{3}$$

or $$\frac{x}{\theta} = \frac{\ell}{3}$$

∴ $$x = \frac{\ell}{3}(\theta) = \frac{\ell}{3} \times (3 \times 2\pi)$$

$$= 2\pi\ell. \qquad \textit{Ans. (b)}$$

TOPIC 7.3: *Rotation and Rolling Motion*

Tips and Tricks for Shortcut Solutions

1. It is better to write $\vec{\tau} = I\vec{\alpha}$ as :

$$\tau_{\text{greater}} - \tau_{\text{smaller}} = I\alpha.$$

The direction of angular acceleration is along greater torque.

2. In the problem having translation and rotation both (for different object) or rolling motion, the acceleration can be obtained as :

$$a = \frac{\text{unbalanced load}}{\text{total inertia}}$$

Here, $\quad$ total inertia = inertia of translation + inertia of rotation

$$= \left(m + \frac{I}{R^2}\right)$$

3. Tension in the string can be obtained as follows :

$$T = m(g \pm a).$$

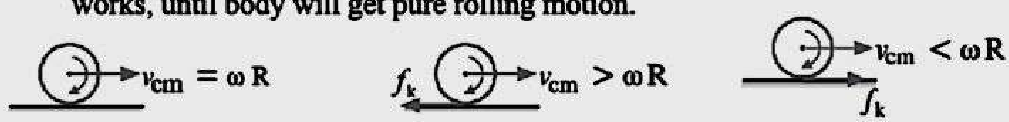
Use '+' sign when block moves up.

4. In the problem, if velocity is asked, energy method is short and easy.

5. (i) In case of pure rolling on rough surface with constant velocity, friction neither acts nor works.

   (ii) In case of accelerated rolling, static friction may acts, but not works.

   (iii) In case of impure rolling $v_{cm} > \omega R$ or $v_{cm} < \omega R$, kinetic friction acts and works, until body will get pure rolling motion.



6. Equations of static equilibrium

   (i) For coplanar-concurrent forces :

   $\Sigma F_x = 0$ and $\Sigma F_y = 0$.

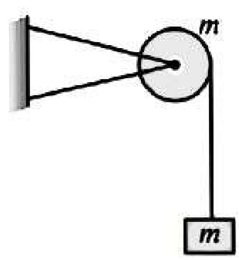
   There may be any number of forces in the system, but unknown shown not be more than **two**.

   (ii) For coplanar non-concurrent forces :

   $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma \tau_z = 0$.

   Here the number of unknown should not be more than **three**.

Illustration 32



A block of mass m is suspended from a pulley in form of a circular disc of mass m and radius R. The system is released from rest, find the angular velocity of disc when block has dropped by height h, (there is no slipping between string and pulley) ***[JEE Main 2020]***

(a) $\frac{1}{R}\sqrt{\frac{4gh}{3}}$ *(b)* $\frac{1}{R}\sqrt{\frac{2gh}{3}}$ *(c)* $R\sqrt{\frac{2gh}{3}}$ *(d)* $R\sqrt{\frac{4gh}{3}}$

Short-cut solution :

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$v = \omega R \text{(no slipping)}$$

$$mgh = \frac{1}{2}m\omega^2 R^2 + \frac{1}{2}\frac{mR^2}{2}\omega^2$$

$$mgh = \frac{3}{4}m\omega^2 R^2$$

$$\omega = \sqrt{\frac{4gh}{3R^2}} = \frac{1}{R}\sqrt{\frac{4gh}{3}} \qquad \textit{Ans. (a)}$$

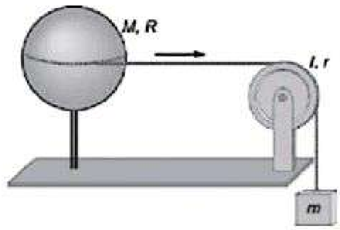
Illustration 33

A uniform spherical shell of mass M and radius R rotates about a vertical axis on frictionless bearing as shown in figure. A massless cord passes around the equator of the shell, over a pulley of rotational inertia I and radius r, and is attached to a small object of mass m that is otherwise free to fall under the influence of gravity. There is no friction of pulley's axle; the cord does not slip on the pulley. What is the speed of the object after it has fallen a distance h from rest ? Use work-energy considerations.

Short-cut solution :

Using a conservation of energy principle, we have

Fall in P.E. of the block = Gain in K.E. of block + rotational K.E. of the pulley + rotational K.E. of the shell.



or

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}\left(\frac{2MR^2}{3}\right)\omega'^2$$

where $\omega = v/r$ and $\omega' = v/R$

After substituting these values in above equation, and solving, we get

$$v^2 = \frac{mgh}{\left(\frac{m}{2} + \frac{I}{2r^2} + \frac{M}{3}\right)}. \qquad \textit{Ans.}$$

Illustration 34

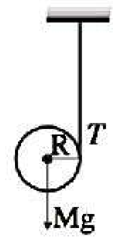
A string is wrapped around a solid cylinder and then the end of the string is held stationary, while the cylinder is released from rest. Find the acceleration of the cylinder and tension in the string.

Short-cut solution :

$$a = \frac{\text{unbalanced load}}{\text{total inertia}}$$

$$= \frac{Mg}{M + \frac{I}{R^2}}$$

$$= \frac{Mg}{M + \frac{MR^2/2}{R^2}} = \frac{2g}{3}. \quad \textbf{Ans.}$$



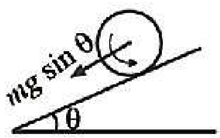
Tension, $T = M(g - a) = M(g - \frac{2g}{3}) = \frac{Mg}{3}.$ ***Ans.***

Illustration 35

A sphere starts rolling on rough inclined plane of inclination θ. Find its acceleration of CM.

Short-cut solution :

Using, $$a_{CM} = \frac{\text{unbalanced load}}{\text{total inertia}}$$



$$= \frac{mg\sin\theta}{m + \frac{I}{R^2}}$$

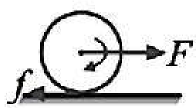
$$= \frac{mg\sin\theta}{m + \frac{2}{5}\frac{mR^2}{R^2}} = \frac{5}{7}g\sin\theta. \quad \textbf{Ans.}$$

Illustration 36

A circular ring is acted by a horizontal force F at its centre and rolls on the plane horizontal surface. Find acceleration of its CM.

Short-cut solution :

$$a_{CM} = \frac{F}{m + \frac{I}{R^2}}$$



$$= \frac{F}{m + \frac{mR^2}{R^2}} = \frac{F}{2m}. \quad \textbf{Ans.}$$

Video Solution

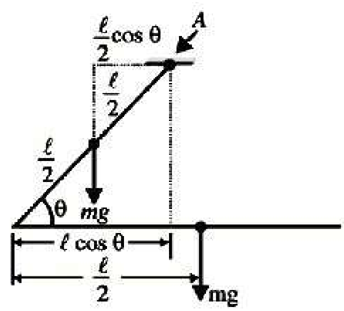
Q. *A cubical block of side 'a' moves down on a rough inclined plane of inclination θ with constant velocity. Find the position of normal reaction.*

To see the video solution, scan the QR code:
OR Visit https://www.youtube.com/watch?v=chtDBZwnPGE

Illustration 37

A rigid body in the shape of a 'V' has two equal arms made of uniform rods. What must the angle between the two rods so that when the body is suspended from one end, the other arm is horizontal. ***[KVPY 2016]***

Short-cut solution :



For the equilibrium of the system of rods

$$\Sigma\vec{\tau}_A = 0$$

$$mg\times\frac{\ell}{2}\cos\theta - mg\times\left(\frac{\ell}{2}-\ell\cos\theta\right) = 0$$

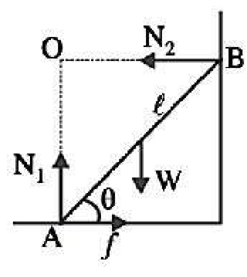
$\therefore$ $$\cos\theta = \frac{1}{3}$$

or $$\theta = \cos^{-1}\left(\frac{1}{3}\right).$$ ***Ans.***

Illustration 38

A uniform ladder of weight 'W' leans against a smooth vertical wall and other end rests on a rough horizontal floor. It makes an angle 'θ' with the horizontal. Find normal reaction and frictional force that the floor exerts on the ladder. ***[JEE Adv. 2014]***

Short-cut solution :



As the ladder is in complete equilibrium. So

$$\Sigma F_y = 0, \text{ or } N_1 = W \ldots\text{(i)}$$

Taking moment of all forces about 'O'

$$\Sigma\tau_O = 0,$$

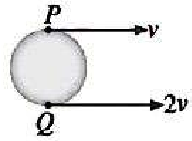
or $$f\times\ell\sin\theta - W\times\ell/2\cos\theta = 0$$

$\therefore$ $$f = \frac{W}{2}\cot\theta.$$

Also $\Sigma F_x = 0$ or $N_2 - f = 0 \Rightarrow N_2 = f$ ***Ans.***

Concept Booster Exercise

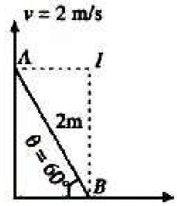
1. Two points P and Q, diametrically opposite on a disc of radius R, have linear velocities v and $2v$ as shown in figure. The angular speed of the disc is :



(a) $\frac{v}{R}$ (b) $\frac{2v}{R}$ (c) $\frac{v}{2R}$ (d) $\frac{v}{4R}$

2. Point A on rod AB of length 2m is moved upwards against a wall with a velocity of 2 m/s. The angular speed of the rod at an instant when $\theta = 60°$ is :

Numeric/Integer



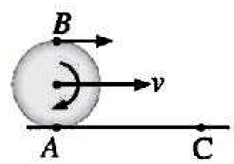
(a) 2.0 rad/s (b) 2.5 rad/s
(c) 3.0 rad/s (d) 1.73 rad/s

3. At a given instant of time the position vector of a particle moving in a circle with a velocity $3\hat{i} - 4\hat{j} + 5\hat{k}$ is $\hat{i} + 9\hat{j} - 8\hat{k}$. Its angular velocity at that time is :

(a) $\frac{13\hat{i} - 29\hat{j} - 31\hat{k}}{\sqrt{140}}$ (b) $\frac{13\hat{i} - 29\hat{j} - 31\hat{k}}{146}$

(c) $\frac{13\hat{i} + 29\hat{j} - 31\hat{k}}{\sqrt{146}}$ (d) $\frac{13\hat{i} + 29\hat{j} + 31\hat{k}}{146}$

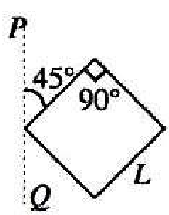
4. A wheel of radius R rolls without slipping with a speed v on a horizontal road. When it is at a point A on the road, a small lump of mud separates from the wheel at its highest point B and drops at a point C on the ground. The distance AC is:



(a) $v\sqrt{\frac{R}{g}}$ (b) $2v\sqrt{\frac{R}{g}}$

(c) $4v\sqrt{\frac{R}{g}}$ (d) $3v\sqrt{\frac{2R}{g}}$

5. A square is made by joining four roads each of mass M and length L. Its moment of inertia about an axis PQ, in its plane and passing through one of its corner is:

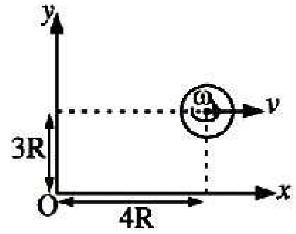


(a) $\frac{4}{3}ML^2$ (b) $5\,ML^2$ (c) $\frac{10}{3}ML^2$ (d) $\frac{8}{3}ML^2$

6. A uniform cylindrical rod of mass 'm' and length 'ℓ' is rotating with an angular velocity ω. The axis of rotation is perpendicular to its axis of symmetry and passes through one of its edge faces. If the room temperature increases by t, the change in its angular velocity is : (α is the coefficient of linear expansion)

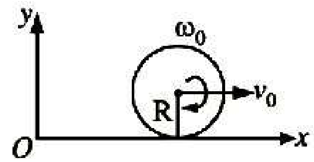
(a) $2\alpha\omega t$ (b) $\alpha\omega t$ (c) $\frac{3}{2}\alpha\omega t$ (d) $\frac{\alpha\omega t}{2}$

7. A disc of mass m and radius R moves in the x-y plane as shown in figure. The angular momentum of the disc about the origin 'O' at the instant shown is: (Given $v = \omega R$)



(a) $\frac{-5}{2}mR^2\omega\hat{k}$ (b) $\frac{7}{2}mR^2m\hat{k}$ (c) $\frac{-9}{2}mR^2\omega\hat{k}$ (d) $\frac{5}{2}mR^2\omega\hat{k}$

8. A uniform sphere of mass m, radius R and moment of inertia 'I' about its CM axis moves along the x-axis is shown in figure. Its centre of mass moves with velocity v_0 and it rotates about its centre of mass with angular velocity ω_0.



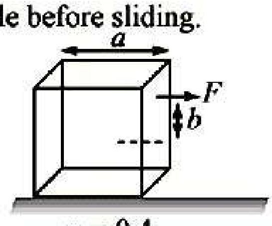
Let $\vec{L} = -(mv_0R + I\omega_0)\hat{k}$. The angular momentum of the body about the origin 'O' is:

(a) $\vec{L}$, only if $v_0 = \omega R$ (b) greater than $\vec{L}$, $v_0 > \omega_0 R$

(c) less than $\vec{L}$, if $v_0 > \omega R$ (d) $\vec{L}$ for all values of v_0 and ω_0

9. A solid cube of side 'a' is shown in the figure. Find maximum value of $100\frac{b}{a}$ for which the block does not topple before sliding. ***[JEE Main 2020]***

Numeric/Integer

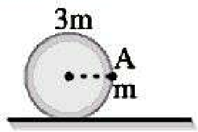


$\mu = 0.4$

10. A tangential force F acts at the top of a thin spherical shell of mass m and radius R. Find the acceleration of the shell if it rolls without slipping.

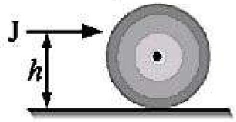
(a) $\frac{6F}{5m}$ (b) $\frac{6m}{5F}$ (c) $\frac{5F}{6m}$ (d) $\frac{3F}{4m}$

11. A particle of mass 'm' is rigidly attached at 'A' to a ring of mass '$3m$' and radius R. The system is released from rest and rolls without sliding. The angular acceleration of ring just after release is:



(a) $\frac{g}{4R}$ (b) $\frac{g}{2R}$ (c) $\frac{g}{8R}$ (d) $\frac{g}{6R}$

12. A solid sphere of radius R is placed on a rough horizontal surface. It is struck by a horizontal cue stick at a height h above the surface. The value of 'h' so that the sphere performs pure rolling motion immediately after it has been stuck is:



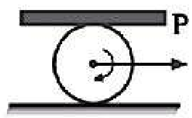
(a) $\frac{2R}{5}$ (b) $\frac{5R}{2}$ (c) $\frac{5R}{7}$ (d) $\frac{7R}{5}$

13. A uniform rod of mass m, hinged at its upper end, is released from rest from a horizontal position. When it passes through the vertical position, the force on the hinge is:

Numeric/Integer

(a) 1 (b) 2 (c) 2.5 (d) 0

14. A plank P is placed on a solid cylinder, which rolls on a horizontal surface. The two are of equal mass. There is no slipping at any of the surfaces in contact. The ratio of the KE of P to that of cylinder is :



(a) 1 : 1 (b) 2 : 1 (c) 1 : 2 (d) 8 : 3

Solutions

1. **(c)** $\omega = \frac{2v - v}{2R} = \frac{v}{2R}$ ***Ans.***

2. **(a)** $\omega = \frac{v}{IA} = \frac{2}{2\cos 60°} = \frac{2}{2 \times \frac{1}{2}} = 2$ rad/s ***Ans.***

3. **(b)** Using $\omega = \frac{(\vec{r} \times \vec{v})}{r^2}$

$$= \frac{(\hat{i} + 9\hat{j} - 8\hat{k}) \times (3\hat{i} - 4\hat{j} + 5\hat{k})}{|\hat{i} + 9\hat{j} - 8\hat{k}|^2}$$

$$= \frac{13\hat{i} - 29\hat{j} - 31\hat{k}}{146}$$ ***Ans.***

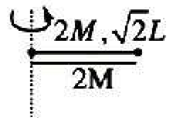
4. **(c)** The velocity of the point B is $2v$. The time taken by mud piece to hit the road,

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(2R)}{g}} = 2\sqrt{\frac{R}{g}}$$

$$\therefore \quad AC = v't = 2v \times 2\sqrt{\frac{R}{g}}$$

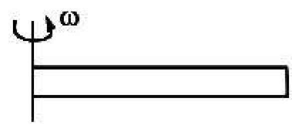
$$= 4v\sqrt{\frac{R}{g}}.$$ ***Ans.***

5. **(d)** The effective system is as shown in figure. So



$$I = 2\frac{2M\left(\sqrt{2}L\right)^2}{3} = \frac{8}{3}ML^2.$$ ***Ans.***

6. **(a)** In the process angular momentum remains constant. So



$$I\omega = \text{const.}$$

or $$\frac{\Delta I}{I} + \frac{\Delta\omega}{\omega} = 0$$

or $$\frac{\Delta\omega}{\omega} = -\frac{\Delta I}{I}$$

As $$I = k\ell^2 \quad (k \text{ constant})$$

$$\therefore \quad \frac{\Delta I}{I} = 2\frac{\Delta\ell}{\ell}$$

or $$\frac{\Delta\omega}{\omega} = \frac{2\Delta\ell}{\ell}$$

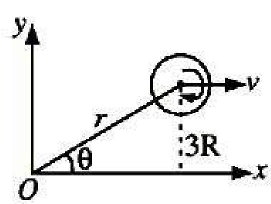
$$= -\frac{2(\ell\alpha t)}{\ell}$$

$$= -2\,\alpha\, t$$

$$\therefore \quad \Delta\omega = -2\,\alpha\,\omega t.$$ ***Ans.***

7. **(a)**

$$\vec{L} = m(\vec{r} \times \vec{v}_{cm}) + I_{CM}\vec{\omega}$$



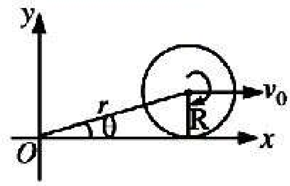
$$\vec{L} = -mvr\sin\theta(\hat{k}) + \frac{mR^2}{2}\omega\hat{k}$$

$$= -mv(3R)\hat{k} + \frac{mR^2}{2}\omega\hat{k}$$

$$= -m(\omega R)3R\hat{k} + \frac{mR^2}{2}\omega\hat{k}$$

$$= -\frac{5}{2}mR^2\omega\hat{k}.$$ ***Ans.***

8. **(d)** $\vec{L} = (mvr\sin\theta)(-\hat{k}) + I\omega(-\hat{k})$



Here $r\sin\theta = R$, so

$\vec{L} = -(mv_0R + I\omega_0)\hat{k}$.

Here all terms are constant, so option (d) is correct.

9. **(50.00)** For no toppling

$$F\left(\frac{a}{2}+b\right) \le mg\frac{a}{2}$$

$$\mu\frac{a}{2}+\mu b \le \frac{a}{2}$$

$$0.2a + 0.4b \le 0.5a$$

$$0.4b \le 0.3a$$

$$b \le \frac{3a}{4}$$

$$b \le 0.75a \quad \text{(in limiting case)}$$

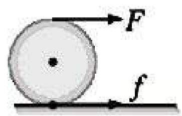
But it is not possible as b can maximum be equal to $0.5a$

$\therefore$ $$\left(100\frac{b}{a}\right)_{max} = 50.00.$$ ***Ans.***

10. **(a)** If a is the acceleration of CM, then

$$F + f = ma$$

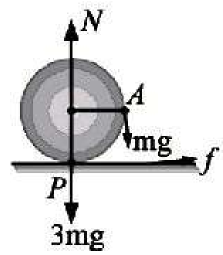
and $$FR - fR = I\alpha$$



For pure rolling, $\alpha = \dfrac{a}{R}$ and $I = \dfrac{2}{3}mR^2$.

On solving above equations, we get $a = \dfrac{6F}{5m}$. ***Ans.***

11. **(c)** Taking moment of force about 'P', we have

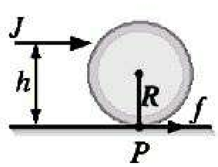


$$mgR = (I_{ring} + I_{particle})_P\,\alpha$$

or $$mgR = \left[\left\{(3m)R^2 + (3m)R^2\right\} + m\left(\sqrt{2}R\right)^2\right]\alpha$$

$\therefore$ $$\alpha = \frac{g}{8R}.$$ ***Ans.***

12. **(d)** Using conservation of momentum about 'P'

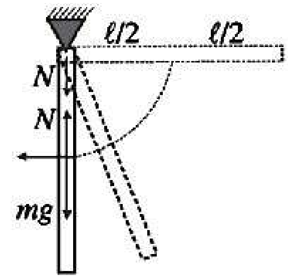


$$mvh = I_p\omega$$

$$= \left(\frac{2}{5}mR^2 + mR^2\right)\frac{v}{R}$$

$$h = \frac{7R}{5}.$$ ***Ans.***

13. **(c)** Using, $mg\frac{\ell}{2} = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$



or $\omega^2 = \frac{3g}{\ell}$

Now $N - mg = m\omega^2\left(\frac{\ell}{2}\right)$

or $N = mg + m\left(\frac{3g}{\ell}\right)\frac{\ell}{2}$

$= \frac{5}{2}mg$. *Ans.*

14. **(d)** If v is the velocity of CM, then $K_{plank} = \frac{1}{2}m(2v)^2 = 2mv^2$

and $K_{cylinder} = \frac{3}{4}mv^2$

$\therefore$ Ratio $= \frac{2mv^2}{\frac{3}{4}mv^2} = \frac{8}{3}$. *Ans.*

Gravitation 8

***TOPIC 8.1:** Kepler's Laws of Planetary Motion, Newton's Law of Gravitation, Gravitational Field, Potential and Potential Energy.*

Review of Formulae

1. Kepler's law of areas : $\frac{\Delta \vec{A}}{\Delta t} = \frac{\vec{L}}{2m}$.
2. Kepler's law of periods : $T^2 \propto a^3$.
3. Newton's law of gravitation, $F = \frac{Gm_1m_2}{r^2}$.
4. Intensity of gravitation field on the earth's surface $E_g = g = \frac{GM}{R^2}$.
5. Mass of the earth, $M = \frac{gR^2}{G}$.
6. The value of g at some height, $g' = \frac{g}{\left(1+\frac{h}{R}\right)^2} \simeq g(1-\frac{2h}{R})$ for $h << R$
7. The value of g at some depth, $g' = g\left(1-\frac{h}{R}\right)$.
8. Effect of rotation, at an latitude λ
   $g' = g - \omega^2 R \cos^2\lambda$
9. At equator, $\lambda = 0$, $g' = g - \omega^2 R$.
10. At equator $g' \to 0$, $\omega = \sqrt{\frac{g}{R}}$ = 17 times present value of rotation
11. Gravitation potential, outside the body $V_g = -\frac{GM}{r}$. Also $E_g = -\frac{dV}{dr}$
12. Gravitational potential energy (earth-body system) $U_g = -\frac{GMm}{r}$.
13. Self energy of earth, $U = -\frac{3}{5}\frac{GM^2}{R}$.

14. Change in potential energy, $\Delta U = \left(\dfrac{mgh}{1+\dfrac{h}{R}}\right) \approx mgh, \quad h << R.$

Tips and Tricks for Shortcut Solutions

Newton's Law of Gravitation

The gravitational force between two particles of masses m_1 and m_2 is given by

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r}$$

where $\hat{r}$ is the unit vector or $F = \dfrac{Gm_1m_2}{r^2}$ from m_1 to m_2.

Two possible motions by gravitational force

(i) *Particles move towards each other due to mutual attraction*

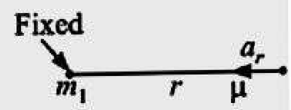
(ii) *Particles rotates about common axis (CM).*

In both the cases, we can fix one of them and move a particle of reduced mass $\mu = \left(\dfrac{m_1m_2}{m_1+m_2}\right)$ towards fixed particle.

Case 1: Particles move towards each other. In this case either we asked the relative acceleration or velocity of approach of the particles.

(a) *Relative acceleration:*

$$a_r = \frac{F}{\mu}$$

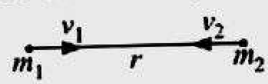


$$= \frac{G\dfrac{m_1m_2}{r^2}}{\left(\dfrac{m_1m_2}{m_1+m_2}\right)}$$

$$= G\left(\frac{m_1+m_2}{r^2}\right).$$

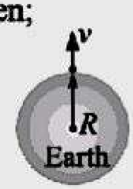
(b) *Velocity of approach:* If particles starts from large distance, then velocity of approach $v_r = (v_1 + v_2)$ at a separation r, can be obtained as;

$$\frac{1}{2}\mu v_r^2 - \frac{Gm_1m_2}{r} = 0$$



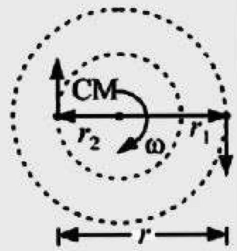
or $$v_r = \sqrt{\frac{2Gm_1m_2}{\mu r}}.$$

(c) *Maximum height attained by a particle:* If a particle is projected vertically upwards with initial velocity v (large enough, but $v < \sqrt{2gR}$), then;



$$\frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}.$$

Case 2: *Rotation of particles about Common-Axis:* Both the particles have same angular velocity, so



$$G\frac{m_1m_2}{r^2} = \mu\omega^2 r$$

$\therefore$ $$\omega = \sqrt{\frac{Gm_1m_2}{\mu r^3}}$$

As $$\mu = \frac{m_1m_2}{m_1+m_2}, \text{ so}$$

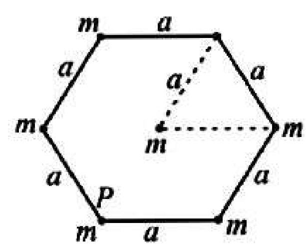
$$\omega = \sqrt{\frac{G(m_1+m_2)}{r^3}}$$

and $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r^3}{G(m_1+m_2)}}.$$

Illustration 1

Six particles each of mass 'm' are placed, one at the centre and five at the corners of a regular hexagon of side 'a'. Find force on the particle placed at the centre due to other particles.

Short-cut solution :



The effective force on the particle is only due to P, so

$$F = \frac{Gmm}{a^2} = \frac{Gm^2}{a^2}$$ *Ans.*

Illustration 2

Assuming the earth to be a sphere of a uniform mass density, how much would a body weigh half way down to the centre of earth if it weighed 250 N on the surface? [g on the surface of the earth = 9.8 m/s²]

Short-cut solution :

Acceleration due to gravity at a depth 'd' is $g' = g\left(1-\frac{d}{R}\right)$

$$\therefore \quad \text{Weight} = mg' = mg\left(1 - \frac{d}{R}\right)$$

$$= 250\left(1 - \frac{R/2}{R}\right) = 125 \text{ N} \qquad \textit{Ans.}$$

Illustration 3

A planet of mass m is in an elliptical orbit around the sun with time period T. The semi major axis and semi minor axis are equal to a and b respectively. The angular momentum of the planet is equal to

(a) $\dfrac{2m\pi ab}{T}$ *(b)* $m\pi abT$ *(c)* $\dfrac{m\pi ab}{2T}$ *(d)* $2\pi abmT$

Short-cut solution :

Area of ellipse, $A = \pi ab,$

We know that $\dfrac{A}{T} = \dfrac{L}{2m}$

$$\Rightarrow \quad L = \frac{2mA}{T} = \frac{2m(\pi ab)}{T}. \qquad \textit{Ans. (a)}$$

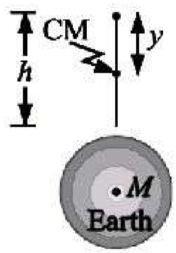
Illustration 4

A heavy ball of mass m = ηM (where M is the mass of the earth and η is a fraction) of a super dense material is kept at height h(h << R). If the size of the ball is negligible as compared to earth, then find the time taken by the ball to reach the earth surface.

Short-cut solution :

Both ball and earth will move towards each other and collide at CM. So distance moved by ball is only y, and assuming value of g constant, so

$$y = \frac{Mh}{m+M} = \frac{Mh}{(\eta M + M)} = \frac{h}{1+\eta}.$$



Now using $y = \dfrac{1}{2}gt^2$, we have

$$t = \sqrt{\frac{2y}{g}} = \sqrt{\frac{2h}{(1+\eta)g}}. \qquad \textit{Ans.}$$

Illustration 5

Two particles of equal mass go round a circle of radius R under the action of their mutual gravitational attraction. The speed of each particle is

(a) $v = \frac{1}{2R}\sqrt{\frac{1}{Gm}}$ (b) $v = \sqrt{\frac{Gm}{2R}}$

(c) $v = \frac{1}{2}\sqrt{\frac{Gm}{R}}$ (d) $v = \sqrt{\frac{4Gm}{R}}$

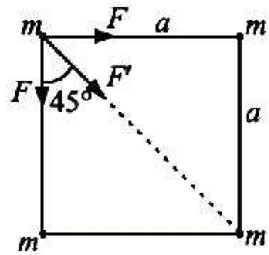
Short-cut solution :

$$\frac{Gm^2}{(2R)^2} = \frac{mv^2}{R} \quad \Rightarrow \quad v = \frac{1}{2}\sqrt{\frac{Gm}{R}}.$$ ***Ans. (c)***

Illustration 6

Four point masses each of mass 'm' are placed at the corners of a square of side 'a'. Find the gravitational force experienced by each particle.

Short-cut solution :



$$F_R = \sqrt{2}F + F'$$

$$= \sqrt{2}\frac{Gm^2}{a^2} + \frac{Gm^2}{(\sqrt{2}a)^2}$$

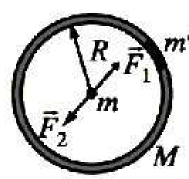
$$= \frac{Gm^2}{2a^2}(2\sqrt{2}+1).$$ ***Ans.***

Illustration 7

A particle of mass m is placed at the centre of a large ring of mass M and radius R. If a small piece of mass m' is removed from the ring, then find the force on the particle due to the remaining ring.

Short-cut solution :

If $\vec{F}_1$ and $\vec{F}_2$ are the forces on the particle due to removed piece and remaining ring, then



$$\vec{F}_1 + \vec{F}_2 = 0$$

or $$\vec{F}_2 = -\vec{F}_1$$

or $$F_2 = F_1$$

$$= \frac{Gmm'}{R}.$$ ***Ans.***

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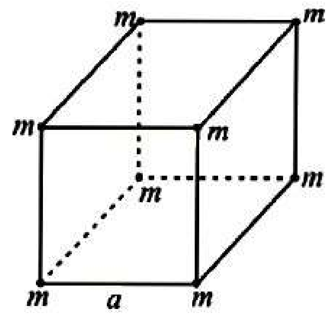
Illustration 8

Eight identical particles each of mass 'm' are situated at the corners of a cube of side 'a'. Calculate gravitation PE of this system.

Short-cut solution :

Total number of pairs: $\frac{n(n-1)}{2}$



$\therefore \qquad N = \frac{8(8-1)}{2} = 28.$

U = 12 pairs of separation a

+ 12 of separation $\sqrt{2}a$

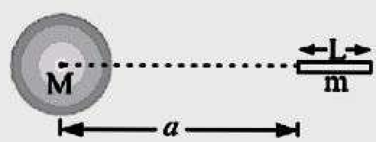
+ 4 of separation $\sqrt{3}a$

$$= -12\frac{Gm^2}{a} - 12\frac{Gm^2}{\sqrt{2}a} - 4\frac{Gm^2}{\sqrt{3}a}$$

$$= -12\frac{Gm^2}{a}\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{3}}\right).$$ ***Ans.***

Tips and Tricks for Shortcut Solutions

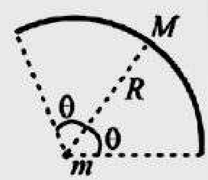
1. Force between a spherical body of mass M and a thin rod of mass m:



$$F = \frac{GMm}{a(a+L)}.$$

2. Force between arc of mass M and a particle of mass 'm':

$$F = \frac{GMm}{R^2}\left(\frac{\sin\theta}{\theta}\right).$$



$\theta \rightarrow$ must be in radian.

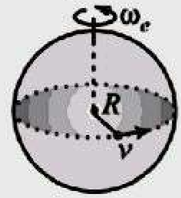
For semicircular arc, $\theta = \frac{\pi}{2}$.

$$F = \frac{GMm}{R^2}\left(\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}\right)$$

$$= \frac{2GMm}{\pi R^2}.$$

3. ***Weight of moving object on the equator of earth:***

If 'W' is the weight of object when at rest (w.r.t. earth), then its weight when in motion



$$W' = W\left(1 \pm \frac{2\omega_e v}{g}\right).$$

Use '–' when object moving in the sense of rotation of earth.

4. For a satellite having gravitational force, $F = \frac{k}{r^n}$

$$v_0 \propto r^{-\left(\frac{n-1}{2}\right)}$$

and $$T \propto r^{\frac{n+1}{2}}.$$

5. Satellite in elliptical orbit

$$\frac{v_{max}}{v_{min}} = \frac{1+e}{1-e}.$$

6. The velocity of a particle falling from infinity will be $\sqrt{2gR}$ on reaching the earth.
7. Time period of spy satellite is nearly 2 hour.
8. Escape speed from height h above earth surface will be

$$v_e' = \sqrt{\frac{2GM}{(R+h)}}.$$

TOPIC 8.2: *Motion of a Satellite, Escape Speed and Orbital Velocity.*

Review of Formulae

1. Orbital velocity of satellite ($r = R + h$);

$v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{gR^2}{r}}$ and time period, $T = 2\pi\sqrt{\frac{r^3}{GM}}$

2. For a satellite close to earth, $h << R, r \simeq R$

$v_0 = \sqrt{gR} \simeq 8$ km/s

and $T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6$ min.

3. Satellite in elliptical orbit : (a is semi-major axis)

$v_{max} = \sqrt{\left(\frac{1-e}{e}\right)\frac{GM}{a}}$, $v_{min} = \sqrt{\left(\frac{1-e}{1+e}\right)\frac{GM}{a}}$.

4. Kinetic energy of a satellite, $K = \frac{GMm}{2r}$.

5. Potential energy, $U = -\frac{GMm}{r}$.

6. Total mechanical energy of satellite $E = K + U = -\frac{GMm}{2r}$.

7. Also $K = -E = -\frac{U}{2}$.

8. Geostationary satellite : $T = 24$ hour, $v_0 \simeq 3$ km/s, $h = 36000$ km from earth surface

9. Escape speed, $v_e = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = 11.2$ km/s.

10. For block hole, $v_e \geq c$,

or $\sqrt{\frac{2GM}{R}} \geq c \Rightarrow R \leq \frac{2GM}{c^2}$.

Video Solution

Q. ***Two masses m_1 and m_2 at an infinite distance from each other are initially at rest, start interacting gravitationally. Find their velocity of approach when they are at a distance r apart.***

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=-q2faqhDjlc

Short-cut solution :

Let v_r is the velocity of approach at the required separation. Then by conservation of mechanical energy.

$$0 + 0 = \frac{1}{2}\mu v_r^2 - \frac{Gm_1 m_2}{r} \qquad \ldots(i)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Substituting the value of μ in equation (i) we get

$$v_r = \sqrt{\frac{2G(m_1 + m_2)}{r}}. \qquad \textit{Ans.}$$

Illustration 9

A satellite close to earth is in orbit above the equator with a period of rotation of 1.5 hrs. If it is above a point P on the equator at some time, after what time it will be above P again ?

Short-cut solution :

Let ω_0 = the angular velocity of earth about its axis = $(2\pi/24)$ rad/hr
Let ω be the angular velocity of the satellite, then $\omega = 2\pi/1.5$
For a satellite rotating from west to east (same as earth), the relative angular velocity $\omega_1 = \omega - \omega_0$

$\therefore$ Time period of rotation relative to earth = $2\pi/\omega_1 = 1.6$ h

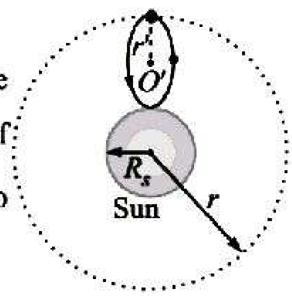
Now, for a satellite rotating from east to west (opposite to earth) the relative angular velocity $\omega_2 = \omega + \omega_0$.

Time period of rotation relative to earth = $2\pi/\omega_2 = (24/17)$ hr. ***Ans.***

Illustration 10

A planet was suddenly stopped in its orbit supposed to be circular. Find the time it takes to fall on to the sun, if its period of revolution around sun is T.

Short-cut solution :



The falling planet can be assumed, a body rotates in a circle of radius r', about O' where $r' \simeq \frac{r}{2}$, neglecting radius of sun in comparison to the radius of orbit. Therefore time to complete a circle T'

$$\frac{T'}{T} = \sqrt{\frac{r'^3}{r^3}}$$

$$= \sqrt{\frac{\left(\frac{r}{2}\right)^3}{r^3}}$$

$$T' = \frac{T}{\sqrt{8}}$$

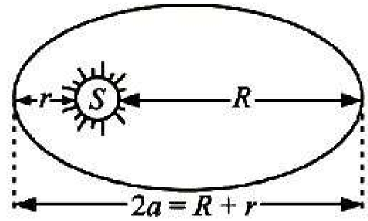
Now time of fall, $$t = \frac{T'}{2} = \frac{\frac{T}{\sqrt{8}}}{2} = \frac{\sqrt{2}T}{8}.$$ ***Ans.***

Illustration 11

A planet moves around the sun along an ellipse so that its minimum distance from the sun is equal to r and the maximum distance to R. Making use of Kepler's laws, find its period of revolution around the sun.

Short-cut solution :

The motion of the planet can be approximated to be in a circle of radius $\left(\frac{R+r}{2}\right)$.



The time period will be given by

$$T = 2\pi\sqrt{\frac{\left[\frac{R+r}{2}\right]^3}{G(M+m)}}$$

Here m is the mass of the planet and M is the mass of sun.
As $m << M$

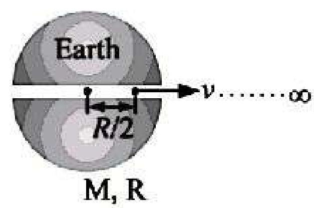
$$\therefore \qquad T = \pi\sqrt{\frac{(R+r)^3}{2GM}}. \qquad \textbf{Ans.}$$

Illustration 12

A very small groove is made in the earth and a particle is placed at $\frac{R}{2}$ distance from the centre. Find escape speed of the particle from that place.

Short-cut solution :

If V is the potential of the point, then



$$\frac{1}{2}mv^2 + mV = 0 + 0$$

$$\frac{1}{2}mv^2 + m\left[-\frac{GM}{2R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)\right] = 0$$

$$\therefore \qquad v = \sqrt{\frac{11GM}{4R}}. \qquad \textbf{Ans.}$$

Illustration 13

A projectile is fired from the surface of the earth at an angle θ = 60° from the vertical. The initial speed is equal to $\sqrt{\frac{GM}{R}}$. How high does the projectile rise? Neglecting earth's rotation and air resistance.

Short-cut solution :

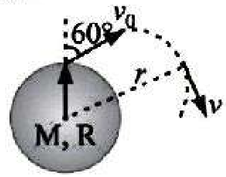
Let projectile goes a distance r from the centre of the earth, then by conservation of angular momentum and ME we have,

$$mv_0 \sin 60° = mvr \qquad ..(i)$$

and

$$\frac{1}{2}mv_0^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad ...(ii)$$

On substituting $v_0 = \sqrt{\frac{GM}{R}}$, and simplifying, we get



$$r = \frac{3R}{2}.$$

So height above earth surface, $h = \frac{3R}{2} - R = \frac{R}{2}$. ***Ans.***

Illustration 14

If a satellite is launched from height 'h' with a speed 'u' horizontally, find the angle with the vertical at which the satellite will hit the earth's surface, if h = R and $u = \sqrt{\frac{GM}{7R}}$.

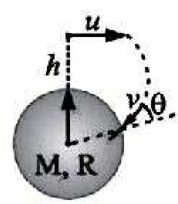
Short-cut solution :

Using conservation of angular momentum and ME, we have

$$mu(R+h) = mvR\sin\theta \quad ...(i)$$

and

$$\frac{1}{2}mu^2 - \frac{GMm}{R+h} = \frac{1}{2}mv^2 - \frac{GMm}{R} \quad ...(ii)$$



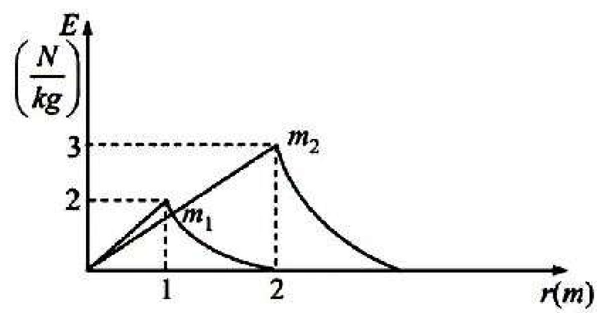
Substituting $h = R$ and $u = \sqrt{\frac{GM}{7R}}$, and simplifying, we get

$$\sin\theta = \frac{1}{\sqrt{2}}$$

or $\theta = 45°$. ***Ans.***

Illustration 15

Two spherical bodies of mass m_1 and m_2 have radii 1m and 2m respectively. The gravitational field of the two bodies with the radial distance from centre is shown below. The value of $\frac{m_1}{m_2}$ is- ***[JEE Main 2020]***



(a) $\frac{1}{6}$ *(b)* $\frac{1}{3}$ *(c)* $\frac{1}{2}$ *(d)* $\frac{1}{4}$

Short-cut solution :

$$3 = \frac{Gm_2}{2^2}$$

$$2 = \frac{Gm_1}{1^2}$$

$$\therefore \quad \frac{3}{2} = \frac{1}{4}\frac{m_2}{m_1}$$

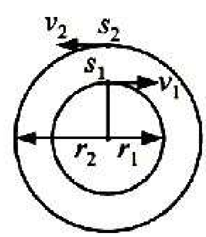
$$\frac{m_1}{m_2} = \frac{1}{6}.$$

Ans. (a)

Illustration 16

Two satellites s_1 and s_2 revolve around a planet in coplanar circular orbits in the opposite sense. The periods of revolutions are T and $8T$ respectively. Find the angular speed of s_2 as observed by an astronaut is s_1, when they are closest to each other.

Short-cut solution :



Using $T^2 \propto r^3$, we have

$$\left(\frac{T}{8T}\right)^2 = \left(\frac{r_1}{r_2}\right)^3$$

$$\therefore \quad r_2 = 4r_1.$$

$$\omega_{21} = \frac{v_2 + v_1}{r_2 - r_1} = \frac{\omega_2 r_2 + \omega_1 r_1}{r_2 - r_1} = \frac{\left(\frac{2\pi}{8T}\right)\times 4r_1 + \left(\frac{2\pi}{T}\right)r_1}{4r_1 - r_1}$$

$$= \frac{\pi}{T}.$$

Ans.

Illustration 17

A stone is dropped from a height equal to 3R, above the earth surface. The velocity of stone on reaching the earth's surface is:

(a) $\sqrt{\frac{gR}{2}}$ *(b)* $\sqrt{2gR}$ *(c)* $\sqrt{\frac{3gR}{2}}$ *(d)* $\sqrt{gR}$

Short-cut solution :

Using, $$\frac{mgh}{\left(1+\frac{h}{R}\right)} = \frac{1}{2}mv^2$$

or $$\frac{mg(3R)}{\left(1+\frac{3R}{R}\right)} = \frac{1}{2}mv^2$$

$\therefore$ $$v = \sqrt{\frac{3gR}{2}}.$$ *Ans. (c)*

Concept Booster Exercise

1. A body of mass m is taken from earth surface to the height equal to radius of earth, the change in potential energy will be :

(a) $mg R$ (b) $\frac{1}{2}$ $mg R$ (c) $2\ mg R$ (d) $\frac{1}{4}$ $mg R$

2. Two planets of masses M and $\frac{M}{2}$ have radii R and $\frac{R}{2}$ respectively. If ratio of escape velocities from their surfaces $\frac{v_1}{v_2}$ is $\frac{n}{4}$, then find n :

(a) 3 (b) 1 (c) 2 (d) 4

3. Imagine a light planet revolving around a very massive star in a circular orbit of radius R with a period of revolution T. If the gravitational force of attraction between the planet and the star proportional to $R^{-5/2}$, then :

(a) $T^2 \propto R^3$ (b) $T^2 \propto R^{7/2}$ (c) $T^2 \propto R^{3/2}$ (d) $T^2 \propto R^{3.75}$

4. Total energy of a satellite revolving in circular path of radius r is –E. The weight of the satellite is

(a) $\frac{E}{r}$ (b) $\frac{E}{2r}$ (c) $\frac{2E}{r}$ (d) $\frac{2E}{r^2}$

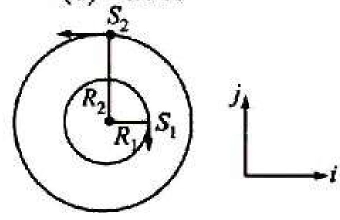
5. If weight of an object at pole is 196 N then weight at equator is [g = 10 m/s^2; radius of earth = 6400 Km] **Numeric/Integer** ***[JEE Main 2020]***

(a) 194.32 N (b) 194.66 N (c) 195.32 N (d) 195.66 N

6. The eccentricity of earth's orbit is e = 0.0167. The ratio of its maximum speed in its orbit to its minimum speed is : **Numeric/Integer**

(a) 2.507 (b) 1.033 (c) 8.324 (d) 1.000

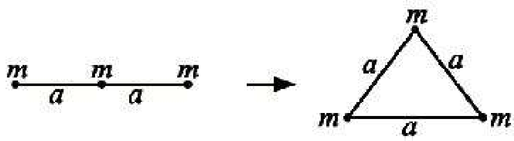
7. Two satellite S_1 and S_2 revolve around a planet in coplanar circular orbits in the opposite sense. The period of revolutions are T and 8T respectively. Find the linear speed of S_2 as observed by an astronaut in S_1 when then angular separation is $\frac{\pi}{2}$ as shown in figure.



(a) $\frac{2\pi R_1}{T_1}\left(-\frac{1}{2}\hat{i}+\hat{j}\right)$ (b) $\frac{2\pi R_1}{T_1}\left(\hat{i}+\frac{1}{2}\hat{j}\right)$

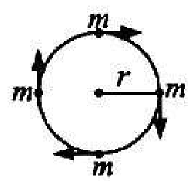
(c) $\frac{2\pi R_1}{T_1}\left(-2\hat{i}+\hat{j}\right)$ (d) $\frac{2\pi R_1}{T_1}\left(\hat{i}+2\hat{j}\right)$

8. Three particles, each of mass 'm' are placed on a line as shown. The work done to put them on an equilateral triangle of side 'a' is :



(a) $-\frac{Gm^2}{a}$ (b) $-\frac{Gm^2}{2a}$ (c) $\frac{-2Gm^2}{a}$ (d) zero

9. Four particles each of mass 'm' are orbiting in a circle of radius r in the same sense under their mutual gravitational force. Speed of each particle is :



(a) $\sqrt{\frac{Gm}{r}}$ (b) $\frac{Gm}{r}\sqrt{(2\sqrt{2}+1)}$

(c) $\sqrt{\frac{Gm}{4r}(2\sqrt{2}+1)}$ (d) $\sqrt{\frac{Gm}{2\sqrt{2}r}}$

10. The angular velocity of the earth's rotation about its axis is 'ω'. An object weighed by a spring balance gives the same reading at the equator as at height 'h' above the poles. The value of h will be :

(a) $\frac{\omega^2 R^2}{g}$ (b) $\frac{\omega^2 R^2}{2g}$

(c) $\frac{2\omega^2 R^2}{g}$ (d) $\frac{2\omega^2 R^2}{3g}$

11. A planet is orbiting the sun is an elliptical orbit. Let U denote the potential energy and K denote the kinetic energy of the planet at an arbitrary point on the orbit.

Choose the correct statement : Numeric/Integer *[KVPY - 2015]*

(a) K < | U | always
(b) K > | U | always
(c) K = | U | always
(d) K = | U | for two positions of the planet in the orbit

12. A satellite moving in a circular orbit of radius r around the earth has a time period T. If its radius slightly increased by Δr, the change in its time period is:

(a) $\left(\frac{T}{r}\right)\Delta r$ (b) $\frac{3}{2}\left(\frac{T}{r}\right)\Delta r$

(c) $\frac{3}{2}\left(\frac{T}{r}\right)^2 \Delta r$ (d) $\frac{3}{2}\left(\frac{T^2}{r}\right)\Delta r$

Solutions

1. (b) $$\Delta U = \frac{mgh}{1+\frac{h}{R}} = \frac{mgR}{1+R/R} = \frac{mgR}{2}$$

2. (d) $$v_e = \sqrt{\frac{2GM}{R}}$$

$\therefore$ $$\frac{v_1}{v_2} = \frac{\sqrt{\frac{2GM}{R}}}{\sqrt{\frac{2GM/2}{R/2}}} = 1 = \frac{n}{4}$$

$\Rightarrow$ $n = 4$

3. (b) $$\frac{mv^2}{R} = \frac{K}{R^{5/2}},$$

$\therefore$ $$v = CR^{-3/4}$$

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{CR^{-3/4}}$$

$\therefore$ $$T^2 \propto R^{7/2}.$$

4. (c) $$-E = -\frac{\text{GM}m}{2r}$$

or $$-E = -\left(\frac{\text{GM}m}{r^2}\right)\frac{r}{2}$$

$$\frac{\text{GM}m}{\text{r}^2} = \frac{2\text{E}}{r}$$

or $$W = \frac{2\text{E}}{r}.$$

5. (c) At pole, weight = mg = 196

$$m = 19.6 \text{ kg}$$

At equator, weight = $mg - m\omega^2 R$

$$= 196 - (19.6)\left[\frac{2\pi}{24 \times 3600}\right]^2 \times 6400 \times 10^3$$

$$= 195.33 \text{ N}$$ ***Ans.***

6. (b) $$\frac{v_{max}}{v_{min}} = \frac{1+e}{1-e} = \left[\frac{1+0.0167}{1-0.0167}\right] = 1.033.$$

7. (a)

$$\frac{T_1^2}{T_2^2} = \left(\frac{R_1}{R_2}\right)^3$$

$$\therefore \quad R_2 = \left(\frac{1}{8}\right)^{\frac{2}{3}} R_1 = 4R_1$$

$$\vec{v}_2 - \vec{v}_1 = \omega_2 R_2(-\hat{i}) - \omega_1 R_1(-\hat{j})$$

$$= -\omega_2 R_2 \hat{i} - \omega_1 R_1 \hat{j}$$

$$= -\frac{2\pi}{T} \times 4R_1 \hat{i} + \frac{2\pi}{T} R_1 \hat{j}$$

$$= \frac{2\pi R_1}{T}\left(-\frac{\hat{i}}{2} + \hat{j}\right)$$

8. (b)

$$W = \Delta U = -\frac{GM^2}{a}\left[1 - \frac{1}{2}\right] = -\frac{GM^2}{2a}.$$ ***Ans.***

9. (c)

$$\frac{Gm^2}{(2r)^2} + \frac{2Gm^2}{(\sqrt{2}r)^2}\cos 45^\circ = \frac{mv^2}{r}.$$ ***Ans.***

10. (b)

$$m(g - \omega^2 R) = mg\left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow \quad h = \frac{\omega^2 R^2}{2g}.$$ ***Ans.***

11. (a) Planet sun system is bounded system

$\therefore$ Total energy of the system is negative $E = K + U$

As E is negative {PE is negative here}

$\therefore$ $K < |U|$ ***Ans.***

12. (b) Using,

$$T = \left(\frac{2\pi}{\sqrt{GM}}\right) r^{3/2}$$

$$\therefore \quad \frac{\Delta T}{T} = \frac{3}{2}\left(\frac{\Delta r}{r}\right).$$ ***Ans.***

Mechanical Properties of Solids 9

***TOPIC**: Hooke's Law, Modulii of Elasticity, Poisson's Ratio, Thermal Stress and Twisting of a Shaft.*

Review of Formulae

1. Elastic force is electromagnetic in nature, but it does not obey Coulomb's law.

2. $$\text{Stress} = \frac{\text{Restoring force}}{\text{Area}}$$

$$\text{Strain} = \frac{\text{Change in dimension}}{\text{Original dimension}}$$

3. **Hooke's law :** Within elastic limit;

$$\text{stress} \propto \text{strain}$$

or $$\frac{\text{stress}}{\text{strain}} = E \text{ (modulus of elasticity)}$$

Modulus of elasticity is the material property which does not depend on size and shape of the body.

4. **Three types of modulii of elasticity.**

(i) Young's modulus,

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$= \frac{F/A}{\Delta l/l} = \frac{Mgl}{\pi r^2 \Delta l}$$

(ii) Bulk modulus,

$$B = \frac{\text{Normal stress}}{\text{Volumetric strain}} = \frac{P}{\left(-\frac{\Delta V}{V}\right)}$$

(iii) Shear modulus,

$$\eta = \frac{\text{shear stress}}{\text{shear strain}} = \frac{F/A}{\theta}$$

$$= \frac{F}{A\theta}$$

Here θ is the shear strain.

5. **Poisson's ratio,**

$$\sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{-\Delta r / r}{\Delta \ell / \ell}$$

Theoretical value of σ lies from -1 to 0.5.

6. $Y = 3B(1 - 2\sigma)$, $Y = 2\eta(1 + \sigma)$

$$Y = \frac{9\eta B}{\eta + 3B}$$

7. Change in volume,

$$\frac{\Delta V}{V} = \frac{\Delta \ell}{\ell}(1 - 2\sigma)$$

Change in density,

$$\rho' = \frac{\rho}{\left(1 - \frac{P}{B}\right)}$$

8. **Thermal stress,** $f_{th} = Y\alpha\Delta T$

9. **Extension due to self weight**

$$\Delta \ell = \frac{\rho g \ell^2}{2Y} = \frac{W\ell}{2AY}$$

10. **Strain energy,** $U = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume}$

Strain energy per unit volume, $u = \frac{f^2}{2Y}$

Strain energy due to shear of the body $u = \frac{f^2}{2\eta}$

11. **Twisting of a shaft :** Torsional rigidity of material of shaft

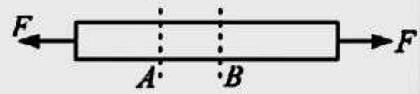
$$\frac{\tau}{\theta} = \frac{\pi \eta r^4}{2\ell}$$

Here θ is the angle of twist.

Tips and Tricks for Shortcut Solutions

1. **Force in the rod:**

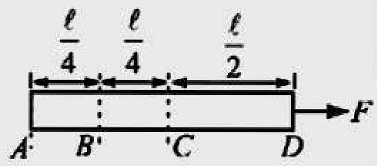
   When equal and opposite forces are applied at its ends, the force at each section is equal to that at its one end.



   $\therefore \qquad F_A = F_B = F$

2. When force (F) is applied at one end of the rod, the force at free end is zero. In this case specimen will not be in equilibrium. Rod will accelerate in the direction of force. The force in the rod varies linearly from zero to F.

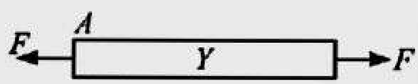
$$F_A = 0, F_B = \frac{F}{4}$$



$$F_C = \frac{F}{2}, F_D = F.$$

3. Hooke's law: When force F acts along the one of the dimensions of the specimen, then

$$\frac{\text{stress}(f)}{\text{strain}(e)} = Y$$



or
$$\frac{F/A}{\Delta\ell/\ell} = Y$$

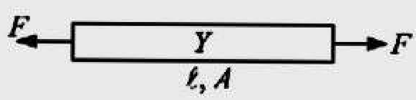
$\therefore$
$$F = \left(\frac{YA}{\ell}\right)\Delta\ell$$

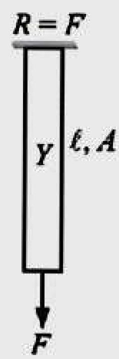
On comparing with native form of Hooke's law, we have

$$F = kx \qquad (x = \Delta\ell)$$

$\therefore$
$$k = \frac{YA}{\ell}.$$

4. Extension/Compression





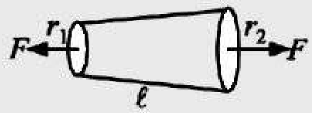
   (i) Due to external force only

$$\Delta\ell = \frac{F\ell}{AY}$$

   (ii) Extension due to self weight

$$\Delta\ell = \frac{W\ell}{2AY} = \frac{1}{2}\frac{\rho g\ell^2}{Y}.$$

5. Extension of non-uniform rod

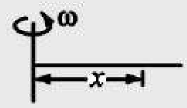


$$\Delta\ell = \frac{F\ell}{\pi r_1 r_2 Y}.$$

6. Strain in the ring of radius r due to radial force,

$$e = \frac{\Delta\ell}{\ell} = \frac{\Delta r}{r}.$$

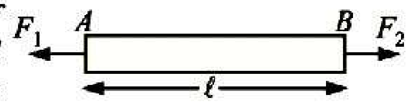
7. Force at any section of a rotating rod



$$F = \frac{m\omega^2}{2\ell}(\ell^2 - x^2)$$

Illustration 1

A uniform rod of length ℓ is made of material of Young's modulus Y is subjected to forces F_1 and F_2 ($F_2 > F_1$) at its ends. Find extension of the rod.



Short-cut solution :

Stress at A, $f_A = \frac{F_1}{A}$

and $f_B = \frac{F_2}{A}$

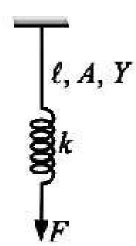
Average stress $f_{av} = \left(\frac{F_1 + F_2}{2A}\right)$

$\therefore$ $\Delta\ell = \frac{F\ell}{AY} = \left(\frac{F_1 + F_2}{2AY}\right)\ell$ ***Ans.***

Illustration 2

A wire of length ℓ, cross-sectional area A, Young's modulus Y is connected to a spring of force constant k. The wire is hanging from the ceiling and a force F is applied on the spring as shown. Find extension of this assymbly.

Short-cut solution :



The force constant of wire, $k' = \frac{YA}{\ell}$.

The equivalent force constant, $k_e = \left(\frac{k \times k'}{k + k'}\right)$

$$= \frac{k\left(\frac{YA}{\ell}\right)}{\left(k+\frac{YA}{\ell}\right)}$$

Using Hooke's law, we have $\quad x = \frac{F}{k_e}$

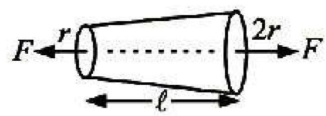
$$= \frac{F\left(k+\frac{YA}{\ell}\right)}{(kYA/\ell)}.$$ ***Ans.***

Illustration 3

A non uniform rod of length ℓ is made of material of Young's modulus Y. The radii of its ends are r and 2r. Find extension of the rod due to a tensile force F.

Short-cut solution :

Using, $\quad \Delta\ell = \frac{F\ell}{\pi r_1 r_2 Y}$



$$= \frac{F\ell}{\pi r(2r)Y} = \frac{F\ell}{2\pi r^2 Y}.$$ ***Ans.***

Illustration 4

A steel ring of radius r and cross-sectional area A is fitted on to a wooden disc of radius R(R > r). If Young's modulus of steel is Y, then the force with which the steel ring is expanded is :

(a) $AY\frac{R}{r}$ *(b)* $AY\frac{R-r}{r}$ *(c)* $\frac{Y}{A}\left(\frac{R-r}{r}\right)$ *(d)* $\frac{Yr}{AR}$

Solution :

Strain, $\quad e = \frac{\Delta r}{r} = \frac{R-r}{r}$

Stress $\quad f = eY = \left(\frac{R-r}{r}\right)Y$

Force needed $\quad F = fA = \left(\frac{R-r}{r}\right)YA.$ ***Ans. (b)***

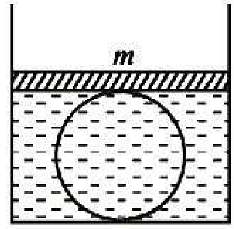
Illustration 5

A solid sphere of radius R made of material of bulk modulus K is surrounded by a liquid in a cylindrical container. A massless piston of area A floats on the surface of the liquid when a mass m is placed on the piston to compress the liquid, the fractional change in the radius of the sphere ΔR / R is :

(a) $\frac{mg}{AK}$ *(b)* $\frac{mg}{3AK}$ *(c)* $\frac{mg}{A}$ *(d)* $\frac{3mg}{AK}$

Short-cut solution :

$$f = \frac{mg}{A}$$



Volumetric strain,

$$\frac{\Delta V}{V} = \frac{\Delta P}{K}$$

or $$3\frac{\Delta R}{R} = \frac{mg/A}{K}$$

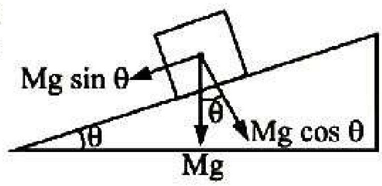
$\therefore$ $$\frac{\Delta R}{R} = \left(\frac{mg}{3KA}\right).$$ ***Ans. (b)***

Illustration 6

A cubical block of mass 'M' and side 'a' is placed on a rough inclined plane of inclination θ. Find direct and shear stresses in the block.

Short-cut solution :

The direct stress is due to $Mg\cos\theta$ and shear stress at the bottom of the cube is due to $Mg\sin\theta$. Therefore



$$f_{\text{direct}} = \frac{Mg\cos\theta}{A} = \frac{Mg\cos}{}$$

and $$f_{\text{shear}} = \frac{Mg\sin\theta}{A} = \frac{Mg\sin\theta}{a^2}.$$ ***Ans.***

Illustration 7

A cylindrical rod is of cross-sectional area 'A', is subjected to tensile force at its ends. Find maximum shear stress at a section inclined θ from the normal cross-section.

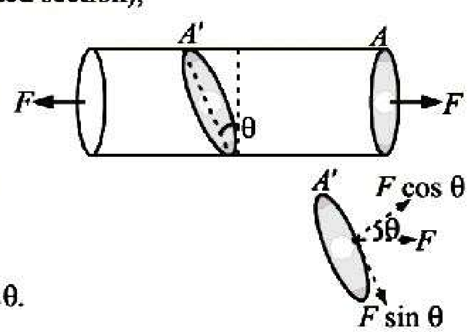
Short-cut solution :

Here $A' = \dfrac{A}{\cos\theta}$ shear stress (parallel to shaded section),

$$f = \frac{F\sin\theta}{A'}$$

$$= \frac{F\sin\theta}{\dfrac{A}{\cos\theta}}$$

$$= \frac{F}{2A}\sin 2\theta.$$

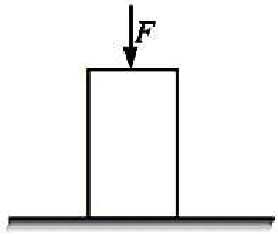


For maximum shear stress, $\sin 2\theta = 1$, $\theta = 45°$

$$\therefore \qquad f_{max} = \frac{F}{2A}. \qquad \textit{Ans.}$$

Illustration 8

A metal cylinder of length L is subjected to a uniform compressive force F as shown in the figure. The material of the cylinder has Young's modulus Y and Poisson's ratio σ. The change in volume of the cylinder is :



(a) $\frac{\mu FL}{Y}$ *(b)* $\frac{(1-\sigma)FL}{Y}$

(c) $\frac{(1+2\sigma)FL}{Y}$ *(d)* $\frac{(1-2\sigma)FL}{Y}$

Short-cut solution :

Using, $$\frac{\Delta V}{V} = \frac{\Delta L}{L}(1-2\sigma).$$

Here $$\Delta L = \frac{FL}{AY},$$

$$\therefore \qquad \frac{\Delta V}{V} = \frac{F}{AY}(1-2\sigma), \text{ and } \Delta V = \frac{F}{AY}(1-2s)AL$$

$$= (1-2\sigma)\frac{FL}{Y}. \qquad \textit{Ans. (d)}$$

Illustration 9

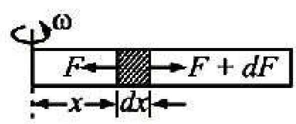
A rod of mass M, length ℓ and cross-sectional area A, made of material of Young's modulus Y is rotated about its one end in horizontal plane with constant angular speed 'ω'. Find its extension.

Short-cut solution :

For the element of mass dm,

$$-(dF) = (dm)\omega^2 x$$

$$\therefore \qquad F = -\int_{\ell}^{x}\left(\frac{M}{\ell}dx\right)\omega^2 x$$



$$= \frac{M}{2\ell}\omega^2(\ell^2 - x^2)$$

Now using, $$d\ell = \frac{F(dx)}{AY}$$

$$= \frac{M\omega^2}{2\ell}\frac{(\ell^2 - x^2)}{AY}dx$$

$$\therefore \qquad \Delta\ell = \frac{M\omega^2}{2YA\ell}\int_0^\ell (\ell^2 - x^2)dx$$

$$= \frac{M\omega^2\ell^2}{3AY}. \qquad \textit{Ans.}$$

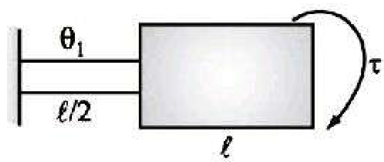
Illustration 10

A rod of length ℓ and radius r is joined to a rod of length $\ell/2$ and radius r/2 of same material. The free end of small rod is fixed to a rigid base and the free end of larger rod is given a twist of $\theta°$, the twist angle at the joint will be

(a) $\theta/4$ *(b)* $\theta/2$ *(c)* $5\theta/6$ *(d)* $8\theta/9$

Short-cut solution :

$$\theta_1 + \theta_2 = \theta_0 \qquad \ldots \text{(i)}$$



We know that

$$\frac{\tau}{\theta} = \frac{\pi\eta r^4}{2\ell}$$

$$\therefore \qquad \frac{\theta_1}{\theta_2} = \frac{\ell_1}{\ell_2} \times \frac{r_2^4}{r_1^4} = \frac{\ell/2 \times r^4}{\ell \times (r/2)^4} = 8 \qquad \ldots \text{(ii)}$$

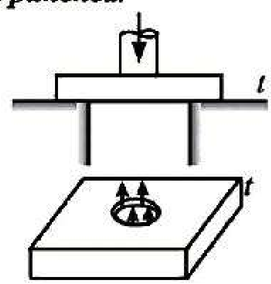
After solving above equations, we get

$$\theta_1 = \frac{8\theta}{9}. \qquad \textit{Ans. (d)}$$

Illustration 11

A hole is punched out of a plate having an ultimate shearing stress f. If the maximum compressive stress on the punch is limited to f_0, then determine the thickness t of the plate from which a hole of radius r can be punched.

Short-cut solution :



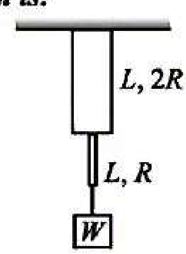
Punching force = shearing resistance

$$f_0 \times \pi r^2 = f \times 2\pi r t$$

$$\therefore \qquad t = \frac{f_0 r}{2f}. \qquad \textit{Ans.}$$

Illustration 12

Two wires of same material (Young's modulus Y) and same length L but radii R and 2R respectively are joined end to end and a weight W is suspended from the combination as shown in figure. The elastic PE stored in the system is:



(a) $\frac{3W^2L}{4\pi R^2 Y}$ *(b)* $\frac{5W^2L}{8\pi R^2 Y}$

(c) $\frac{3W^2L}{8\pi R^2 Y}$ *(d)* $\frac{W^2L}{8\pi R^2 Y}$

Short-cut solution :

Using, $U = \frac{1}{2}kx^2 = \frac{k}{2}\left(\frac{F}{k}\right)^2 = \frac{F^2}{2k}$.

We have, $U = U_1 + U_2$

$$= \frac{F^2}{2k_1} + \frac{F^2}{2k_2} \qquad (F = W)$$

$$= \frac{W^2}{2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \qquad \left[k_1 = \frac{YA}{L} = \frac{Y\pi(2R)^2}{L}, k_2 = \frac{Y\pi R^2}{L}\right]$$

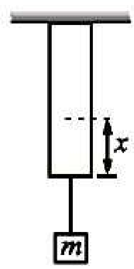
$$= \frac{W^2}{2}\left(\frac{L}{4\pi R^2 Y} + \frac{L}{\pi R^2 Y}\right)$$

$$= \frac{5W^2 L}{8\pi R^2 Y}.$$

Ans. (b)

Illustration 13

One end of a uniform wire of length 'L' and mass 'M' is attached rigidly to a point in the roof and a load of mass 'm' is suspended from its lower end. If A is the area of cross-section of the wire then stress in the wire at a height 'x' from its lower end (x < L) is



(a) $\frac{Mg}{A} + \frac{mxg}{AL}$ *(b)* $\frac{mg}{A} + \frac{Mxg}{AL}$

(c) $\frac{mg}{AL} + \frac{Mxg}{A}$ *(d)* $\frac{mg}{A} - \frac{Mxg}{AL}$

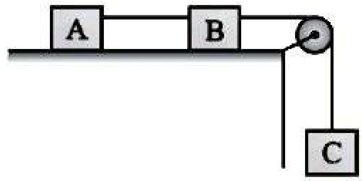
Short-cut solution :

Stress, $f = \frac{\text{Force below section}}{A} = \left[\frac{mg + \frac{Mgx}{L}}{A}\right]$ ***Ans. (b)***

Illustration 14

Each of three blocks shown in figure has a mass 3kg. The wire connecting, blocks A and B has area of cross-section 0.005 cm² and Young's modulus of elasticity Y = 2 × 10¹¹ N/m² neglect friction. Find the elastic PE stored per unit volume (energy density) in wire connecting blocks A and B in steady state. (Take g = 10 m/s²)

Short-cut solution :



Acceleration, $a = \frac{3g}{9} = \frac{g}{3}\ m/s^2$

Tension in the wire between A and B

$$T = ma = 3\times\frac{g}{3} = g = 10\ N.$$

Using, $u = \frac{f^2}{2Y} = \frac{(T/A)^2}{2Y} = \frac{T^2}{2A^2Y}$

$$= \frac{10^2}{2\times(0.005\times10^{-4})^2\times2\times10^{11}}$$

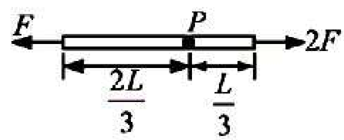
$$= 1000\ J/m^3.$$ ***Ans.***

Concept Booster Exercise

1. A wire of length 1m and radius 1 mm is subjected to a load. The extension is 'x'. The wire is melted and then drawn into a wire of square cross-section of side 1 mm. What is the extension under the same load?

(a) $\frac{\pi^2}{x}$ (b) πx^2 (c) $\pi^2 x$ (d) πx

2. A uniform cylindrical rod of length L, cross-sectional area A and Young's modulus Y is acted upon by the forces as shown in figure. The rod is hinged at point P, $\frac{L}{3}$ from the right end. Find extension of the rod



(a) $\frac{4FL}{3AY}$ (b) $\frac{3FL}{8AY}$

(c) $\frac{2FL}{5AY}$ (d) $\frac{3FL}{5AY}$

3. A material has Poisson's ratio 0.5. If a uniform rod of it suffers a longitudinal strain of 2×10^{-3} the percentage change in its volume is: **Numeric/Integer**

(a) 5% (b) 10% (c) 12% (d) 0%

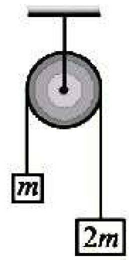
4. A body of mass 10 kg is attached to a wire of length 0.3 m. Its breaking stress is $4.8\times10^7\ N/m^2$. The area of cross-section of wire is $10^{-6}\ m^2$. The maximum angular velocity with which it can be rotated in a horizontal circle without breaking **Numeric/Integer**

(a) 2 rad/s (b) 3 rad/s (c) 4 rad/s (d) 5 rad/s

5. A copper wire of negligible mass, length (ℓ) and cross-sectional area (A) is kept on a smooth horizontal table with one end fixed, a ball of mass 'm' is attached at other end. The wire and the ball are rotated with angular speed 'ω'. If wire elongates by $\Delta\ell$, then Young's modulus of wire is: ($\Delta\ell << \ell$)

(a) $\frac{m\omega^2\ell^2}{A(\Delta\ell)}$ (b) $\frac{m\ell}{\omega^2 A(\Delta\ell)}$ (c) $\frac{m\omega^2\ell}{A(\Delta\ell)}$ (d) $\frac{m\ell\omega^2}{3A(\Delta\ell)}$

6. Two blocks of masses 'm' and '$2m$' are connected through a wire of breaking stress f_0 passing over a frictionless pulley. The maximum radius of wire to be used so that the wire may not break is:



(a) $\sqrt{\frac{3mg}{4\pi f_0}}$ (b) $\sqrt{\frac{4mg}{3f_0}}$

(c) $\sqrt{\frac{mg}{2\pi f_0}}$ (d) $\sqrt{\frac{4mg}{3\pi f_0}}$

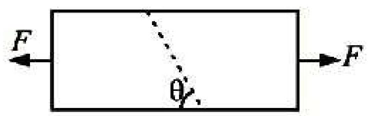
7. A uniform pressure 'P' is exerted on all sides of a solid cube at temperature t°C. By what amount should the temperature of the cube be raised in order to bring back to its original volume, it had before the pressure was applied? The bulk modulus and coefficient of volume expansion of material are B and γ respectively.

(a) $\frac{P}{\gamma B}$ (b) $\frac{\gamma B}{P}$

(c) $\frac{2P}{\gamma B}$ (d) None of these

8. Consider a long steel bar under a tensile force F acting at the edges along the length of the bar. Consider a plane making an angle θ with the length. For what angle is the tensile stress is maximum?

Numeric/Integer



(a) 30° (b) 45°

(c) 60° (d) 90°

Solutions

1. (c) $\pi(1)^2 \times 1 = 1^2 \times \ell \Rightarrow \ell = \pi$

Using, $\Delta\ell = \frac{F\ell}{AY}$

$\therefore \quad \frac{\Delta\ell_1}{\Delta\ell_2} = \frac{\ell_1}{\ell_2} \times \frac{A_2}{A_1} = \frac{1}{\pi} \times \frac{1^2}{\pi(1)^2}$

or $\Delta\ell_2 = \pi^2(\Delta\ell_1) = \pi^2 x.$ ***Ans.***

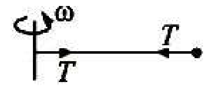
2. (a) Rod will extend on both sides of the hinge, so

$$\Delta\ell = \frac{2F\left(\frac{L}{3}\right)}{AY} + \frac{F\left(\frac{2L}{3}\right)}{AY} = \frac{4FL}{3AY}$$ ***Ans.***

3. (d) $\frac{\Delta V}{V} \times 100 = \frac{\Delta\ell}{\ell}(1-2\sigma)$

$= \frac{\Delta\ell}{\ell}(1-2\times 0.5) = 0\%$ ***Ans.***

4. **(c)** The tansion in the wire at the axis of rotation



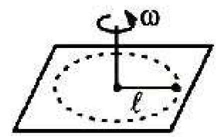
$$T = m\omega^2\ell$$

or $$f_A = m\omega^2\ell$$

or $$4.8 \times 10^7 \times 10^{-6} = 10 \times \omega^2 \times 0.3$$

$\therefore$ $$\omega = 4 \text{ rad/s.}$$ **Ans.**

5. **(a)** Using, $$\Delta\ell = \frac{F\ell}{AY}$$



$\therefore$ $$Y = \frac{F\ell}{A(\Delta\ell)} = \frac{(m\omega^2\ell)\ell}{A(\Delta\ell)}$$

$$= \frac{m\omega^2\ell^2}{A(\Delta\ell)}.$$ **Ans.**

6. **(d)** Here tension in wire, $$T = \frac{2m_1m_2g}{m_1+m_2}$$

$$= \frac{2(2m)mg}{2m+m} = \frac{4mg}{3}.$$

Using, $$T = f_0A$$
$$= f_0 \times \pi r^2$$

$\therefore$ $$r = \sqrt{\frac{T}{\pi f_0}}$$

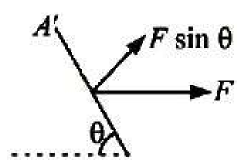
$$= \sqrt{\frac{4mg}{3\pi f_0}}.$$ **Ans.**

7. **(a)** $$B = \frac{\Delta P}{\frac{\Delta V}{V}} \Rightarrow \Delta V = \frac{\Delta V}{B}$$

or $$V\gamma\Delta t = \frac{PV}{B}$$

$\therefore$ $$\Delta t = \frac{P}{\gamma B}.$$ **Ans.**

8. **(d)** If A is the area of cross-section of the bar, then



$$f = \frac{F\sin\theta}{\left(\frac{A}{\sin\theta}\right)} = \frac{F}{A}\sin^2\theta$$

f will be maximum for $\sin\theta = 1$. or $\theta = 90°$. **Ans.**

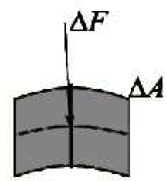
Fluid Mechanics 10

TOPIC 10.1: *Pressure, Archimedes Principle and Accelerating Liquid.*

Review of Formulae

1. **Pressure :**

   Average pressure, $P = \frac{\Delta F}{\Delta A}$



   Pressure at any point is defined as :

$$P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

   or $$P = \frac{dF}{dA}$$

   SI unit of pressure is $\frac{N}{m^2} \cdot \frac{1N}{m^2} = 1Pa.$ 1 bar = $10^5 N/m^2$.

2. **Variation of pressure with depth.**

$$P = \rho gh$$

   here h is the depth of the point from free surface of the liquid.

3. **Atmospheric pressure :**

   1 atmospheric pressure = 76 cm of mercury height
   = 10.3 m of water height
   = $1.013 \times 10^5 \ N/m^2$

   Gauge pressure = Absolute pressure–Atmospheric pressure

   or $$P_{gauge} = P_{absolute} - P_a$$

   Assuming isothermal atmosphere, pressure at any height h is given by

$$P = P_0 e^{-\rho_0 gh/P_0}$$

   here P_0 is the pressure at ground level.

4. **Manometer :** It is used to measure the gauge pressure of a gas.
   If h is the difference of levels of liquid in the arms of manometer tube, then

$$P_{gauge} = \rho gh$$

5. **Archimedes' principle :**
   Buoyant force on the immersed body

$$F_b = V\rho g$$

   Apparent weight of the body

$$= W - F_b = W\left(1 - \frac{\rho}{\sigma}\right)$$

6. **For floating body :**
   weight of the body = buoyant force on the body

7. If V' and V are the submerged and total volume of the body, then

$$\frac{V'}{V} = \frac{\sigma}{\rho}$$

here $\sigma \leq \rho$.

8. **Equilibrium of a submerged body :** For completely submerged floating body;
   (i) If C.G. his below centre of buoyancy, then there will be stable equilibrium.
   (ii) If C.G. lies above centre of buoyancy, then there will be unstable equilibrium.
   (iii) If C.G. concides with the centre of buoyancy, then there will be neutral equilibrium

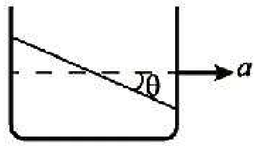
9. **Accelerating liquid :**
   (i) Pressure difference between two points at a vertical height h in a liquid accelerating upwards is given by

$$P_2 - P_1 = \rho(g + a)h$$

   (ii) Buoyant force,

$$F_b = V\rho(g + a)$$

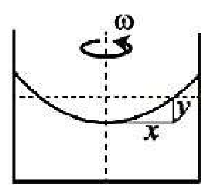
   (iii) When liquid is subjected to horizontal acceleration :



$$\tan 0 = \frac{a}{g}$$

   (iv) When liquid is subjected to rotation :
   The difference in elevation between the axis and at a distance x,

$$y = \frac{\omega^2 x^2}{2g}$$



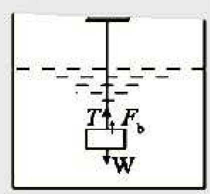
Tips and Tricks for Shortcut Solutions

1. The relative density of the body ($\rho_{body} < \rho_{\omega}$) can be obtained by floating in water as :
   RD = % of volume of body inside water.

2. Relative density, RD = $\frac{\rho_{substance}}{\rho_{water}} = \frac{W_{atr}}{W_{air} - W_{water}}$.

3. Also, $\frac{\rho_{substance}}{\rho_{liquid}} = \frac{W_{air}}{W_{air} - W_{liquid}}$

4. Tension in the string connected to a submerged body.



   (i) When system has zero acceleration

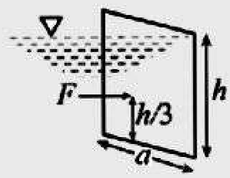
$$T = W - F_b$$
$$= W - V\rho_{\ell}g$$

(ii) When system has constant acceleration in vertical direction

$T = W - V\rho_\ell(g \pm a)$. Use '+' for acceleration up or retardation down.

(iii) If T_0 is the tension in the string when system at rest, when accelerated up, $T = T_0\,(1 + a/g)$.

5. Force on the vertical walls (any shape) of the container

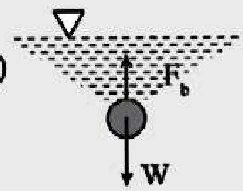


$$F = P_{av} \times ah$$

$$= \left(\frac{\rho g h}{2}\right) ah = \frac{\rho g a h^2}{2}.$$

The line of action is at a height $\frac{h}{3}$ from bottom.

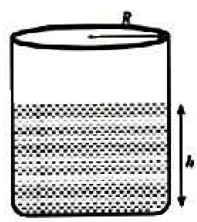
6. Acceleration of body inside non-viscous liquid ($\rho_b < \rho_\ell$)



$$a = \left[\frac{\rho_\ell - \rho_b}{\rho_b}\right] g = \left[\frac{\rho_\ell}{\rho_b} - 1\right] g.$$

Illustration 1

The liquid is filled in a cylindrical container of radius R. What should be the height of the liquid, so that force at the wall of the container equal to force at the bottom.



Short-cut solution :

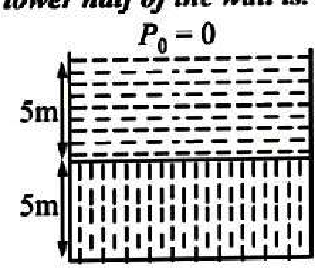
$$F_{wall} = F_{bottom}$$

or $\left(\frac{\rho g h}{2}\right) \times 2\pi R h = (\rho g h)\pi R^2$

$\therefore$ $h = R$. ***Ans.***

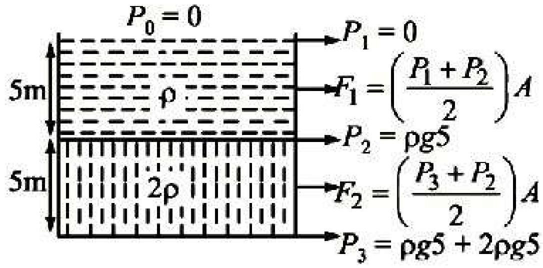
Illustration 2

Two liquid columns of same height 5m and densities ρ and 2ρ are filled in a container of uniform cross sectional area. Then ratio of force exerted by the liquid on upper half of the wall to lower half of the wall is. ***[JEE Main 2020]***



(a) $\frac{1}{4}$ *(b)* $\frac{1}{2}$ *(c)* $\frac{1}{3}$ *(d)* $\frac{2}{3}$

Short-cut solution :



$$\frac{F_1}{F_2} = \frac{1}{4}$$ ***Ans. (a)***

Illustration 3

A tube 1 cm² in cross-section is attached to the top of a vessel 1 cm high and of cross-section 100 cm². Water is pouered into the system, filling it to a depth of 100 cm above the bottom of the vessel.

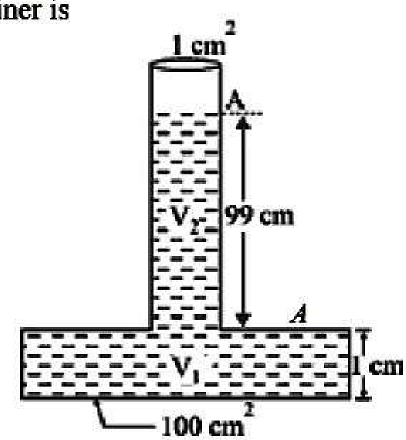
(a) What is the force exerted by the water against the bottom of the vessel?

(b) What is the weight of the water in the vessel?

(c) Explain why (a) and (b) are not equal.

Solution :

(a) Intensity of pressure at the bottom of the container is



$P = h\rho g$

$= 1 \times 1000 \times 10$

$= 1 \times 10^4$ N/m²

Force exerted by water at the bottom

$F = P \times A$

$= (1 \times 10^4) \times (100 \times 10^{-4})$

$= 100$ N

(b) Weight of the water in the container

$W = (V_1 + V_2)\rho g$

$= [100 \times 10^{-4} \times 1 \times 10^{-2} + 1 \times 10^{-4} \times 99 \times 10^{-2}] \times 1000 \times 10$

$= 1.99 \text{ N} \simeq 2\text{N}$

(c) As we have seen in (a) and (b) the thrust of water at the bottom of the vessel is greater than the weight of the water. It is because of the force exerted by the top face of the vessel on the water. Water transfer this force to the bottom of the vessel.

Force exerted by wall (A) on the liquid = 100 – 2 = 98 N. ***Ans.***

Illustration 4

A body weighs 250 N in air, 200 N in water and 150 N in liquid. Find density of liquid. (density of water 1000 kg/m³)

Short-cut solution :

Using, $\frac{\rho_b}{\rho_w} = \frac{W_{air}}{W_{air} - W_{water}}$

$= \frac{250}{250-200} = 5$

or $\rho_b = 5\rho_w = 5000 \text{ kg/m}^3$

Now $\frac{\rho_b}{\rho_\ell} = \frac{250}{250-150} = 2.5$

$\therefore$ $\rho_\ell = \frac{\rho_b}{2.5} = \frac{5000}{2.5} = 2000 \text{kg/m}^3$. ***Ans.***

Illustration 5

A body floats in a liquid with one fourth volume out of liquid. The same body floats in water with its one third volume out of water. Find density of liquid.

Short-cut solution :

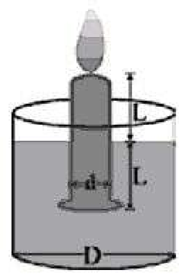
Density of body, $\rho_b = \frac{3}{4}\rho_\ell$

and $\rho_b = \frac{2\rho_w}{3} = \frac{2000}{3} \text{kg/m}^3$

$\therefore$ $\rho_\ell = \frac{4}{3}\rho_b = \frac{4}{3} \times \frac{2000}{3} = 889 \text{ kg/m}^3$. ***Ans.***

Illustration 6

A candle of diameter d is floating on a liquid in a cylindrical container of diameter D (D > > d) as shown in figure. If it is burning at the rate of 2 cm/hour, then the top of the candle will



(a) remain at the same height
(b) fall at the rate of 1 cm/hour
(c) fall at the rate of 2 cm/hour
(d) go up the rate of 1 cm/hour

Short-cut solution :

Initially, $mg = (AL)P_w g$

or $\rho(A \times 2L)g = (AL)\rho_w g$ ... (i)

After one hour,

$\rho[A \times (2L - 2)g = (A\ell)\rho_w g$ ... (ii)

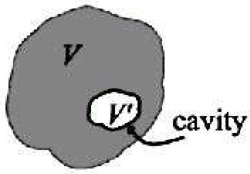
Dividing (ii) by (i), we get

$\ell = (L - 1)$. ***Ans. (b)***

Illustration 7

A metal weighs 20 g in air and 12 g in water. If relative density of metal is 5, find volume of cavity inside metal.

Short-cut solution :



Volume of metal, $V = \frac{m}{\rho} = \frac{20}{5} = 4\text{cm}^3$

Loss in weight, $(V + V')\rho_w g = (20 - 12)\text{g}$

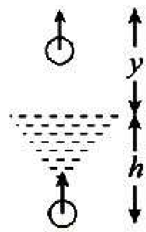
or $V + V' = 8 \qquad (\rho_w = 1\text{g/cm}^3)$

$\therefore \quad V' = 4\ \text{cm}^3.$ ***Ans.***

Illustration 8

A rubber ball of mass m and density ρ is immerged in a liquid of density 3ρ to a depth h and released. To what height will the ball jump above the surface due to buoyant force of liquid on the ball? Neglecting the resistance of liquid and air.

Short-cut solution :



Acceleration, $a = \left[\frac{\rho_\ell - \rho_b}{\rho_b}\right]g = \left[\frac{3\rho - \rho}{\rho}\right]g = 2g$

Velocity of the ball when it comes out of the surface of liquid,

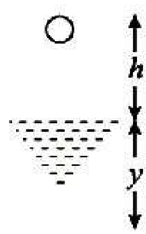
$v = \sqrt{2ah} = \sqrt{2 \times 2gh} = 2\sqrt{gh}.$

$\therefore$ The ball jumps out a height, $y = \frac{v^2}{2g} = \frac{(2\sqrt{gh})^2}{2g} = 2h.$ ***Ans.***

Illustration 9

A ball of RD 0.8 falls into water from a height of h. Find the depth to which the ball will sink. (Neglect viscous forces).

Short-cut solution :

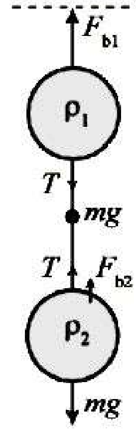


Retardation, $a = \left[\frac{\rho_\ell - \rho_b}{\rho_b}\right]g$

$= \left[\frac{1000 - 800}{800}\right]g = \frac{g}{4}\ \text{m/s}^2$

$\therefore \quad y = \frac{v^2}{2a} = \frac{(\sqrt{2gh})^2}{2 \times \frac{g}{4}} = 4h.$ ***Ans.***

Illustration 10



Two spheres of mass 'm' but of densities ρ_1 and ρ_2 ($\rho_2 > \rho_1$) are connected by a string and the combination is immersed in a liquid (ρ_ℓ). Find the tension in the string.

Short-cut solution :

For the equilibrium

$$T = F_{b1} - mg \quad \ldots(i)$$

and

$$T = mg - F_{b2} \quad \ldots(ii)$$

From above equations, we get

$$T = \frac{F_{b1} - F_{b2}}{2}$$

$$= \frac{V_1\rho_\ell g - V_2\rho_\ell g}{2}$$

$$= \frac{\left[\frac{m}{\rho_1}\rho_\ell g - \frac{m}{\rho_2}\rho_\ell g\right]}{2}$$

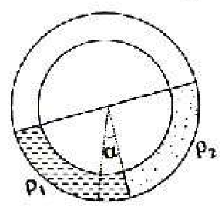
$$= \frac{m\rho_\ell g}{2}\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right].$$

Ans.

Illustration 11

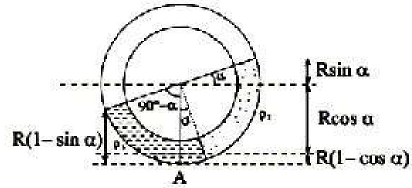
There is a circular tube in vertical plane. Two liquids which do not mix and of densities ρ_1 and ρ_2 are filled in the tube. Each liquid subtends 90° angle at centre. Radius joining their interface makes an angle α with the vertical. Ratio $\frac{\rho_1}{\rho_2}$ is :

[JEE Main 2014]



(a) $\frac{1+\sin\alpha}{1-\sin\alpha}$ *(b)* $\frac{1+\cos\alpha}{1-\cos\alpha}$ *(c)* $\frac{1+\tan\alpha}{1-\tan\alpha}$ *(d)* $\frac{1+\sin\alpha}{1-\cos\alpha}$

Short-cut solution :



In static liquid, net pressure at A,

$$P_A = 0$$

or $\rho_2 g[R\sin\alpha + R\cos\alpha] + \rho_1 gR(1-\cos\alpha)$

$$= \rho_1 gR(1-\sin\alpha)$$

On simplifying, we get

$$\frac{\rho_1}{\rho_2} = \frac{1+\tan\alpha}{1-\tan\alpha}.$$

Ans. (c)

***TOPIC 10.2**: Rate of Flow, Equation of Continuity, Bernoulli's Theorem, Toricelli's Theorem and Efflux Velocity.*

Review of Formulae

1. **Rate of flow or discharge**

$$Q = Av$$

2. **Reynolds number**

$$R_e = \frac{\rho v D}{\eta}.$$

If $R_e \leq 2000$, the flow will be laminar. If $R_e > 3000$, the flow is turbulent. If R_e lies between 2000 and 3000, the flow is unstable.

3. **Equation of continuity :**

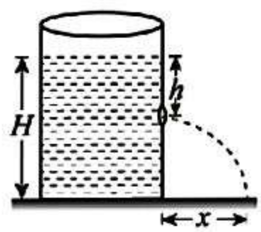
$$Q = Av = \text{constant}$$

4. **Bernoulli's equation :**

(i) $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$

(ii) $\frac{P}{\rho g} + \frac{v^2}{2g} + h = \text{constant}$

5. **Speed of efflux :** The speed of the liquid emerging from a small hole under head h is given by



$$v_e = \sqrt{2gh}$$

$$x = 2\sqrt{h(H-h)}$$

For maximum x, $h = \frac{H}{2}$, and $x_{max} = H$.

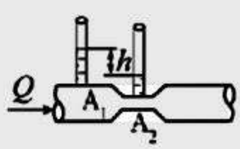
6. **Time of employing a tank :**

$$t = \frac{A\sqrt{2}}{a\sqrt{g}}\left(\sqrt{h_1} - \sqrt{h_2}\right)$$

Tips and Tricks for Shortcut Solutions

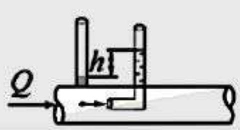
1. Rate of flow of liquid in horizontal venturimeter with simple manometer :

$$Q = A_1 A_2 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}} = A_1 A_2 \frac{\sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}.$$



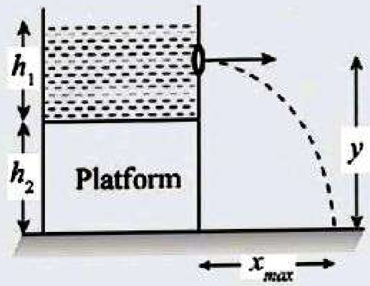
Also $h = \left[\frac{v_2^2 - v_1^2}{2g}\right].$

2. Pitot tube : $Q = Av = A\sqrt{2gh}$



3. (i) **Toricelli's Theorem :**

In a large open container, velocity of efflux under liquid head h, $v_e = \sqrt{2gh}$, and maximum range of liquid is found when height of hole is H/2 from bottom.



If h_2 is the height of platform and h_1 is the height of liquid in the container then,

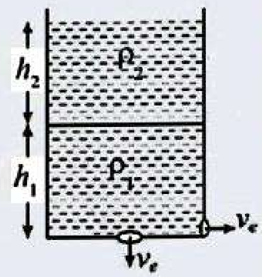
$$y = \frac{h_1 + h_2}{2}.$$

and $\quad x_{max} = \quad (h_1 + h_2).$

(ii) Backward force on the container

$$F = \rho Q v_e = \rho A v_e^2 = 2\rho g h A \qquad (A \rightarrow \text{area of hole})$$

4. Efflux velocity when there are two liquids in the container

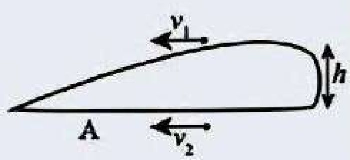


$$\frac{1}{2}\rho_1 v_e^2 = \rho_1 g h_1 + \rho_2 g h_2$$

or $$v_e = \sqrt{\frac{2g}{\rho_1}(\rho_1 h_1 + \rho_2 h_2)}.$$

$$= \sqrt{2g(h_1 + \frac{\rho_2}{\rho_1} h_2)}.$$

5. Lift force on aeroplane wing



$$F = (P_2 - P_1)A = \left[\frac{1}{2}\rho_{air}(v_1^2 - v_2^2) + \rho_{air} g h\right] A$$

$$\simeq \left[\frac{1}{2}\rho_{air}(v_1^2 - v_2^2) A\right].$$

Neglecting gravitational head in comparison to other head.

6. In a close container, if P is the pressure over the liquid then

$$v_e = \sqrt{2gh + \frac{2(P - P_a)}{\rho}}.$$

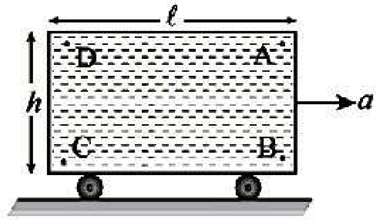
7. If there is vacuum over water in a closed container, then water will come out from the hole, when it is at a depth, $h \geq \frac{Pa}{\rho g} = 10.3m$, from free surface of water.

8. In open container with addition pressure P over the liquid, then

$$v_e = \sqrt{2gh + \frac{2P}{\rho}}.$$

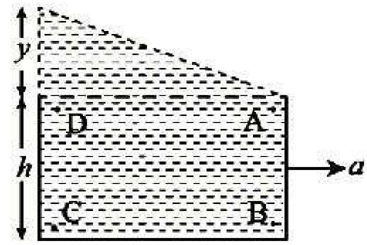
Illustration 12

A rectangular tank is completely filled with a liquid of density ρ. The length and height of the tank are ℓ and h respectively. The tank is given a constant horizontal acceleration a. Determine the gauge pressure at points A, B, C and D.



Short-cut solution :

The effect of acceleration can be realised by adding the head of liquid as shown in figure. If y is the maximum head, then



$$\tan\theta = \frac{a}{g} = \frac{y}{\ell}$$

$$\therefore \quad y = \frac{a\ell}{g}.$$

Therefore pressure :

$$P_A = 0$$

$$P_B = \rho gh$$

$$P_C = \rho g(h+y) = \rho g\left(h + \frac{a\ell}{g}\right) = \rho gh + \rho a\ell$$

$$P_D = \rho gy = \rho g\left(\frac{a\ell}{g}\right) = \rho a\ell.$$ ***Ans.***

Illustration 13

The velocities on lower and upper surface of an aeroplane wing are 10 m/s and 20 m/s respectively. If area of the wing is 10 m^2, find lift force on the aeroplane. Density of air 1.2 kg/m^3.

Short-cut solution :

Using,

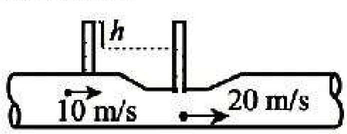
$$F = \frac{1}{2}\rho_{air}(v_1^2 - v_2^2)A$$

$$= \frac{1}{2} \times 1.2(20^2 - 10^2) \times 10$$

$$= 1800 \text{ N.}$$ ***Ans.***

Illustration 14

The water is flowing in a pipe as shown in figure. What is the value of h difference in pressure head from the data given.



Short-cut solution :

$$h = \frac{v_2^2 - v_1^2}{2g} = \frac{20^2 - 10^2}{2 \times 10}$$

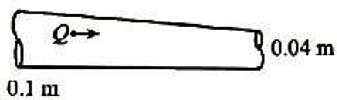
$$= 15 \text{ m.} \qquad \textit{Ans.}$$

Illustration 15

Calculate rate of flow of glycerin of density 1.25×10^3 kg/m^3 through the conical section of a horizontal pipe, if the radii of its ends are 0.1 m and 0.04 m and pressure drop across its length is 10 N/m^2.

Short-cut solution :

$A_1 = \pi r_1^2 = \pi(0.1)^2 = 3.14 \times 10^{-2}$ m^2,
$A_2 = \pi r_2^2 = \pi(0.04)^2 = 5.02 \times 10^{-4}$ m^2



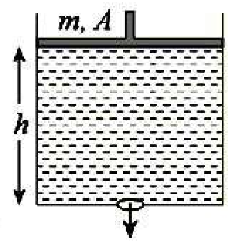
and $P_1 - P_2 = 10$ N/m^2

Using, $$Q = A_1 A_2 \sqrt{\frac{2(P_1 - P_2)}{\rho(A_1^2 - A_2^2)}}$$

On substituting the values we get, $Q = 6.28 \times 10^{-4}$ m^3/s. ***Ans.***

Illustration 16

A cylindrical vessel contains a liquid of density ρ upto a height 'h'. The cylinder is closed by a piston of mass 'm' and area of cross-section A. There is a small hole at the bottom of the vessel. Find the speed 'v' with which the liquid is comes out of the hole.



Short-cut solution :

Using, $$v_e = \sqrt{2gh + \frac{2P}{\rho}}$$

$$= \sqrt{2gh + \frac{2(mg/A)}{\rho}}. \qquad \textit{Ans.}$$

Illustration 17

A cylindrical tank has a hole of 1 cm^2 at its bottom. If the water is allowed to flow into the tank from a tube above it at the rate of 70 cm^3/s, then find the maximum height upto which water can rise in the tank.

Short-cut solution :

As well as height of water in the tank increases, the efflux velocity and hence rate of flow of emerging water also increases. At a certain height h the output become equal to input and the level of water becomes constant.

$$\therefore \quad Q_{in} = Q_{out}$$

$$Q = a\sqrt{2gh}$$

$$h = \frac{Q^2}{2ga^2}$$

$$= \frac{(70\times10^{-6})^2}{2\times9.8\times(1\times10^{-4})^2}$$

$$= 2.5\times10^{-2}\text{ m.}$$ ***Ans.***

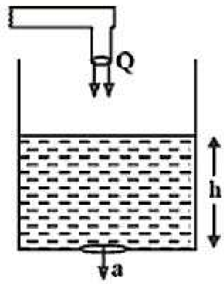


Illustration 18

A liquid is poured into a vessel at rest with the hole in a wall closed by a valve. It is filled by liquid upto height h above the valve. What horizontal acceleration 'a' should the vessel moved, so that liquid does not come out when valve is opened?

Short-cut solution :

The vessel is given an acceleration of such a value so that level of liquid at valve become zero. Let a be the acceleration of the vessel towards right, then

$$\tan\theta = \frac{a_x}{g}$$

or $$\frac{2h}{\ell} = \frac{a}{g}$$

$$\Rightarrow \quad a = \frac{2gh}{\ell}.$$ ***Ans.***

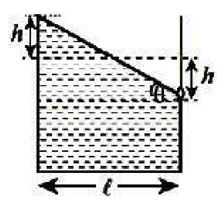


Illustration 19

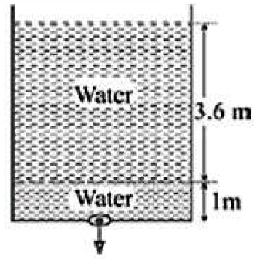
A tank filled with water (density ρ_ω =1000 kg/m^3) and oil of (density ρ_{oil} = 900 kg/m^3). The height of water is 1.00m and of the oil is 4.00m. Find the velocity of efflux through a hole at the bottom of the tank.

Short-cut solution :

Height of water which exerts the same pressure on interface, whatever oil exerts, let it is h.

$$\therefore \quad h\rho_w g = 4\times\rho_{oil}\times g$$

or $h = \dfrac{4 \times 900}{1000} = 3.6\text{m}$



Effective height of water over the hole

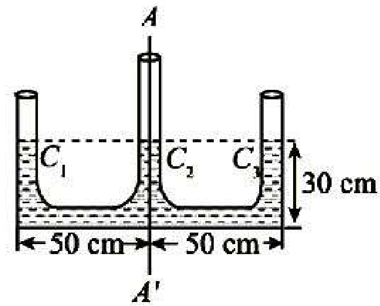
$$H = 1 + 3.6 = 4.6 \text{ m}$$

$$\therefore \quad v_e = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 4.6}$$

$$= 9.5 \text{ m/s.} \qquad \textbf{Ans.}$$

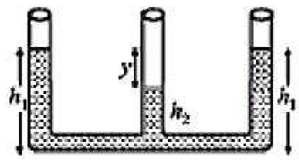
Illustration 20

W–tube system as shown is rotated about an axis AA' at 10 rpm. Determine the levels in columns C_1, C_2 and C_3 in the new position of equilibrium.



Short-cut solution :

The rise in the tubes C_1 or C_3 rises with respect to C_2, is



$$y = \frac{\omega^2 x^2}{2g} = \frac{\left[2\pi\left(\frac{10}{60}\right)\right]^2 \times 0.5^2}{2 \times 9.8}$$

$$= 1.4 \times 10^{-2} \text{ m}$$

If h_1 and h_2 be the heights of the liquid in tubes, then

$$h_1 - h_2 = 1.4 \qquad .....(i)$$

and $$2h_1 + h_2 = 30 + 30 + 30$$

$$= 90 \qquad .....(ii)$$

After solving above equations, we get

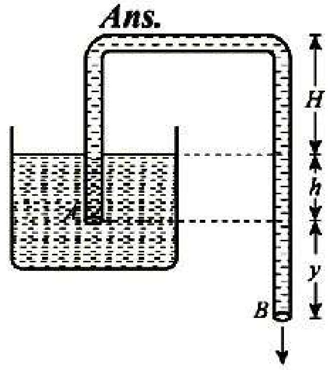
$$h_1 = 30.47 \text{ cm}$$

and $$h_2 = 29.06 \text{ cm.}$$

Ans.

Illustration 21

A syphon tube is used to remove liquid from a container as shown in figure. In order to operate the syphon tube, it must initially be filled with the liquid. Determine the speed of the liquid through the tube.



Short-cut solution :

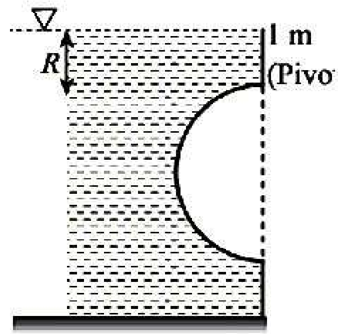
Using Bernoulli's equation between points A and B, we have

$$P_a + \rho gh = P_a + \frac{1}{2}\rho v^2 - \rho gy$$

$$\therefore \quad v = \sqrt{2g(h+y)}. \qquad \textit{Ans.}$$

Illustration 22

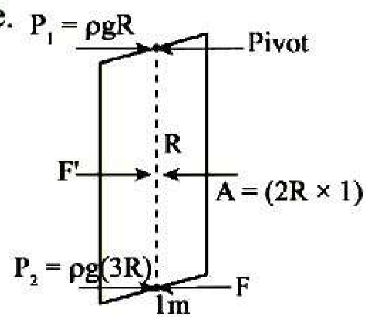
The figure shows a semi-cylindrical massless gate of unit length perpendicular to the plane of the page and is pivoted at the point 'O' holding a stationary liquid of density ρ. A horizontal force F is applied at its lowest position to keep it stationary. The magnitude of force is :



(a) $\frac{3}{2}\rho gR^2$ *(b)* $\frac{9}{2}\rho gR^2$ *(c)* ρgR^2 *(d)* $2\rho gR^2$

Short-cut solution :

The vertical projection of gate is shown in figure. Force exerted by liquid on the gate,



$$F' = \left(\frac{P_1+P_2}{2}\right)A$$

$$= \left[\frac{\rho gR + \rho g(3R)}{2}\right]\times(2R\times 1)$$

$$= 4\rho gR^2$$

The line of action of this force

$$= \frac{2h}{3} \text{ from top} = \frac{2\times 3R}{3} = 2R.$$

So distance of this force from pivot will be R.

Now using, $\Sigma\tau_{pivot} = 0$

$$F'\times R = F\times 2R$$

$$\therefore \quad F = \frac{F'}{2}$$

$$= \frac{4\rho gR^2}{2} = 2\rho gR^2. \qquad \textit{Ans. (d)}$$

TOPIC 10.3: *Surface Tension, Surface Energy, Angle of Contact, Excess Pressure and Capillarity.*

Review of Formulae

1. **Surface tension,**

$$T = \frac{F}{l}$$

or $$F = Tl$$

(i) Surface tension force on a wire of length l, placed on water

$$F = 2Tl$$

(ii) Surface tension force on circular disc of radius r,

$$F = 2\pi rT$$

(iii) Surface tension force on a ring of radius R.

$$F = 2(2\pi R \times T)$$

2. Surface means a thin layer of approximately 10-15 molecular diameters.

3. **Surface energy :** A molecule in the surface has greater potential energy than a molecule which inside the liquid. The extra energy that a surface film has is called the surface energy.

4. **Work done in increasing the area of the surface film,**

$$W = T\Delta A$$

(i) W.d. in breaking a liquid drop

$$W = T[n \times 4\pi r^2 - 4\pi R^2]$$

here $$r = \frac{R}{n^{1/3}}$$

(ii) Work done in blowing a soap bubble from zero to radius R.

$$W = T \times 2(4\pi R^2) = 8\pi TR^2$$

5. **Pressure difference :**

(i) In a liquid drop $P_i - P_0 = \dfrac{2T}{R}$.

(ii) In a soap bubble $P_i - P_0 = \dfrac{4T}{R}$.

(iii) In general, for one free surface

$$P_i \sim P_0 = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

6. **Angle of contact :**
   (i) Angle of contact is the property of the materials in contact.
   (ii) It decreases with increase in temperature.
   (iii) It decreases with the addition of soap and detergent.
   (iv) It increases with the addition of sugar and salt.
   (v) Angle of contact of water with glass is 8°, and for mercury in glass is 140°.

7. **Capillary rise :** If r is the radius of the capillary tube, then

$$h = \frac{2T\cos\theta}{r\rho g}$$

In terms of radius of curvature $R = \frac{r}{\cos\theta}$, $h = \frac{2T}{R\rho g}$

8. In case of square tube of side a,

$$h = \frac{4T\cos\theta}{a\rho g}$$

9. **Capillary tube of insufficient length** :

If h is the free rise of the liquid and l is the length of tube, being $l < h$, then $hR = lR'$ here R' is the radius of curvature of the meniscus of the liquid at the top of the tube.

The liquid will not spillout.

As $l < h$, so $R' > R$.

10. **Apparent angle of contact :** If θ and θ' are the true and apparent angle of contact, then $\cos\theta' = \frac{l}{h}\cos\theta$

11. Force required to pull the plates apart having some liquid between them :

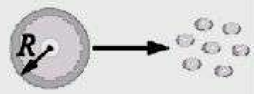
$$F = (P_0 - P_i)A = \frac{2TA}{d}$$

Here d is the separation between the plates and A is the area of each plate.

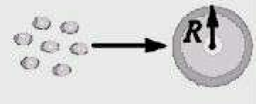
TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. Work done in breaking a liquid drop into 'n' identical drops.

$$W = 4\pi R^2 T\left[n^{1/3} - 1\right]$$



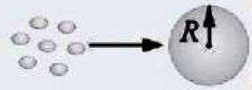
Ratio of their surface energies; $\frac{U_{bigger}}{U_{small}} = \frac{4\pi R^2 T}{4\pi r^2 nT} = n^{\frac{1}{3}}$.



2. Work done in merging 'n' identical drops.

$$W = 4\pi r^2 T\left[n - n^{\frac{2}{3}}\right]$$

$$= 4\pi R^2 T\left[n^{\frac{1}{3}} - 1\right]$$



If V is the total volume of all the drops, then

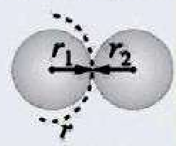
$$W = \frac{3VT}{R}\left(n^{\frac{1}{3}} - 1\right) = 3VT\left[\frac{1}{r} - \frac{1}{R}\right].$$

3. When soap bubble of radii $r_1, r_2, ..., r_n$ are coalesced isothermally to form a single bubble, its radius

$$r^2 = r_1^2 + r_2^2 + ... + r_n^2$$

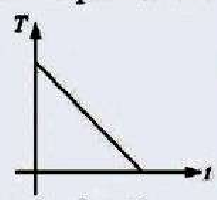
4. When two soap bubbles of radii r_1 and r_2 are in contact, the radius of curvature of contact point

$$r = \left(\frac{r_2 - r_1}{r_1 r_2}\right)$$



5. If a capillary tube is held in satellite, the rise of liquid will equal to the length of the tube (what so ever the length).

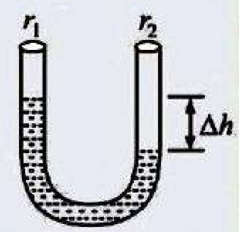
6. If T_0 is the surface tension of liquid at 0°C, then at any temperature



$$T = T_0(1 - \alpha\,\Delta t);$$

Here $\alpha \rightarrow$ temperature coefficient of surface tension and Δt is the change in temperature.

7. The difference of heights in a liquid in U-tube



$$\Delta h = \frac{2T}{\rho g}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

8. Mass of the rise of liquid in capillary tube is, $m \propto r$.

Illustration 23

When a wire of length ℓ and cross-sectional radius r ($r << \ell$) is kept floating on surface of a liquid. Find maximum radius of wire so that it may not sink. Surface tension of liquid is T.

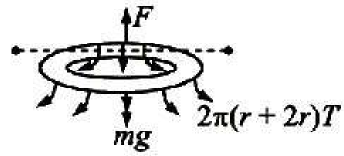
Short-cut solution :

$$Mg = 2T\ell$$

or $$\rho(\pi r^2 \ell)g = 2T\ell \quad \Rightarrow \quad r = \sqrt{\frac{2T}{\pi\rho g}}.$$ ***Ans.***

Illustration 24

An annular metal ring of inner radius r and outer radius 2r and of mass 'm' is floating on the surface of a liquid of ST is T. Calculate the force required to lift it from liquid surface.



Short-cut solution :

$F = mg + 2\pi(r + 2r)T$ ***Ans.***

Illustration 25

A large number of liquid drops each of radius 'r' merge to form a single spherical drop of radius R. If the energy released in the process is converted into KE of the big drop formed. Find the speed of the big drop (ρ is the density of the liquid).

Short-cut solution :

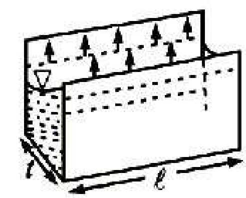
Using, Energy released $= \frac{1}{2}Mv^2$

or $3VT\left[\frac{1}{r}-\frac{1}{R}\right] = \frac{1}{2}(\rho V)v^2$

$\therefore \quad v = \sqrt{\frac{6T}{\rho}\left(\frac{1}{r}-\frac{1}{R}\right)}$. ***Ans.***

Illustration 26

Two rectangular plates are placed at a small gap 't'. Find rise of liquid between them. Surface tension of liquid is T, and angle contact 0°.



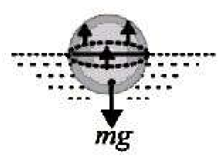
Short-cut solution :

$$T \times 2\ell = mg$$
$$= \rho V g = \rho(\ell t h)g$$

$\therefore \quad h = \frac{2T}{\rho t g}$. ***Ans.***

Illustration 27

A drop of liquid of density ρ_1 is floating with half immerged in a liquid of density ρ_2. If T is the surface tension of the liquid. Find radius of the drop.
[JEE Main 2020]



Short-cut solution :

$$F_{ST} + F_b = \text{mg}$$
$$2\pi r T + \frac{2}{3}\pi r^3 \rho_2 g = \frac{4}{3}\pi r^3 \rho_1 g$$

$\therefore \quad r = \sqrt{\frac{3T}{(2\rho_1 - \rho_2)g}}$. ***Ans.***

***TOPIC 10.4**: Viscosity, Stoke's Law, Terminal Velocity, Poiseuille's Equation.*

Review of Formulae

1. **Viscosity** is the resistance force between the adjacent layers of fluid.
2. Viscosity of liquids decreases with increase in temperature and viscosity of gases increases with increase in temperature.
3. **Newton's law of viscosity :** For liquid layer of area A with velocity gradient $\frac{dv}{dy}$ with the adjacent layer, the viscous force

$$F_v = \eta A\left(\frac{dv}{dy}\right)$$

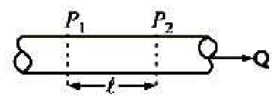
SI unit of η is $N - s/m^2$.

4. **Stoke's law (experimental law) :** Viscous force on a spherical body moving in an infinite liquid is given by $Fv = 6\pi\eta rv$
5. Viscous force on a spherical body by stoke's is little different from that obtained by Newton's law.
6. **Terminal velocity :** A constant velocity in a viscous fluid is given by

$$v_t = \frac{2}{9}r^2\frac{(\sigma-\rho)g}{\eta}$$

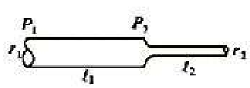
7. **Poiseuille's equation :** Rate of flow of a viscous liquid in a circular pipe is given by

$$Q = \frac{P_1 - P_2}{R}$$



Here R is the resistance of pipe, which is,

$$R = \frac{8\eta l}{\pi r^4}$$



$$\therefore \qquad Q = \frac{\pi(P_1 - P_2)r^4}{8\eta l}$$

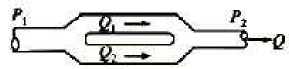
8. **Pipes in series :**

Total resistance $\quad R = R_1 + R_2 = \left(\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}\right)$

and rate of flow, $\quad Q = \left[\frac{P_1 - P_2}{R}\right]$

9. Pipes in parallel :

Total resistance, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$



$$Q = Q_1 + Q_2 = \left(\frac{P_1 - P_2}{R}\right)$$

Illustration 28

Two identical drops of water are falling through air with a constant speed 'v' each. If the drops coalesce to form a single drop, what is the new terminal velocity?

Short-cut solution :

$$2\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

or $$R = 2^{\frac{1}{3}} r$$

Using, $$v = \frac{2}{9} r^2 \frac{(\rho_w - \rho_{air})g}{\eta}$$

We have $$\frac{v}{v'} = \frac{r^2}{R^2}$$

$$= \frac{r^2}{\left(2^{\frac{1}{3}} r\right)^2}$$

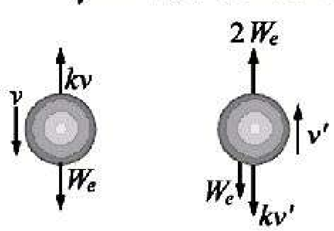
$\therefore$ $$v' = 2^{\frac{2}{3}} v.$$ **Ans.**

Illustration 29

A small steel ball falls through a syrup at a constant speed of 1 m/s. If the steel ball is pulled upward with a force equal to twice its effective weight, how fast will it move upward?

Short-cut solution :

Viscous force, $F_v = kv$, and effective weight W_e, then



Initially, $W_e = kv$ ...(i)

and finally $2W_e = W_e + kv'$

or $W_e = kv'$ ...(ii)

From above equations, we have

$$v' = v = 1 \text{ m/s.}$$ ***Ans.***

Illustration 30

A 16 cm³ of water flows per second through a capillary tube of radius r cm and length ℓ cm, when connected to a pressure head of h cm of water. If a tube of the same length and radius $\frac{r}{2}$ is connected to the same pressure head. Find the mass of water flowing per minute through the tube.

Short-cut solution :

Using,
$$Q = \frac{\pi p r^4}{8\eta \ell}$$

$$\therefore \quad \frac{Q'}{Q} = \frac{\left(\frac{r}{2}\right)^4}{r^4}$$

$$Q' = \left(\frac{1}{16}\right)Q = \frac{1}{16}\times 16 = 1 \text{ cm}^3\text{/s.}$$

$$\text{Mass} = \rho Q' = 1 \times 1 = 1 \text{ g/s}$$
$$= 60 \text{ g/min.}$$ ***Ans.***

Illustration 31

Three capillary tubes of same radius but of lengths ℓ_1, ℓ_2 and ℓ_3 are fitted horizontally to the bottom of a long vessel containing a liquid at constant pressure and flowing through these. What is the length of a single tube which can replace the three capillaries?

Short-cut solution :

$$Q = Q_1 + Q_2 + Q_3$$

or
$$\frac{\pi P r^4}{8\eta \ell} = \frac{\pi P r^4}{8\eta}\left[\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3}\right]$$

or
$$\frac{1}{\ell} = \frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3}.$$ ***Ans.***

Video Solution

Q. A wire forming a loop is dipped in to soap solution and taken out so that a film of soap solution is formed. A loop of ℓ long thread is gently put on the film and the film is pricked with a needle inside the loop. The thread loop takes the shape of a circle. Find the tension in the thread. Surface tension of soap solution is T.

To see the video solution, scan the QR code:

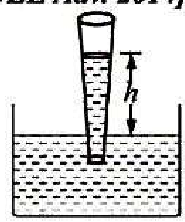
OR Visit https://www.youtube.com/watch?v=30_-wQXtgq0

Illustration 32

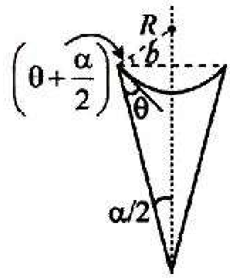
A glass capillary tube is of the shape of truncated cone with an apex angle α so that its two ends have cross-section of different radii. When dipped in water vertically, water rises in it to a height h, where the radius of its cross-section is b. If the surface tension of water is T, its density ρ, and its contract angle with glass is θ, the value of h will be (g is the acceleration due to gravity) [JEE Adv. 2014]

(a) $\frac{2\pi}{b\rho g}\cos(\theta-\alpha)$ (b) $\frac{2\pi}{b\rho g}\cos(\theta+\alpha)$

(c) $\frac{2\pi}{b\rho g}\cos\left(\theta-\frac{\alpha}{2}\right)$ (d) $\frac{2\pi}{b\rho g}\cos\left(\theta+\frac{\alpha}{2}\right)$



Short-cut solution :



From geometry $R = \frac{b}{\cos\left(\theta+\frac{\alpha}{2}\right)}$

$$\Delta P = \frac{2T}{R} = \rho g h$$

or $\frac{2T}{\left[b/\cos\left(\theta+\frac{\alpha}{2}\right)\right]} = \rho g h \Rightarrow h = \frac{2T}{b\rho g\cos\left(\theta+\frac{\alpha}{2}\right)}$ *Ans. (d)*

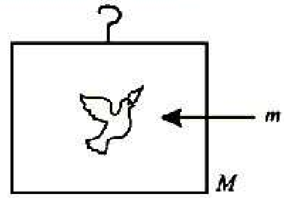
Concept Booster Exercise

FLUID MECHANICS

1. There is a liquid of density ρ in a container upto height h. Now a block of mass M and base area 'a' is placed on the liquid, on result of which the level of liquid in the container becomes h'. If A is the area of base of the container, then pressure at the bottom of the container.

(a) $\rho g h$ (b) $\rho g h'$ (c) $\rho g h + \frac{Mg}{a}$ (d) $\rho g h' + \frac{Mg}{A}$

2. A bird of mass 'm' is inside a close cage of mass M, starts flying vertically up with constant acceleration a. The reading of cage is:

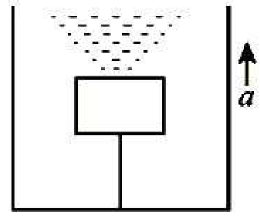


(a) $(m + M)g$ (b) Mg
(c) $Mg + m(g + a)$ (d) $(m + M)a$

3. A solid body is found floating in water with $\left(\frac{\alpha}{\beta}\right)^{th}$ of its volume submerged. The same solid body is found floating in a liquid with $\left(\frac{\alpha}{\beta}\right)^{th}$ of its volume above the liquid surface. The specific gravity of the liquid is:

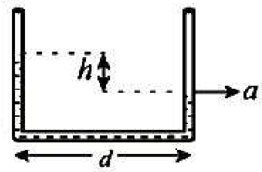
(a) $\frac{\beta-\alpha}{\alpha}$ (b) $\frac{\alpha-\beta}{\beta}$ (c) $\frac{\alpha}{\beta-\alpha}$ (d) $\frac{\beta}{\alpha-\beta}$

4. A tank accelerates upwards with acceleration 1 m/s^2 contain water. A block of mass 1 kg and density 0.8 g/cm^3 is held stationary inside the tank with the help of the string as shown in figure. The tension in the string is : (Given, density of water = 1000 kg/m^3). **Numeric/Integer**



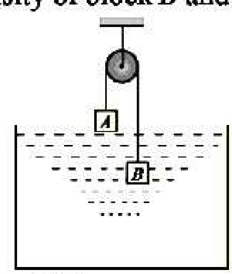
(a) 2.2 N (b) 2.75 N (c) 3 N (d) 2.4 N

5. Figure shows a U-tube of uniform cross-sectional area A accelerated with acceleration 'a' as shown. If 'd' is the separation between the limbs, then the difference in the levels of the liquid in the U-tube is :



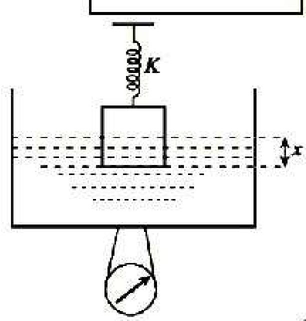
(a) $\frac{ad}{g}$ (b) $\frac{g}{ad}$ (c) adg (d) $\frac{adg}{2}$

6. In the arrangement shown in figure $\frac{m_A}{m_B} = \frac{2}{3}$ and the ratio of density of block B and the liquid is 2 : 1. The system is released from rest. Then



(a) block B will oscillate but not simple harmonically
(b) block B will oscillate simple harmonically
(c) the system will remain in equilibrium
(d) None of these

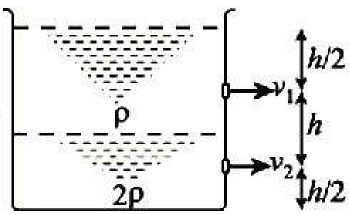
7. A vessel filled with water is kept on a weighing pan and the scale adjusted to zero. A block of mass 10 kg is suspended by a massless spring of force constant $k = 100$ N/m. This block is submerged inside into water in the vessel such that elongation in spring is $x = 10$ cm. The reading of the scale is : **Numeric/Integer**



(a) 100 N (b) 110 N
(c) 90 N (d) 120 N

8. A man in a boat which is floating in water is having some wooden planks inside the boat. He unloaded the planks into water. The level of water in the pond
(a) increases (b) decreases
(c) remains same (d) none of these

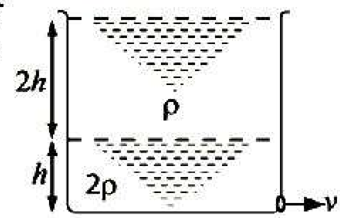
9. Equal volumes of two immiscible liquids of densities ρ and 2ρ are filled in a vessel as shown in figure. Two small holes are punched at depth $\frac{h}{2}$ and $\frac{3h}{2}$ from surface of lighter liquid. If v_1 and v_2 are the velocities of efflux at these two holes, then $\frac{v_1}{v_2}$ is



Numeric/Integer

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{\sqrt{2}}$

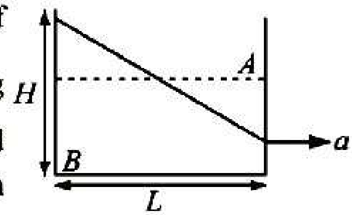
10. The velocity of the liquid coming out of a small hole of a vessel containing two different liquids of densities 2ρ and ρ as shown in figure is:



(a) $\sqrt{gh}$ (b) $2\sqrt{gh}$
(c) $2\sqrt{2gh}$ (d) $\sqrt{6gh}$

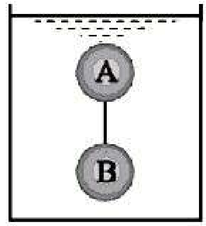
11. The Bhagirathi and the Alaknanda merge at Deoprayag to form the Ganga with their speeds in the ratio 1 : 1.5. The cross-sectional areas of the Bhagirathi, the Alaknanda and the Ganga are in the ratio 1 : 2 : 3. Assuming streamline flow, the ratio of the speed of Ganga to that of the Alaknanda is : *[KVPY-2017]*
(a) 7 : 9 (b) 4 : 3 (c) 8 : 9 (d) 5 : 3

12. A vessel of height H and length L contains a liquid of density ρ upto height $\frac{H}{2}$. The vessel starts accelerating horizontally with acceleration 'a' towards right. Find pressure difference between points B and A as shown in figure (assume liquid does not spill).



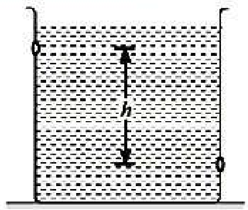
(a) $\frac{\rho}{2}(gH + aL)$ (b) $\frac{\rho}{2}(\rho H - aL)$
(c) $2\rho(gH - aL)$ (d) $\frac{3\rho}{2}(gH + aL)$

13. The solid spheres A and B of equal volumes but different densities ρ_A and ρ_B are connected by a string. They are fully immersed in a fluid of density ρ_F. They get arranged into equilibrium state as shown with tension in the string. The arrangement is possible only if:



(a) $\rho_A < \rho_F$ (b) $\rho_B > \rho_F$
(c) $\rho_A > \rho_F$ (d) $\rho_A + \rho_B = 2\rho_F$

14. There are two identical small holes of area of cross-section a on the opposite sides of a tank containing a liquid of density ρ. The difference in height between the holes is h. Tank is resting on a smooth horizontal surface. Horizontal force which will has to be applied on the tank to keep it in equilibrium is



(a) $gh\rho a$ (b) $\frac{2gh}{\rho a}$ (c) $2\rho agh$ (d) $\frac{\rho gh}{a}$

SURFACE TENSION AND VISCOSITY

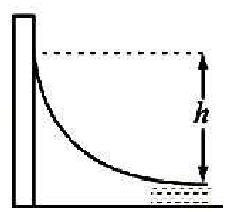
15. The lower end of a capillary tube of radius r is placed vertically in water, then with the rise of water in the capillary heat evolved is: (h is capillary rise and J mechanical equivalent of heat)

(a) $\frac{\pi r^2 h^2 \rho g}{2J}$ (b) $\frac{\pi r^2 h^2 \rho g}{J}$ (c) $-\frac{\pi r^2 h^2 \rho g}{2J}$ (d) $-\frac{\pi r^2 h^2 \rho g}{J}$

16. When a solid ball of volume V is falling through a viscous liquid, a viscous force F acts on it. If another ball of volume $2V$ of the same material is falling through the same liquid then the viscous force experienced by it will be: (both fall with terminal velocities)

(a) F (b) $\frac{F}{2}$ (c) $2F$ (d) $\frac{F}{4}$

17. Water in a clean aquarium forms a meniscus, as shown in figure. The difference in height h between the centre and the edge of the meniscus is: (The surface tension of water is T and density of liquid is ρ)



(a) $\sqrt{\frac{2T}{\rho g}}$ (b) $\sqrt{\frac{T}{\rho g}}$ (c) $\sqrt{\frac{4T}{\rho g}}$ (d) none of these

18. Two soap bubbles of radii R_1 and R_2 are kept in vacuum at constant temperature, the ratio of masses of air inside then is:

(a) $\frac{R_1}{R_2}$ (b) $\frac{R_1^3}{R_2^3}$ (c) $\frac{R_1^2}{R_2^2}$ (d) $\frac{R_2^3}{R_1^3}$

19. A liquid is filled into a semi elliptical cross-section with a as semi-major axis and b as semi-minor axis. The ratio of surface tension forces on the curved part and plane part of the tube in vertical position will be:

(a) $\frac{\pi(a+b)}{4b}$ (b) $\frac{2\pi a}{b}$ (c) $\frac{\pi a}{4b}$ (d) $\frac{\pi(a-b)}{4b}$

20. The work done in blowing a soap bubble of volume 'V' is W. The work done in blowing a soap bubble of volume '$2V$' is

(a) W (b) $2W$ (c) $2^{\frac{2}{3}}W$ (d) $3^{\frac{2}{3}}W$

Solutions

1. **(b)** The height of liquid above the bottom of container becomes h', so

$$P = \rho g h'.$$ ***Ans.***

2. **(c)** When bird flies up with constant acceleration, so it need an upward force, $F = m(g + a)$, whose reaction will transfer at the bottom of the case through air.

So $R = Mg + m(g + a)$. ***Ans.***

3. **(c)**

$$\frac{\alpha}{\beta} = \frac{\rho_b}{\rho_w} \text{ and } 1 - \frac{\alpha}{\beta} = \frac{\rho_b}{\rho_\ell}$$

On solving above equations, we get

$$\frac{\rho_\ell}{\rho_w} = \frac{\alpha}{\beta - \alpha}.$$ ***Ans.***

4. **(b)** Tension,

$$T = m(g+a)\left(\frac{\rho}{\rho_b} - 1\right)$$

$$= 1(10+1)\left(\frac{1}{0.8} - 1\right)$$

$$= 2.75 \text{ N}$$ ***Ans.***

5. **(a)** Using,

$$\tan\theta = \frac{a}{g}$$

or

$$\frac{h}{d} = \frac{a}{g}$$

∴

$$h = \frac{ad}{g}.$$ ***Ans.***

6. **(a)** When block inside liquid, its acceleration [$m_A = 2$, $m_B = 3$] in upward,

$$a_1 = \frac{(m_A g + F_b) - m_B g}{m_A + m_B} = \frac{2g + \frac{3g}{2} - 3g}{2+3} = \frac{g}{10} \text{ m/s}^2$$

When block is outside liquid

$$a_2 = \frac{(3-2)g}{2+3} = \frac{g}{5} \text{ m/s}^2.$$

So its motion is up and down periodically. ***Ans.***

7. **(c)** The reading,

$$R = mg - kx$$

$$= 10 \times 10 - 100 \times 0.1$$

$$= 90 \text{ N}$$ ***Ans.***

8. **(c)** If M is the mass of the boat + man and m is the mass of the planks, then

$$V\rho_{\omega g} = (M+m)g$$

or

$$V = \frac{(M+m)}{\rho_\omega}.$$

When planks are unloaded into water,

$$V' = \text{vol. displaced by (boat + man)} + \text{vol. displaced by plank}$$

$$= \frac{M}{\rho_\omega} + \frac{m}{\rho_\omega}.$$

So $V' = V$. ***Ans.***

9. **(d)** Using Bernoulli's equation

$$P_0 + \rho g\frac{h}{2} = P_0 + \frac{1}{2}\rho v_1^2 \Rightarrow v_1 = \sqrt{gh}$$

and $$P_0 + \rho gh + (2\rho)g\frac{h}{2} = P_0 + \frac{1}{2}(2\rho)v_2^2 \Rightarrow v_2 = \sqrt{2gh}$$

$$\therefore \frac{v_1}{v_2} = \frac{1}{\sqrt{2}}.$$ ***Ans.***

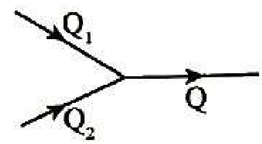
10. **(b)** $$P_0 + \rho g(2h) + (2\rho)gh = \frac{1}{2}(2\rho)v^2 + P_0$$

$$\therefore v = 2\sqrt{gh}.$$ ***Ans.***

11. **(c)** By equation of continuity

Area of Bhagirathi = A ; Area of Alaknanda = 2A

Area of Ganga = 3A

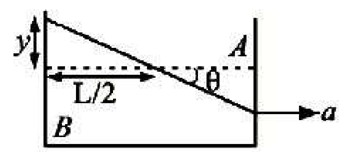


$Q_1 + Q_2 = Q$

$1 \times 1 + 2 \times 1.5 = 3v$

or $v = \frac{4}{3}$ $\therefore \frac{v_{ganga}}{v_{alakhnanda}} = \frac{4/3}{1.5} = \frac{8}{9}$. ***Ans.***

12. **(a)** $$P_B = P_A = \rho g\left(\frac{H}{2} + y\right)$$

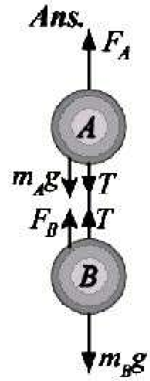


Using, $$\tan\theta = \frac{y}{\frac{L}{2}} = \frac{a}{g}$$

$$\therefore y = \frac{aL}{2g}.$$

Now $$P_B - P_A = \rho g\left(\frac{H}{2} + \frac{\rho L}{2g}\right) = \frac{\rho}{2}(gH + aL).$$ ***Ans.***

13. **(a, b)** For the whole system, $\vec{F}_{net} = 0$



or $$F_A + F_B = (m_A + m_B)g \quad \text{...(i)}$$

As volumes of both one equal, so

$$F_A = F_B = V\rho_F g$$

$$\therefore (2V)\rho_F g = V(\rho_A + \rho_B)g$$

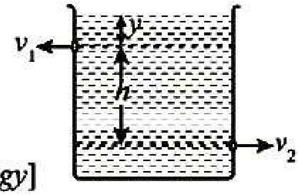
or $$2\rho_F = \rho_A + \rho_B$$

For the system like this

$$\rho_B > \rho_F$$

$\therefore$ $\rho_A < \rho_F$. **Ans.**

14. (c) Force $F = \rho Q v = \rho a v^2$



$= \rho a \times 2gh$

Thus net force $= F_2 - F_1$

$= \rho a v_2^2 - \rho a v_1^2$

$= \rho a(v_2^2 - v_1^2) = \rho a[2g(y + h) - 2gy]$

$= \rho a \times 2gh$. **Ans.**

15. (a) $Q = \dfrac{mgh}{2J} = \dfrac{(\pi r^2 h)\rho h}{2J} = \dfrac{\pi r^2 h^2 \rho g}{2J}$. **Ans.**

16. (c) We know, $F_v = 6\pi\eta r v$

$\therefore$ and $v \propto r^2$ $\dfrac{F_1}{F_2} = \dfrac{r_1^3}{r_2^3}$

$= \dfrac{V}{2V}$

or $F_2 = 2F$. **Ans.**

17. (a) $P_{av} \times (\ell h) = T\ell$

or $\dfrac{\rho g h}{2} \times \ell h = T\ell$

$\therefore$ $h = \sqrt{\dfrac{2T}{\rho g}}$. **Ans.**

18. (c) Using $PV = nRT = \dfrac{m}{M} RT$

$\therefore$ $\dfrac{m_1}{m_2} = \dfrac{P_1 V_1}{P_2 V_2} = \dfrac{\left(\dfrac{4T}{R_1}\right) \times \dfrac{4}{3}\pi R_1^3}{\left(\dfrac{4T}{R_2}\right) \times \dfrac{4}{3}\pi R_2^3}$

$= \dfrac{R_1^2}{R_2^2}$. **Ans.**

19. (a) $\dfrac{F_1}{F_2} = \dfrac{T \times \pi \dfrac{(a+b)}{2}}{T \times 2b} = \dfrac{\pi(a+b)}{4b}$. **Ans.**

20. (c) $\dfrac{W_1}{W_2} = \dfrac{8\pi r_1^2 T}{8\pi r_2^2 T} = \dfrac{r_1^2}{r_2^2}$

As $V = \dfrac{4}{3}\pi r^3$, $\therefore$ $\left(\dfrac{r_1}{r_2}\right)^2 = \left(\dfrac{V}{2V}\right)^{\frac{2}{3}}$. **Ans.**

Thermal Properties of Matter 11

TOPIC 11.1: *Temperature Scales, Thermal Expansion of Solids, Liquids and Gases.*

Review of Formulae

1. **Thermometer :** If X is the property which varies linearly, then temperature

$$t = \left(\frac{X_t - X_0}{X_{100} - X_0}\right) \times 100 \text{ degree}$$

Here property X may be length of mercury column, resistance of metal etc.

2. **Temperature scales :**

$$\frac{C}{5} = \frac{F-32}{9} = \frac{K-273.15}{5}$$

1 div of C = 1.8 div of F = 1 div of K.

3. Triple point of water is 273.16 K at 4 mm of mercury.

4. Absolute temperature $T = T_{tr}\dfrac{P}{P_{tr}}$

5. **Radiation pyrometer :** It is based on Stefan's law.

$$T = \left[\frac{E}{\sigma}\right]^{\frac{1}{4}} \text{kelvin}$$

6. **Expansion of solids :**

(i) Coefficient of linear expansion

$$\alpha = \frac{\Delta L}{L_0 \Delta t}$$

(ii) Coefficient of superficial expansion

$$\beta = \frac{\Delta A}{A_0 \Delta t}$$

(iii) Coefficient of volume expansion

$$\gamma = \frac{\Delta V}{V_0 \Delta t}$$

(iv) $\beta \simeq 2\alpha;\ \gamma \simeq 3\alpha$

7. Density of material at any temperature

$$\rho_t = \frac{\rho_0}{(1+\gamma t)}$$

8. If ρ_1 and ρ_2 are the densities at t_1 and t_2 respectively, then coefficient of volume expansion

$$\gamma = \frac{\rho_1 - \rho_2}{\rho_2(t_2 - t_1)}$$

9. **Expansion of liquids :** If γ_a and γ_g are the apparent coefficient of expansion of liquid and volume coefficient of expansion of container, then

$$\gamma_r = \gamma_a + \gamma_g$$

(i) $$\gamma_a = \frac{V_t - V_0}{V_0 t}$$

(ii) $$\gamma_a = \frac{M_0 - M}{Mt}$$

10. **Expansion of gases :**

$$\gamma_v = \frac{1}{273} / ^\circ C$$

$$\gamma_p = \frac{1}{273} / ^\circ C$$

11. Fractional change in M.I. of the rod due to small change in temperature ΔT

$$\frac{\Delta I}{I} = 2\alpha\Delta T.$$

For rotating rod, $\frac{\Delta\omega}{\omega} = -\frac{\Delta I}{I} = -2\alpha\Delta T.$

TIPS & TRICKS Tips and Tricks for Shortcut Solutions

1. If X is any temperature scale (may be faulty), then

$$\frac{X - L.F.P.}{U.F.P. - L.F.P.} = \frac{C-0}{100-0} = \frac{F-32}{212-32} = \frac{K-273}{373-273}.$$

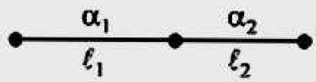
Also $\frac{\Delta C}{5} = \frac{\Delta F}{9} = \frac{\Delta K}{5}$, OR $\frac{\Delta F}{\Delta C} = \frac{5}{9}$ and $\frac{\Delta K}{\Delta C} = 1.$

2. The temperature at which celcius and Fahrenheit read same is –40. The temperature at which Kelvin and Fahrenheit read same is 574.6.
3. The coefficient of volume expansion in differential form is defined as:

$$\gamma = \frac{1}{V}\left(\frac{dV}{dT}\right).$$

4. If two rods are connected in series, then equivalent coefficient linear expansion

$$\alpha = \left[\frac{\ell_1\alpha_1 + \ell_2\alpha_2}{\ell_1 + \ell_2}\right]$$



5. Increase in circumference of a ring $\Delta\ell = \ell\,\alpha\,\Delta T = (2\pi R)\,\alpha\,\Delta T$.
6. For anisotropic solids, α_1, α_2 and α_3 are the coefficients of linear expansions along x, y and z axes respectively, then coefficient of volume expansion

$$\gamma = \alpha_1 + \alpha_2 + \alpha_3.$$

7. The difference of length of two rods remains same at all temperature, if $\ell_1\alpha_1 = \ell_2\alpha_2$.
8. Volume of liquid overflows from the vessel

$$\Delta V = V_0(\gamma_\ell - 3\alpha)\Delta T.$$

If $\gamma_\ell = 3\alpha$, $\Delta V = 0 \Rightarrow$ no overflows.

9. Apparent weight of body in a liquid at any temperature

$$W_t = W - F_0[1 - (\gamma_\ell - \gamma_s)t];\ F_0 = V_0\rho_0 g.$$

The apparent weight of body in liquid will increase with increase in temperature.

10. The change in time period of a pendulum of metal wire

$$\Delta T = \left(\frac{\alpha t}{2}\right)T_0.$$

11. For bimetallic strip, mean radius

$$R = \frac{t}{(\alpha_2 - \alpha_1)\Delta T}.$$

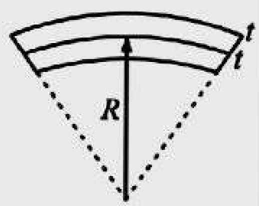


Illustration 1

An accurate celcius thermometer and a faulty Fahrenheit thermometer register 60° and 141.5° respectively when placed in the same constant temperature enclosure. What is the error in the Fahrenheit thermometer?

Short-cut solution :

Using, $F = \frac{9}{5}C + 32$

$= \frac{9}{5} \times 60 + 32 = 140°\text{F}$

Error $= 141.5 - 140 = 1.5°\text{F}$ ***Ans.***

Illustration 2

A faulty thermometer reads freezing point and boiling point of water as 5°C and 95°C respectively. What is the correct value of temperature as it reads 60°C on faulty thermometer?

Short-cut solution :

If X is the value of temperature on faulty thermometer, then

$$\frac{X - F.P.}{B.P. - F.P.} = \frac{C - 0}{100 - 0}$$

or

$$\frac{60-5}{95-5} = \frac{C}{100}$$

After solving, we get

$$C = 61.11\ °C. \qquad \textit{Ans.}$$

Illustration 3

What length of brass and iron at 0°C must be used, if difference between their lengths is always x. The coefficient of linear expansions are α_1 and α_2 respectively.

Short-cut solution :

Using, $\ell_2 - \ell_1 = x$ and $\ell_1\alpha_1 t = \ell_2\alpha_2 t$

On solving, we get $\ell_1 = \frac{x\alpha_2}{\alpha_1 - \alpha_2}$, and $\ell_2 = \frac{x\alpha_1}{\alpha_1 - \alpha_2}$. *Ans.*

Illustration 4

A blacksmith fixes iron ring on the rim of the wooden wheel of bullock cart. The diameter of the rim and the iron ring are 5.24 m and 5.23 m respectively at 27°C. Find the temperature to which the ring should be heated so as to fit the rim on the wheel. ($\alpha_{iron} = 1.20 \times 10^{-5}/°C$)

Short-cut solution :

Using, diameter $D_t = D_0[1 + \alpha(t_2 - t_1)]$

or $5.24 = 5.23[1 + 1.20 \times 10^{-5}(t_2 - 27)]$

$\therefore$ $t_2 = 218°C$ *Ans.*

Illustration 5

A clock with a metallic pendulum is 5 second fast each day at a temperature of 15°C and 10 second slow each day at a temperature of 30°C. Find coefficient of linear expansion of the metal.

Short-cut solution :

Using, $\Delta T = \left(\frac{\alpha \Delta t}{2}\right) T$

or $5 = \frac{\alpha}{2}(T_0 - 15) \times 86400$ ...(i)

and $10 = \frac{\alpha}{2}(30 - T_0) \times 86400$ ...(ii)

On solving, we get $\alpha = 2.31 \times 10^{-5}/°C$. ***Ans.***

Illustration 6

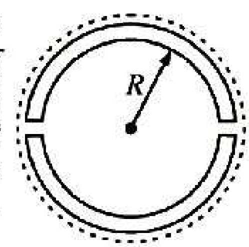
A bimetallic strip having each strip of thickness 2cm consists of zinc and silver rivetted together. The approximate radius of curvature of the strip when heated through 50°C will be: ($\alpha_{zinc} = 32 \times 10^{-6}/°C$, $\alpha_{silver} = 19 \times 10^{-6}/°C$)

Short-cut solution :

Using, $R = \frac{t}{(\alpha_2 - \alpha_1)\Delta T}$

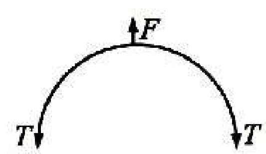
$= \frac{2 \times 10^{-2}}{(32-19) \times 10^{-6} \times 50} = 30.77$ m. ***Ans.***

Illustration 7



A wooden wheel of radius R is made of two semi-circular parts (see figure). The two parts held together by a ring made of metal strip of cross-sectional area A and length L. L is slightly less than 2πR. To fit the ring on the wheel, it is heated so that its temperature rises by ΔT and is just steps over the wheel. As it cools down to surrounding temperature. It presses the semi-circular parts together. If the coefficient of linear expansion of the metal is α and its Young's modulus is Y, then the force that one part of the wheel applies on the other part is: ***[JEE Main 2012]***

Short-cut solution :



Using, $f_{th} = Y\alpha(\Delta T)$

$\therefore$ Force $T = f_{th} A = YA\alpha(\Delta T)$

The required force, $F = 2T = 2YA\alpha(\Delta T)$. ***Ans.***

Illustration 8

A long cylindrical metal vessel, having a linear coefficient (α), is filled with a liquid upto a certain level. On heating, it is found that the level of liquid in the cylinder remains the same. What is the volume coefficient of expansion of the liquid?

Short-cut solution :

$$\Delta V_{vessel} = \Delta V_{liquid} \quad ...(i)$$

As level of liquid is const, so

$$\Delta V_{vessel} = (\Delta A)\ell$$

Now from equation (i)

$$A\beta(\Delta T)\ell = (A\ell)\gamma_\ell \Delta T$$

$$\therefore \quad \gamma_\ell = \beta$$

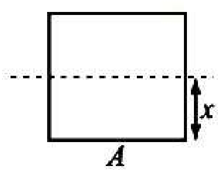
$$= 2\alpha.$$

Ans.

Illustration 9

A cube of coefficient of linear expansion α is floating in a bath containing a liquid of coefficient of volume expansion γ_ℓ. When the temperature is raised by ΔT, the depth upto which the cube is submerged in the liquid remains the same. Find relation between α and γ_ℓ.

Short-cut solution :



Before heating, $Mg = (Ax)\rho_\ell g$

After heating, $Mg = (A'x)\rho_\ell' g$

From above equations, we have

$$A'\rho_\ell' = A\rho_\ell$$

or $$A(1 + \beta\Delta T)\frac{\rho_\ell}{(1+\gamma_\ell \Delta T)} = A\rho_\ell$$

$$\therefore \quad \gamma = \beta \quad \text{or} \quad \gamma = 2\alpha.$$

Ans.

Illustration 10

The volume of an air bubble becomes 'η' times when it rises from bottom of a water lake to its surface. If the water barometer reads H, the depth of the lake is:

(a) ηH *(b)* $(\eta - 1)H$ *(c)* $\frac{H}{\eta}$ *(d)* *none of these*

Short-cut solution :

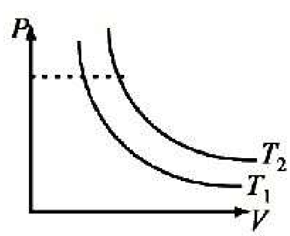
Using, $$P_1V_1 = P_2V_2$$

or $$(H + h)V = H \times (\eta V)$$

$$\therefore \quad h = (\eta - 1)H.$$

Ans. (b)

Illustration 11

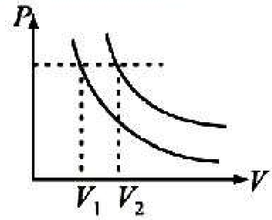


P-V diagram of some mass of a gas are drawn at two different temperatures T_1 and T_2. Then

(a) $T_1 = T_2$ *(b)* $T_1 < T_2$

(c) $T_2 > T_1$ *(d)* *No relation b/w T_1 and T_2*

Short-cut solution :



As $V_2 > V_1$; and $V \propto T$

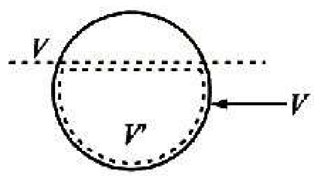
$\therefore$ $T_2 > T_1$. *Ans. (c)*

Illustration 12

A solid sphere with coefficient of linear expansion 'α' floats on a liquid whose coefficient of volume expansion is γ. On heating the system, find ratio of submerged to total volume $\left(\frac{V'}{V}\right)$.

Short-cut solution :

Using, $F_b = mg$ or $V'\rho_\ell g = V\rho_s g$ or $\frac{V'}{V} = \frac{\rho_s}{\rho_\ell}$



Initially, $f = \frac{V'}{V} = \frac{\rho_s}{\rho_\ell}$

On heating, $f' = \frac{V'}{V} = \frac{\rho'_s}{\rho'_\ell}$

$$= \left(\frac{\rho_s}{1+3\alpha\Delta T}\right)\left(\frac{1+\gamma_\ell \Delta T}{\rho_\ell}\right)$$

$$= f\left(\frac{1+\gamma_\ell \Delta T}{1+3\propto \Delta T}\right)$$

(i) The solid submerged as before, if $\gamma_\ell = 3\alpha$

(ii) The solid submerged more, if $\gamma_\ell > 3\alpha$.

(iii) The solid submerged less, if $\gamma_\ell < 3\propto$. *Ans.*

TOPIC 11.2: Specific Heat, Latent Heat, Mechanical Equivalent of Heat, Law of Mixture.

Review of Formulae

1. Mechanical equivalent of heat

$$J = \frac{W}{Q}$$

$$1 \text{ cal} = 4.2 \text{ J}$$

2. If c is the specific heat of substance, then

$$Q = mc\Delta T$$

If c is the function of temperature, then

$$Q = \int_{T_1}^{T_2} mc\,dT$$

3. **Specific heat of gas :**
   (i) If C_v is the specific heat of gas at constant volume, then
   $$Q_v = nC_v\Delta T$$
   (ii) If C_p is the specific heat at constant pressure
   $$Q = nC_p\Delta T$$
4. $C_p - C_v = R$
   $$C_v = \frac{R}{\gamma - 1} \text{ and } C_p = \frac{\gamma R}{\gamma - 1}$$
5. Latent heat : $Q = mL$
   Latent heat of fusion of ice = 80 cal/g or 336 kJ/kg.
   Latent heat of vapourisation of water 540 cal/g or 2259 kJ/kg at one atmospheric pressure.
6. **Law of mixture :** Heat lost = heat gained.

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. Two liquids of masses m_1 and m_2 and specific heats C_1 and C_2 respectively are mixed, then the specific heat of the mixture.
   $$C = \left[\frac{m_1C_1 + m_2C_2}{m_1 + m_2}\right].$$
2. When two substances of masses m_1 and m_2, specific heats C_1 and C_2 are mixed. If their temperatures are T_1 and T_2, then temperature of the mixture (no change of states of any substance)
   $$T = \left[\frac{m_1C_1T_1 + m_2C_2T_2}{m_1C_1 + m_2C_2}\right].$$
3. When equal amount of ice and steam (at 100°C) are taken together, the final temperature of them remains 100°C.
4. When 1g ice (0°C) is taken together with 1g water ($t \le 80$°C) the final temperature becomes 0°C.

Illustration 13

'n' number of liquids of masses m, 2m, 3m, ... having specific heats C, 2C, 3C, ... are at temperatures t, 2t, 3t, ...are mixed. The resultant temperature of the mixture is:

(a) $\left(\frac{3n}{2n+1}\right)t$ (b) $\frac{3n(n+1)}{2(2n+1)}t$ (c) $\frac{2n(n+1)}{3(2n+1)}t$ (d) $\frac{3n(n+1)}{(2n+1)}t$

Short-cut solution :

Using,
$$t = \frac{m_1C_1t_1 + m_2C_2t_2 + m_3C_3t_3 + ...}{m_1C_1 + m_2C_2 + m_3C_3 + ...}$$
$$= \left[\frac{mct + 2m(2c)2t + 3m(3c)3t}{mct + 2m(2c) + 3m(3c)}\right]$$
$$= \frac{(1^3 + 2^3 + 3^3 + ...)}{(1^2 + 2^2 + 3^2 + ...)}t$$
$$= \frac{3n(n+1)}{2(2n+1)}t.$$
Ans. (b)

Illustration 14

A piece of ice (heat capacity –2100 J/kg °C, and latent heat = 3.36 × 10⁵ J/kg) of mass m gram is at –5°C at atmospheric pressure. It is given 420 J of heat so that the ice starts melting. Finally when the ice-water mixture is in equilibrium, it is found that 1 gm of ice has melted. Assuming there is no other heat exchange in the process. Find the value of m.

Short-cut solution :

Using, heat lost = heat gained

or $420 = (m \times 2100 \times 5 + 1 \times 3.36 \times 10^5) \times 10^{-3}$

∴ $m = 8$ gm. ***Ans.***

Illustration 15

6 gm of steam at 100°C is mixed with 6 gm of ice at 0°C. Find the mass of steam left uncondensed.

Short-cut solution :

Using, heat lost = heat gained

$m \times 540 = 6 \times 80 + 6 \times 1 \times (100 - 0)$

or $m = 2$ gm.

∴ mass of steam left uncondensed = 6 – 2 = 4 gm. ***Ans.***

Illustration 16

A tap water at 10°C and another tap at 100°C. How hot water must be taken so that we get 20 kg of water at 35°C ?

(a) $\frac{40}{9}$ *kg* *(b)* $\frac{50}{9}$ *kg* *(c)* $\frac{20}{9}$ *kg* *(d)* $\frac{130}{9}$ *kg*

Short-cut solution :

If m kg hot water is taken, then

$$(20 - m) \times 1 \times (35 - 10) = m \times 1 \times (100 - 35)$$

or $$m = \frac{50}{9} kg.$$ ***Ans. (b)***

Illustration 17

The amount of steam at 100°C that should be passed into 600g of water at 10°C to make the final temperature as 40°C will be:

(a) ***40 g*** ***(b)*** ***30 g*** ***(c)*** ***20 g*** ***(d)*** ***45 g***

Short-cut solution :

$$m_{steam} \times L_v + m_{steam} C_w(100° - 40°) = m_{water} C_w(40° - 10°)$$

or $$m_{steam} \times 540 + m_{steam} \times 1 \times 60 = 600 \times 1 \times 30$$

or $$m_{steam} = 30 \text{ g}.$$ ***Ans. (b)***

TOPIC 11.3: *Rate of Flow of Heat in Conduction, Thermal Resistance and Thermal Conductivity.*

Review of Formulae

1. **Rate of heat flow in conduction**

$$H = KA\frac{(T_1 - T_2)}{L} = -KA\frac{\Delta T}{L}$$

2. **Thermal resistance,** $R_H = \dfrac{L}{KA}$

3. **Equivalent thermal conductivity :** When two rods of thermal conductivities K_1 and K_2 are placed in

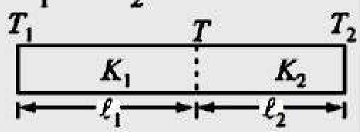
(i) Series : $$K = \left(\frac{2K_1K_2}{K_1 + K_2}\right)$$

(ii) Parallel : $$K = \left(\frac{K_1A_1 + K_2A_2}{A_1 + A_2}\right)$$

Tips and Tricks for Shortcut Solutions

1. The temperature of interface of walls of thickness ℓ_1 and ℓ_2

$$T = \frac{(K_1T_1)\ell_2 + (K_2T_2)\ell_1}{K_1\ell_2 + K_2\ell_1}$$



2. In thermal conduction, temperature varies linear from T_1 to T_2 along the length, so temperature of its mid point,

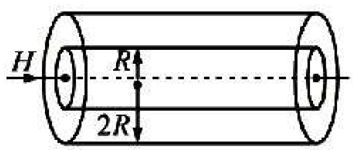
$$T = \frac{T_1 + T_2}{2}$$

Therefore, length of the rod at any temperature

$$L_t = L_0\left[1 + \alpha\left(\frac{T_1 + T_2}{2}\right)\right]$$

Illustration 18

A cylinder of radius R made of material of thermal conductivity K is surrounded by cylindrical shell of inner radius R and outer radius 2R made of a material of thermal conductivity 2K. The two ends of the combined system are maintained at two different temperatures. There is no loss of heat across the cylindrical surface and system is in steady state. Find the effective thermal conductivity of the system.



Short-cut solution :

Using, $K_e = \dfrac{K_1A_1 + K_2A_2}{A_1 + A_2}$

Here $A_1 = \pi R^2$ and $A_2 = \pi(4R^2 - R^2) = 3\pi R^2$

$\therefore$ $K_e = \dfrac{K \times \pi R^2 + (2K) \times 3\pi R^2}{\pi R^2 + 3\pi R^2} = -$ ***Ans.***

Illustration 19

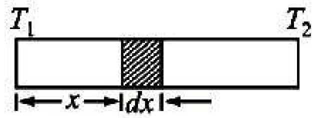
A rod of length L with sides fully insulated is made of a material whose thermal conductivity K varies with temperature as $K = \dfrac{\alpha}{T}$, where α is a constant. The ends of rod are at temperature T_1 and T_2 ($T_2 < T_1$). Find the rate of heat flow.

Short-cut solution :

$$H = \frac{dQ}{dt} = -k_A \frac{dt}{dx}$$

or $$H = -\left(\frac{\alpha}{T}\right)A\frac{dT}{dx}$$

or $$\int_0^{\ell} Hdx = -\alpha A\int_{T_1}^{T_2}\left(\frac{dT}{T}\right)$$

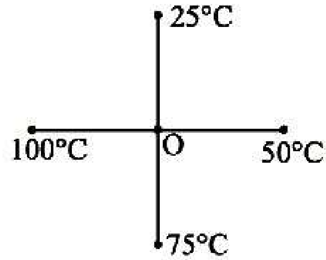


$\Rightarrow$ $$H = -\frac{\alpha A}{L}\ell n\left(\frac{T_2}{T_1}\right)$$

$\therefore$ $$H = \frac{\alpha A}{L}\ell n\left(\frac{T_1}{T_2}\right).$$ ***Ans.***

Illustration 20

Four identical rods are joined at their ends. The free ends are maintained at constant temperatures as indicated. Find temperature of the junction.



Short-cut solution :

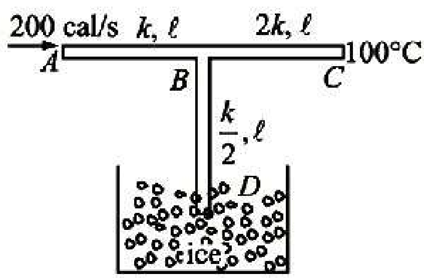
If T is the temperature of junction, then

$$\frac{100-T}{R} = \frac{T-25}{R}+\frac{T-50}{R}+\frac{T-75}{R}$$

$\therefore$ $T = 62.5°C.$ ***Ans.***

Illustration 21

Three rods AB, BC and BD of same length ℓ and cross-sectional area A are arranged as shown. The end D is immersed in ice whose mass is 440 gm. Heat is being supplied at constant rate 200 cal/s from the end A. Time in which whole ice will melt is: (latent heat of ice is 80 cal/g, K = 100 cal/m/s/°C A = 10 cm², ℓ = 1m)



(a) $\frac{40}{3}$ **min** *(b)* *700 s* *(c)* $\frac{20}{3}$ **min** *(d)* *not possible*

Short-cut solution :

If T is the temperature of junction, then

$$200 = \frac{2kA(T-100)}{\ell}+\frac{\left(\frac{k}{2}\right)A(T-0)}{\ell}$$

After putting the values and solving

$$\therefore \qquad T = 880°C.$$

If t is the required time, then

$$Ht = mL$$

or

$$\frac{\left(\frac{k}{2}\right)A(880-0)}{\ell}t = 440 \times 80$$

Here $A = 10\ cm^2, \ell = 100\ cm.$

On solving, we get $t = \frac{40}{3}$ min. ***Ans. (a)***

TOPIC 11.4: *Radial Flow of Heat, Radiation, Kirchhoff's Law, Stefan-Boltzmann Law, Wein's-Displacement Law and Newton's Law of Cooling.*

Review of Formulae

1. **In radial flow of heat** $K = \dfrac{H(r_2 - r_1)}{4\pi r_1 r_2 (T_1 - T_2)}$

2. **Cylindrical flow of heat** $K = \dfrac{H \ell n\left(\frac{r_2}{r_1}\right)}{2\pi\ell(T_2 - T_1)}$

3. **Formation of ice on pond**

$$t = \frac{\rho L}{2KT}\left(y_2^2 - y_1^2\right)$$

4. **Radiation** is the universal and fastest mode of heat transfer.

5. **Kirchoff's law :** $\dfrac{e_\lambda}{a_\lambda} = \text{constant}$

6. **Stefan's-Boltzmann law :** Net loss of heat

$$E_{net} = \varepsilon\sigma(T^4 - T_0^4)\ W/m^2$$

7. **Newtons's law of cooling**

$$\frac{dT}{dt} = -k(T - T_0)$$

here $(T - T_0)$ is small.

8. **Wien's displacement law**

$$\lambda_m T = \text{constant}$$

9. **Solar constant,** $S = \sigma T^4 \left(\dfrac{R_s}{r}\right)^2$

TIPS & TRICKS! Tips and Tricks for Shortcut Solutions

1. For spherical body, $A = 4\pi r^2 \therefore H \propto r^2$

2. Rate of cooling, $-\frac{dT}{dt} \propto \frac{A}{mc}(T - T_0)$. Thus two bodies of same material and equal surface areas, one solid other hollow ($m_s > m_H$), then rate of cooling of solid body will be less.

3. For spherical body, $\frac{A}{mc} = \left[\frac{4\pi r^2}{\frac{4}{3}\pi r^3 \rho c}\right] \propto \left(\frac{1}{r}\right)$.

   $\therefore$ Rate of cooling of body is inversely proportional to its radius, provided other things remain constant.

4. If temperature of the body falls from T_1 and T_2 in time t, then we can write

   $T = \frac{T_1 + T_2}{2}$,

   and $$\frac{T_1 - T_2}{t} = k\left[\frac{T_1 + T_2}{2} - T_0\right]$$

Illustration 22

Two spheres are made of same material have their radii in the ratio 1 : 3. They are heated to the same temperature and kept in the same surroundings at a moderate temperature. Find rate of cooling, if they are cooled by natural convection and radiation.

Short-cut solution :

Using, $$C = \left(-\frac{dT}{dt}\right) \propto \frac{1}{r}$$

$\therefore$ $$\frac{C_1}{C_2} = \frac{r_2}{r_1} = \frac{3}{1}.$$ ***Ans.***

Illustration 23

A body cools from 80°C to 50°C in 5 minute. Calculate the time it takes to cool from 60°C to 30°C. The temperature of the surroundings is 20°C.

Short-cut solution :

Using, $$\frac{T_1 - T_2}{t} = k\left[\frac{T_1 + T_2}{2} - T_0\right]$$

or $\frac{80-50}{5} = k\left[\frac{80+50}{2}-20\right]$ ...(i)

and $\frac{60-30}{t} = k\left[\frac{60+30}{2}-20\right]$ ...(ii)

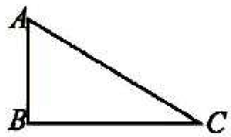
From above equations, we get

$t = 9$ min. **Ans.**

Concept Booster Exercise

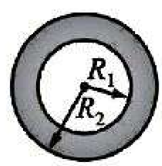
THERMAL EXPANSION

1. ABC is a right angled triangle made of metal rod bent as shown in figure. If it is heated, the angle ABC :



(a) increases (b) decreases
(c) remains same (d) becomes 180°

2. In the given figure there is metallic annular disc when temperature is increased, which of the following increases

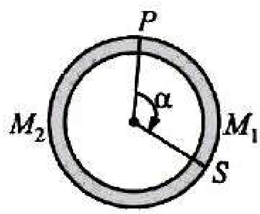


(a) R_1 (b) R_2
(c) $R_2 - R_1$ (d) All

3. A thin cylindrical metal rod is bent into a ring with a small gap as shown in figure. On heating the system

(a) θ decreases, r and d increases (b) θ increases
(c) θ is constant (d) d and r increases

4. A ring shaped tube contains two ideal gases with equal masses and atomic mass numbers $M_1 = 32$ and $M_2 = 28$. The gases are separated by one fixed partition P and another movable conducting partition S which can move angle 'α' as shown is figure in equilibrium, then find α. **Numeric/Integer**



(a) 291° (b) 219°
(c) 125° (d) 168°

5. When m gram of steam at 100° C is mixed with 200 gm of ice at 0°C, it results in water at 40°C. Find the value of m in gram. *[JEE Main 2020]*

(given: Latent heat of fusion (L_f) = 80 cal/gm, Latent heat of vaporisation (L_v) = 540 cal/gm. specific heat of water (C_w) = 1 cal/gm/°C) **Numeric/Integer**

6. At the bottom of a lake where temperature is 7°C the pressure is 2.8 atmosphere. An air bubble of radius 1 cm at the bottom rises to the surface, where the temperature is 27°C. Radius of air bubble at the surface is: **Numeric/Integer**

(a) $3^{\frac{1}{3}}$ (b) $4^{\frac{1}{3}}$ (c) $5^{\frac{1}{3}}$ (d) $6^{\frac{1}{3}}$

CALORIMETRY

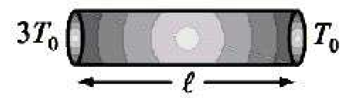
7. Two liquids A and B of equal volumes have their specific heats in the ratio 2 : 3. If they have same thermal capacity, then the ratio of their densities is: **Numeric/Integer**
(a) 1 : 1 (b) 2 : 3 (c) 3 : 2 (d) 5 : 6

8. 1 gm of ice at 0°C is mixed with 1 gm steam at 100°C. The mass of water formed is :
(a) $\frac{4}{3}$ gm (b) $\frac{2}{3}$ gm (c) $\frac{5}{3}$ gm (d) 1 gm
Numeric/Integer

9. A liquid of mass 'm' and specific heat C is at temperature $2t$. If another liquid of thermal capacity of 1.5 times, at a temperature of $\frac{t}{3}$ is added t_0 it the resultant temperature will be:
(a) $\frac{4t}{3}$ (b) t (c) $\frac{t}{2}$ (d) $\frac{2t}{3}$

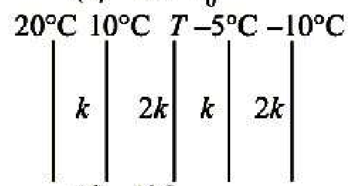
HEAT TRANSFER

10. Two ends of a rod of uniform cross-sectional area are kept at temperature $3T_0$ and T_0 as shown. Thermal conductivity of rod varies as $K = \alpha T$, where α is a constant and T is the absolute temperature. In steady state, the temperature of the middle section of the rod is:



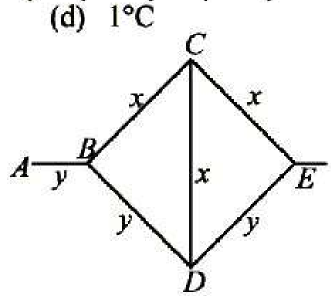
(a) $\sqrt{7}\ T_0$ (b) $\sqrt{5}\ T_0$ (c) $2\ T_0$ (d) $\sqrt{3}\ T_0$

11. The figure shows the face and interface temperature of a composite slab containing of four layers of two materials having identical thickness. Under steady state condition, the value of T is: **Numeric/Integer**



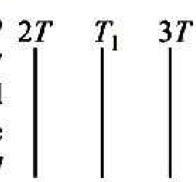
(a) 5°C (b) 2°C (c) 4°C (d) 1°C

12. Three rods of material 'x' and three rods of material y are connected as shown in figure. All rods are of identical length and cross-section. If the end A is maintained at $2T_0$ and the junction E at T_0, find the effective thermal resistance. Given length of each rod is ℓ, area A and thermal conductivity of x is K and that of y is $2K$.



(a) $\frac{4\ell}{3KA}$ (b) $\frac{7\ell}{6KA}$ (c) $\frac{4KA}{3\ell}$ (d) $\frac{7KA}{3\ell}$

13. Three very large plates of same area are kept parallel and close to each other. They are considered as ideal black surface and have very high thermal conductivity. The first and third plates are maintained at temperature $2T$ and $3T$ respectively. The temperature of the middle plate under steady state is: *[JEE Main 2012]*



(a) $\left(\frac{97}{2}\right)^{\frac{1}{4}} T$ (b) $\left(\frac{49}{2}\right)^{\frac{1}{2}}$ T (c) 81 T (d) none of these

14. Two spherical bodies A (radius 6 cm) and B (radius 18 cm) are at temperature T_1 and T_2 respectively. The maximum intensity in the emission spectrum of A is at 500 nm and in that of B is at 1500 nm. Considering them to be black bodies, what will be the ratio of the rate of total energy radiated by A to that of B? ***[JEE Main 2010]***

(a) 2 : 1 (b) 3 : 2 (c) 9 : 4 (d) 9 : 1

Solutions

1. **(c)** Each side of the triangle expands in equal proportion, so angle ABC remains same.

2. **(d)** $\Delta R_1 = R_1\,\alpha\,\Delta T$; $\Delta R_2 = R_2\,\alpha\,\Delta T$. Also $R_2' - R_1' = (R_2 - R_1)\,\alpha\,\Delta T$.

3. **(c, d)** Each part of the ring expands in same proportion.

4. **(d)** Using, $PV = \dfrac{m}{M}RT,$

we have $\dfrac{M_2}{M_1} = \dfrac{V_1}{V_2}$

$= \dfrac{A\ell_1}{A\ell_2} = \dfrac{\alpha}{2\pi-\alpha}$

or $\dfrac{28}{32} = \dfrac{\alpha}{2\pi-\alpha}$

On solving, we get $\alpha = 168°$. ***Ans.***

5. **(40)** $M_{ice}L_f + m_{ice}(40-0)\,C_w = m_{steam}L_v + m_{steam}(100-40)\,C_w$

$\Rightarrow \quad 200[80 + 40(1)] = m[540 + 60(1)]$

$\Rightarrow \quad 200(120) = m(600)$

$m = 40$ gm. ***Ans.***

6. **(a)** Using, $\dfrac{P_1V_1}{T_1} = \dfrac{P_2V_2}{T_2}$. ***Ans.***

7. **(c)** Given, $m_1C_1 = m_2C_2$ or $V\rho_1C_1 = V\rho_2C_2$

$\therefore \quad \dfrac{\rho_1}{\rho_2} = \dfrac{C_2}{C_1} = \dfrac{3}{2}$. ***Ans.***

8. **(a)** The amount of steam condensed to raise the temperature of ice by 100°C after melting,

$m \times 540 = 1 \times 80 + 1 \times 1 \times (100-0) = 180°$

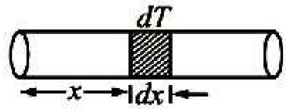
or $m = \dfrac{1}{3}$ gm

$\therefore \quad$ water $= 1 + \dfrac{1}{3} = \dfrac{4}{3}$ gm. ***Ans.***

9. **(b)** Using, $T = \dfrac{m_1C_1T_1 + m_2C_2T_2}{m_1C_1 + m_2C_2} = \dfrac{m \times C \times 2t + 1.5(mC) \times t/3}{mC + 1.5(mC)}$

$= t.$ ***Ans.***

10. **(b)** In steady state



$$H = -KA\left(\frac{dT}{dx}\right)$$

or
$$\int_0^{\ell} Hdx = -\int_{3T_0}^{T_0} \alpha TA \, dT \quad \text{...(i)}$$

And
$$H\int_0^{\ell/2} dx = -\alpha A\int_{3T_0}^{T} TdT \quad \text{...(ii)}$$

After simplifying above equations, we get

$$T = \sqrt{5}T_0.$$ ***Ans.***

11. **(a)** For first two layers

$$H_1 = H_2$$

$$kA\frac{(20-10)}{\ell} = 2kA\frac{(10-T)}{\ell}$$

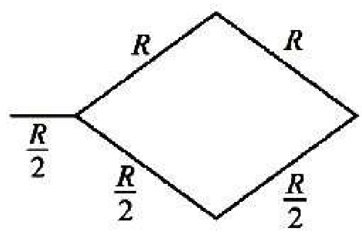
$\therefore$ $T = 5°C.$ ***Ans.***

12. **(b)** In this arrangement of rods, there is no flow of heat in CD. So effective arrangement is as follows

As $R \propto \frac{1}{k}$ so resistance of each rod x is R and that of rod y is $\frac{R}{2}$.

$$R_e = \frac{2R\times R}{2R+R}+\frac{R}{2}$$

$$= \frac{7R}{6} = \frac{7\ell}{6kA}.$$ ***Ans.***



13. **(a)** At steady state

Rate of emission = rate of absorption

or $\sigma(2A)T_1^4 = \sigma A(2T)^4 + \sigma A(3T)^4$

$\therefore$ $T_1 = \left(\frac{97}{2}\right)^{\frac{1}{4}} T$ ***Ans.***

14. **(d)** We known that, $\lambda_m T = \text{constant}$

$\therefore$ $\frac{T_A}{T_B} = \frac{\lambda_B}{\lambda_A} = \frac{1500}{500} = 3.$

Using, $H = A\sigma T^4$, we have

$$\frac{H_1}{H_2} = \frac{A_1T_1^4}{A_2T_2^4} = \frac{4\pi r_1^2}{4\pi r_2^2}\times\left(\frac{T_1}{T_2}\right)^4 = \left(\frac{6}{18}\right)^2\times 3^4$$

$$= 9.$$ ***Ans.***

Thermodynamics and Kinetic Theory 12

***TOPIC 12.1**: Boyle's, Charle's and Gay-Lussac's Law, Ideal Gas Equation, Vander wall's Equation for Real Gas, Kinetic Energy of Gas and Degrees of Freedom.*

Review of Formulae

1. **Boyle's law**

$$PV = \text{constant}$$

2. **Charle's law**

$$\frac{V}{T} = \text{constant}$$

3. **Gay-Lussac's law**

$$\frac{P}{T} = \text{constant}$$

4. **Ideal gas equation**

$$PV = nRT$$

5. **Vander Waal's equation for real gases**

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT$$

6. **Pressure,** $P = \dfrac{nmv_{rms}^2}{3V}$

7. **Kinetic energy of gas**

(i) Translational K.E. $= \dfrac{3}{2}RT$ per mole for each gas

(ii) For diatomic gas

Translational + rotational K.E. $= \dfrac{5}{2}RT$ per mole

8. If n_1 moles of gas with degrees of freedom f_1 and n_2 moles with f_2 are mixed, then effective degrees of freedom

$$f = \frac{n_1 f_1 + n_2 f_2}{n_1 + n_2}.$$

9. **Dalton's law of partial pressure**

$$P = P_1 + P_2 + \ldots\ldots$$

10. Degrees of freedom :

(i) Manoatomic gas

Degrees of freedom $f = 3$ translational

Kinetic energy per mole = $\frac{3}{2}RT$

$$C_v = \frac{\partial U}{\partial T} = \frac{3}{2}R$$

$$C_p = C_v + R = \frac{5}{2}R$$

$$\gamma = \frac{C_p}{C_v} = \frac{5}{3}$$

(ii) Diatomic gas

Degree of freedom = 5(3 translational + 2 rotational)

Kinetic energy per mole = $\frac{5RT}{2}$

$$C_v = \frac{5R}{2}$$

$$C_p = \frac{7R}{2}$$

$$\gamma = \frac{7}{5}$$

(iii) Triatomic or polyatomic gas

Degrees of freedom = 6 (3 translational + 3 rotational)

Kinetic energy per mole = $3RT$

$$C_v = 3RT$$

$$C_p = 4R$$

$$\gamma = \frac{4}{3}$$

11. In general, $\gamma = 1 + \frac{2}{f}$

12. Mean free path, $\lambda = \frac{kT}{\sqrt{2}\pi d^2 P}$

13. γ of the mixture :

(i) When C_v and C_p are given : For two gases with n_1 and n_2 moles

$$[C_v]_{mix} = \frac{n_1 C_{v_1} + n_2 C_{v_2}}{n_1 + n_2}$$

$$[C_p]_{mix} = \frac{n_1 C_{p_1} + n_2 C_{p_2}}{n_1 + n_2}$$

$$\gamma_{mix} = \frac{[C_p]_{mix}}{[C_v]_{mix}}$$

(ii) For two gases of γ_1 and γ_2 with n_1 and n_2 moles

$$\frac{(n_1+n_2)}{\gamma_{mix}-1} = \frac{n_1}{\gamma_1-1}+\frac{n_2}{\gamma_2-1}$$

Tips and Tricks for Shortcut Solutions

1. For an ideal gas with 'f' degrees of freedom (translational and rotation), its translational KE is $\frac{3}{2}RT$ per mole and total kinetic energy will be $\frac{3}{2}fRT$ per mole.
2. When n_1 moles of a gas with molar mass M_1, are mixed with n_2 moles of another gas with molar mass M_2, the equivalent molar mass of the mixture
$$M=\left[\frac{n_1M_1+n_2M_2}{n_1+n_2}\right].$$
3. When n_1 moles of an ideal gas at temperature T_1 are mixed with n_2 moles of an another gas at temperature T_2, the temperature of mixture
$$T=\left[\frac{n_1T_1+n_2T_2}{n_1+n_2}\right].$$
4. The relation $C_p - C_v = R$ is same for all ideal gases. Here C_p and C_v are molar specific heat at constant pressure and constant volume respectively.
5. $C_v=\frac{R}{\gamma-1}$ and $C_p=\frac{\gamma R}{\gamma-1}$. Here $\gamma=\frac{C_p}{C_v}$.
6. If v is the speed of sound in gas, then
$$v=\sqrt{\frac{\gamma}{3}}\,v_{rms}.$$
7. $v_{av} = 0.92\, v_{rms}$ and $v_{mp} = 0.816\, v_{rms}$.
8. If two vessels (P_1,V_1,T_1) and (P_2,V_2,T_2) with different gases are connected together, then they will get a common pressure and $n_1+n_2=n'_1+n'_2$.

Illustration 1

A light vessel containing one mole of an ideal gas of molar mass M moves with constant speed 'v' on a smooth horizontal surface. If vessel suddenly stops, then determine the increase in temperature of the gas. The adiabatic exponent of the gas is 'γ'.

Short-cut solution :

When vessel stops, the external KE will convert into internal energy and so

$$\frac{1}{2}Mv^2 = nC_v(\Delta T)$$

As $n = 1$, and $C_v = \frac{R}{\gamma - 1}$

$$\therefore \quad \frac{1}{2}Mv^2 = \frac{R}{(\gamma-1)}\Delta T$$

or $$\Delta T = \frac{(\gamma-1)Mv^2}{2R}. \qquad \textit{Ans.}$$

Illustration 2

Two moles of a monoatomic gas are mixed with 'n' moles of a polyatomic gas. If mixture behaves like diatomic gas, then find the value of n.

Short-cut solution :

Using, $$f = \frac{n_1 f_1 + n_2 f_2}{n_1 + n_2}$$

or $$5 = \frac{2\times 3 + n\times 6}{2+n}$$

$$\therefore \quad n = 4. \qquad \textit{Ans.}$$

Illustration 3

Two container of equal volume contain the same gas at pressures P_1 and P_2 and absolute temperatures T_1 and T_2 respectively. On joining the containers, the gas reaches a common pressure P and temperature T, then find the value of $\frac{P}{T}$.

Short-cut solution :

Using, $$n_1 + n_2 = n'_1 + n'_2$$

$$\frac{P_1V}{RT_1} + \frac{P_2V}{RT_2} = \frac{P\times 2V}{RT}$$

$$\therefore \quad \frac{P}{T} = \frac{1}{2}\left[\frac{P_1}{T_1} + \frac{P_2}{T_2}\right]. \qquad \textit{Ans.}$$

Illustration 4

Find the minimum heat energy required to cause complete dissociation of n-moles of H_2 gas at constant temperature.

Short-cut solution :

When n-moles of H_2 dissociates, there becomes $2n$-moles of H. So

, $=$ n[energy of two moles of H – energy of one mole of H_2]

$$= 2n \times \frac{3}{2}RT - n \times \frac{5}{2}RT$$

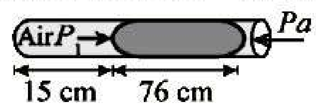
$$= \frac{nRT}{2}.$$ ***Ans.***

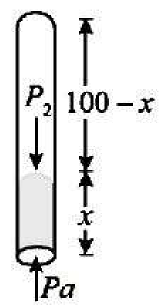
Illustration 5

A metre long narrow bore held horizontally (and closed at one end) contains a 76 cm long mercury thread, which traps a 15 cm column of air. What happens if the tube is held vertically with the open end at the bottom ?

Solution :

Suppose the air column in the tube becomes $(15 + x)$, then





$$V_1 = A \times 15,$$
$$P_1 = P_a$$
$$V_2 = A\,(15 + x),$$
$$P_2 + x = P_a$$
$$\therefore \quad P_2 = (P_a - x)$$

By using Boyle's law, we have $P_1V_1 = P_2V_2$

or $\quad P_a \times (15A) = (P_a - x)\,(100 - x)\,A$

or $\quad 76 \times 15 = (76 - x)\,(100 - x)$

or $\quad x^2 - 176x + 6460 = 0$

After solving, $\quad x = 52.17$ cm

Thus mercury thread will decrease in length by 23.8 cm. ***Ans.***

TOPIC 12.2: First Law of Thermodynamics, Thermodynamic Processes, Polytropic Process, Entropy, Carnot's Engine and Refrigerator.

Review of Formulae

1. **Internal energy** of an ideal gas is due to its kinetic energy, which is the function of temperature.

$$U = \frac{3}{2}RT \text{ for monoatomic gas}$$

$$= \frac{5}{2}RT \text{ for diatomic gas}$$

Also $\Delta U = nC_V\Delta T$

If $\Delta T = 0, \Delta U = 0,$

In cyclic process,

$$\Delta U = 0.$$

2. **Work done,** $W = \int_{V_i}^{V_f} P dV$

3. **First law of thermodynamics**

$$Q = \Delta U + W$$

In diferential form, it can be written as

$$dQ = dU + W$$

4. **Thermodynamic processes :**

PV = const for isothermal; PV^r = constant for adiabatic.

(i) Work done in isobaric process,

$$W = P\Delta V = nR\Delta T$$

(ii) Work done in isochoic process

$$W = 0$$

Also $Q = \Delta U + W = \Delta U + 0 = nC_V\Delta T$

(iii) Work done in isothermal process

$$W = nRT\ell n\frac{V_f}{V_i} = nRT\ell n\frac{P_i}{P_f}$$

Isothermal elasticity,

$$E_{iso} = P$$

(iv) Work done in adiabatic process

$$W = \frac{(P_iV_i - P_fV_f)}{(\gamma - 1)}$$

or $W = \frac{nR}{\gamma - 1} = (T_i - T_f)$

Adiabatic elasticity,

$$E_{ad} = \gamma P$$

$\therefore$ $E_{ad} = \gamma E_{iso}$

5. **Polytropic process :**

$$PV^r = 1, r \neq 1 \text{ or } \gamma.$$

Work done, $W = \frac{nR}{r-1}[T_i - T_f]$

Specific heat, $C = C_V - \frac{R}{r-1}$

6. The change in entropy,

$$\Delta S = \frac{\Delta Q}{T}.$$

For a system with variable T

$$\Delta S = \int_{S_i}^{S_f} \frac{dQ}{T}$$

7. Efficiency of heat engine

$$\eta = \frac{W}{Q_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

8. Carnot heat engine :

$$V_1 V_3 = V_2 V_4$$

$$\eta = 1 - \frac{T_2}{T_1}$$

9. Refrigerator or heat pump: Coefficient of performance

$$\beta = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2} = \frac{1}{\frac{Q_1}{Q_2} - 1}$$

$$= \frac{1}{\frac{T_1}{T_2} - 1} = \frac{\eta}{1-\eta}$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions in Thermodynamics

1. If Q heat is given to an ideal gas at constant pressure, then change in internal energy (ΔU).

$$\frac{\Delta U}{Q} = \frac{1}{\gamma}$$

and $$\frac{W}{Q} = \frac{Q - \Delta U}{Q} = 1 - \frac{\Delta U}{Q} = 1 - \frac{1}{\gamma}.$$

If 'f' be the degrees of freedom, then

$$\frac{\Delta U}{Q} = \frac{f}{f+2} \text{ and } \frac{W}{Q} = \frac{2}{f+2}.$$

2. For adiabatic process, PV^γ = constant

Also $PT^{\frac{\gamma}{1-\gamma}}$ = constant and $TV^{\gamma-1}$= constant.

3. In free expansion. $Q = 0, W = 0, \Delta U = 0, \Delta T = 0$.
4. If two engines are put in series, having first source temperature T_1 and last sink temperature T_3, then efficiency of engine, $\eta = 1-\frac{T_3}{T_1}$.

   If efficiencies of engines are η_1 and η_2, then net efficiency, $\eta = \eta_1 \times \eta_2$.
5. On increasing the pressure ΔP, the change in volume ΔV, in isothermal process and ΔV_2 in adiabatic pressure, then $\Delta V_1/\Delta V_2 = \gamma$.
6. For the same volume expansion, final pressure in adiabatic process is smaller than isothermal process. But for same volume compression, the final pressure in adiabatic process will be greater than isothermal process.
7. $\frac{P}{\rho} = \frac{RT}{M}$.

Illustration 6

Figure shows two identical cylinders A and B which contain equal amounts of an ideal gas with adiabatic exponent γ. The piston of cylinder A is free and that of cylinder B is fixed. If the temperature rise in cylinder A is T_0, then find the temperature raised in cylinder B.

Short-cut solution :

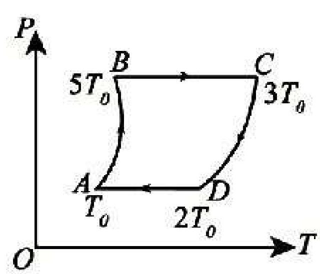
$$\frac{\Delta T_B}{\Delta T_A} = \frac{\Delta U_B}{\Delta U_A} = \frac{Q}{Q/\gamma} = \gamma$$

or $\Delta T_B = \gamma(\Delta T_A)$

$= \gamma T_0$. ***Ans.***

Illustration 7

The P-T diagram of a thermodynamic heat engine cycle is shown in figure. Two curved processes are adiabatic. The work done for one mole of a monoatomic gas in one cycle is given by



(a) $1.25\ RT_0$ (b) $2RT_0$
(c) $-2RT_0$ (d) $-1.25\ RT_0$

Short-cut solution :

$$W = W_{AB} + W_{BC} + W_{CD} + W_{DA}$$

$$= \frac{R(T_0 - 1.5T_0)}{(5/3-1)} + \frac{5R}{2}(3T_0 - 1.5T_0) + \frac{R(3T_0 - 2T_0)}{(5/3-1)} + \frac{5R}{2}(T_0 - 2T_0)$$

$= 2RT_0$. *Ans.*

Illustration 8

A diatomic ideal gas is heated at constant volume until the pressure is doubled and again heated at constant pressure until volume is doubled. The average molar heat capacity for whole process is :

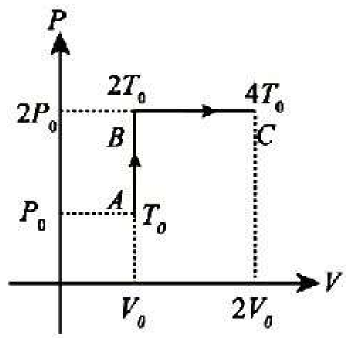
(a) $\frac{13R}{6}$ *(b)* $\frac{19R}{6}$ *(c)* $\frac{23R}{6}$ *(d)* $\frac{17R}{6}$

Short-cut solution :

$$\Delta Q = C_v(\Delta T) + C_p(\Delta T')$$

$$= \frac{5R}{2}(2T_0 - T_0) + \frac{7R}{2}(4T_0 - 2T_0)$$

$$= \frac{19}{2}RT_0$$



Total change in temperature from A to C

$$\Delta T = 4T_0 - T_0 = 3T_0$$

$$\therefore \quad C = \frac{\Delta Q}{\Delta T} = \frac{19RT_0/2}{3T_0} = \frac{19}{6}R.$$ *Ans. (b)*

Illustration 9

The pressure of an ideal gas varies according to the law $P = P_0 - AV^2$, where P_0 and A are positive constants. Find the highest temperature that can be attained by the gas.

Solution :

The ideal gas equation

$$PV = nRT$$

Given $$P = P_0 - AV^2,$$

$$\therefore \quad (P_0 - AV^2)V = nRT$$

or $$P_0V - AV^3 = nRT \quad .....(i)$$

Differentiating above equation w.r.t. V, we have

$$P_0 - 3AV^2 = nR\left(\frac{dT}{dV}\right).$$

For highest value of T, $dT/dV = 0$

or $P_0 - 3AV^2 = 0$

$\therefore$ $V = \sqrt{\frac{P_0}{3A}}$

Now from equation (i), we have

$$\left(P_0 - A \times \frac{P_0}{3A}\right)\left(\sqrt{\frac{P_0}{3A}}\right) = nRT$$

$\therefore$ $T_{max} = \frac{2P_0}{3nR}\sqrt{\frac{P_0}{3A}}.$ ***Ans.***

Illustration 10

The efficiency of a Carnot cycle is $\frac{1}{6}$. If on reducing the temperature of the sink by 65°C, the efficiency becomes $\frac{1}{3}$, find the initial and final temperatures between which the cycle is working.

Solution :

If T_1 and T_2 are the temperatures of source and sink respectively, then

$$\eta_1 = 1 - \frac{T_2}{T_1}$$

or $\frac{1}{6} = 1 - \frac{T_2}{T_1}$ ...(i)

When temperature of sink reduces by 65°C, then $\eta_2 = \frac{1}{3}$

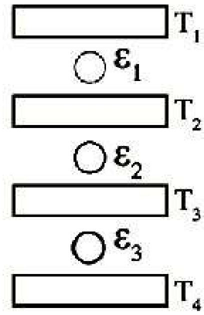
$\therefore$ $\frac{1}{3} = 1 - \frac{T_2 - 65}{T_1}$ ...(ii)

Solving equations (i) and (ii), we get

$T_1 = 390\ K$ and $T_2 = 325\ K.$ ***Ans.***

Illustration 11

Three Carnot engines operate in series between a heat source at a temperature T_1 and a heat sink at temperature T_4 (see figure). There are two other reservoirs at temperature T_2 and T_3, as shown, with $T_1 > T_2 > T_3 > T_4$ The three engines are equally efficient if: ***[JEE Main 2019]***



(a) $T_2 = (T_1 T_4)^{1/2}; T_3 = (T_1^2 T_4)^{1/3}$

(b) $T_2 = (T_1^2 T_4)^{1/3}; T_3 = (T_1 T_4^2)^{1/3}$

(c) $T_2 = (T_1 T_4^2)^{1/3}; T_3 = (T_1^2 T_4)^{1/3}$

(d) $T_2 = (T_1^3 T_4)^{1/4}; T_3 = (T_1 T_4^3)^{1/4}$

Short-cut solution :

According to question, $\eta_1 = \eta_2 = \eta_3$

$$\therefore 1 - \frac{T_2}{T_1} = 1 - \frac{T_3}{T_2} = 1 - \frac{T_4}{T_3}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3}$$

$$\Rightarrow T_2 = \sqrt{T_1 T_3} = \sqrt{T_1 \sqrt{T_2 T_4}}$$

$$T_3 = \sqrt{T_2 T_4}$$

$$T_2^{3/4} = \sqrt{T_1^{1/2} T_4^{1/4}} \qquad T_2 = T_1^{2/3} T_4^{1/3}.$$ ***Ans. (b)***

Illustration 12

One mole of an ideal gas is contained within a cylinder by a frictionless position and is initially at temperature T. The pressure of the gas is kept constant while it is heated and its volume doubles. If R is the molar gas constant, the work done by the gas in increasing the volume is

(a) $RT\ln 2$ ***(b)*** $\frac{RT}{2}$ ***(c)*** RT ***(d)*** $\frac{3}{2}RT$

Short-cut solution :

We know $PV = RT$, or $P\Delta V = R\Delta T$

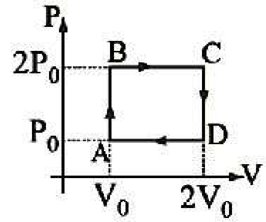
At constant pressure $V \propto T$, when volume doubles, temperature also doubles, therefore.

$$\Delta T = 2T - T = T.$$

$$\therefore \quad W = P\Delta V = R\Delta T = RT.$$ ***Ans.***

Video Solution

Q. ***Helium gas goes through a cycle ABCDA (consisting of two isochoric and isobaric lines) as shown in figure. Find efficiency of this cycle. (Assume the gas to be close to an ideal gas)*** **[JEE Main 2012]**

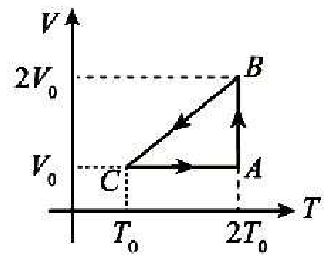


To see the video solution, scan the QR code:
OR Visit https://www.youtube.com/watch?v=-D28GvblNEA

Illustration 13

Find the efficiency of an ideal gas with adiabatic exponent 'γ' for the cyclic process shown in figure.

Short-cut solution :



Work done $\quad W_{AC} = nR(2T_0)\ell n\left(\frac{2V_0}{V_0}\right)$

$$= 2nRT_0\ell n2$$

and $\quad W_{BC} = P\Delta V$

$$= nR(\Delta T) = -nRT_0$$

Total work done

$$W = W_{AC} + W_{BC}$$

$$= 2nRT_0\ell n2 - nRT_0$$

$$= nRT_0(2\ell n\,2 - 1)$$

Heat extracted

$$Q_1 = Q_{BC} = nC_P\Delta T$$

$$= n\left(\frac{\gamma R}{\gamma - 1}\right)T_0$$

Hence efficiency

$$\eta = \frac{W}{Q_1} = \frac{nRT_0(2\ell n2 - 1)}{\frac{nrR}{(r-1)}T_0}$$

$$= \frac{(2\ell n2 - 1)(\gamma - 1)}{\gamma}. \qquad \textit{Ans.}$$

Concept Booster Exercise

1. Two gases Ar (40) and Xe (131) at same temperature have same number density. Their diameters are 0.07 nm and 0.10 nm respectively. Find the ratio of their mean free time. *[JEE Main 2020]*
(a) 1.03 (b) 2.04 (c) 3.04 (d) 2.40

2. Two identical containers joined by a small pipe initially contain the same gas at pressure P_0 and absolute temperature T_0. One container is now maintained at the same temperature while other is heated to $2T_0$. The common pressure of the gas will be:
(a) $\frac{3}{2}P_0$ (b) $\frac{4P_0}{3}$ (c) $\frac{5P_0}{3}$ (d) $2P_0$

3. For a gas $\frac{R}{C_V} = 0.67$. This gas is made up of molecules which are
(a) diatomic
(b) mixture of diatomic and polyatomic molecules
(c) monoatomic
(d) polyatomic

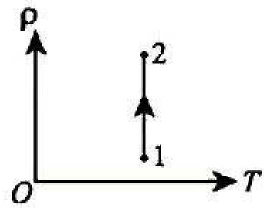
4. Two moles of helium are mixed with n with moles of hydrogen. If $\frac{C_P}{C_V} = \frac{3}{2}$ for the mixture, then the value of n is **Numeric/Integer**
(a) 3/2 (b) 2 (c) 1 (d) 3

5. A gas mixture consists of 2 moles of oxygen and 4 moles of argon at temperature T. Neglecting all vibrational modes, the total internal energy of the system is :
(a) $4RT$ (b) $15RT$ (c) $9RT$ (d) $11RT$

6. In a thermodynamic process helium gas obeys the law $TP^{-2/5}$ = constant, what is the heat given to the gas when the temperature of 2 moles of the gas is raised from T to $4T$? (R is the universal gas constant)
(a) $9R$ (b) $18RT$ (c) zero (d) none of these

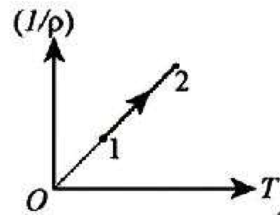
7. A diatomic gas obeys the law PV^n = constant. For what value of x, it has negative molar heat?
(a) $n > 1.4$ (b) $n < 1.4$ (c) $1 < n < 1.4$ (d) $0 < n < 1$

8. The figure shows the variation of density ρ with temperature T for an ideal gas. Choose the incorrect alternative



(a) $P_1V_1 = P_2V_2$ (b) $P_1 < P_2$
(c) W_{12} is negative (d) $\Delta U_{12} = 0$

9. The figure shows (1/ρ) versus T graph for an ideal gas. Choose the correct alternative(s)



(a) the graph represents isobaric expansion
(b) large the slope of straight line higher the pressure
(c) internal energy of gas increases
(d) work done during the process is positive

10. A gas undergoes a process in which its pressure P and volume V are related as VP^n = constant. The bulk modulus for the gas in the process is :

(a) P (b) nP (c) P^n (d) $\frac{P}{n}$

11. The average degree of freedom per molecule for a gas is 6. The gas performs 25 J of work when it expands at constant pressure. The heat absorbed by the gas is:

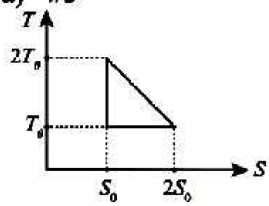
Numeric/Integer

(a) 75 J (b) 100 J (c) 150 J (d) 125 J

12. During an adiabatic process, the pressure of a gas is found to be proportional to the cube of its absolute temperature. The ratio $\frac{C_P}{C_V} = \gamma$ for the gas is: **Numeric/Integer**

(a) 2 (b) 3/2 (c) 5/3 (d) 4/3

13. The temperature-entropy diagram of a reversible engine cycle is given in the figure. Its efficiency is *[JEE Main 2005]* **Numeric/Integer**



(a) 1/3 (b) 2/3

(c) 1/2 (d) 1/4

14. A system is taken from state A to state B along two different paths 1 and 2. The heat absorbed and work done by the system along these two paths are Q_1 and Q_2 and W_1 and W_2 respectively, then

(a) $Q_1 = Q_2$ (b) $W_1 = W_2$

(c) $Q_1 - W_1 = Q_2 - W_2$ (d) $Q_1 + W_1 = Q_2 + W_2$

15. Cascaded Carnot engine is an arrangement in which heat sink of one engine is source for other. If high temperature for one engine is T_1, low temperature for other engine is T_2 (Assume work done by both engine is same) Calculate lower temperature of first engine. *[JEE Main 2020]*

(a) $\frac{T_1T_2}{T_1\ \ T_2}$ (b) $\frac{T_1+T_2}{2}$ (c) 0 (d) $\sqrt{T_1T_2}$

Solutions

1. **(b)** Mean free time $= \frac{1}{\sqrt{2}n\pi d^2 v_{rms}}$

$$\frac{t_{Ar}}{t_{Xe}} = \frac{d_{Xe}^2}{d_{Ar}^2}$$

$$= \left(\frac{0.1}{0.07}\right)^2 = \left(\frac{10}{7}\right)^2 = 2.04 \quad \textit{Ans.}$$

2. **(b)**

$$2\frac{P_0V_0}{RT_0} = \frac{PV_0}{RT_0} + \frac{PV_0}{R(2T_0)}$$

or

$$P = \frac{4P_0}{3}. \quad \textit{Ans.}$$

3. (c) $\frac{R}{C_v} = 0.67 \Rightarrow C_v = \frac{R}{0.67} = \frac{3R}{2}$.

Thus gas is monoatomic.

4. (b) Using formula,

$$\gamma_{\text{mixture}} = \left(\frac{C_p}{C_v}\right)_{\text{mix}} = \frac{\frac{n_1\gamma_1}{\gamma_1 - 1} + \frac{n_2\gamma_2}{\gamma_2 - 1}}{\frac{n_1}{\gamma_1 - 1} + \frac{n_2}{\gamma_2 - 1}}$$

Putting the value of $n_1 = 2, n_2 = n, \left(\frac{C_p}{C_v}\right)_{\text{mix}} = \frac{3}{2}$

$\gamma_1 = \frac{5}{3}, \gamma_2 = \frac{7}{5}$ and solving we get, $n = 2$. ***Ans.***

5. (d) $U = 2 \times \frac{5RT}{2} + 4 \times \frac{3}{2}RT = 11RT$. ***Ans.***

6. (c) Using, $PV = nRT \Rightarrow T = \frac{PV}{nR}$

Given $TP^{-2/5} = \text{constant}$

$\frac{PV}{nR}P^{-2/5} = \text{constant}$

or $PV^{5/3} = \text{constant}$

On comparing with $PV^r = \text{constant}$, we have

$r = 5/3$.

Also $C = \frac{R}{\gamma - 1} - \frac{R}{r - 1} = \frac{R}{\frac{5}{3} - 1} - \frac{R}{\frac{5}{3} - 1} = 0$

So $Q = nC(\Delta T) = 0$. ***Ans.***

7. (c) For diatomic gas, $\gamma = 1.4$ and

$C = \frac{R}{\gamma - 1} - \frac{R}{r - 1} = \frac{R}{1.4 - 1} - \frac{R}{r - 1} = \frac{R}{0.4} - \frac{R}{r - 1}$

For negative value of C,

$(r - 1) < 0.4$

or $r < 1.4 \Rightarrow n < 1.4$. ***Ans.***

8. (a) Using, $\frac{P}{\rho} = \frac{RT}{M} \Rightarrow \rho \propto \frac{P}{T}$

As ρ increases from 1 to 2 so $P_2 > P_1$. Also T is constant, so V constant,

$\therefore P_1V_1 \neq P_2V_2$ ***Ans.***

9. (a, c, d) We know that $\frac{P}{\rho} = \frac{RT}{M} \Rightarrow \frac{1}{\rho} \propto T$. As $\left(\frac{1}{\rho}\right)$ increases, so T increases and hence U increases. Also ρT is constant, so P is constant. ***Ans.***

10. **(d)** $\frac{d}{dV}(VP^n)=0$ or $V\times nP^{n-1}\left(\frac{dP}{dV}\right)+P^n\times 1=0$

$\therefore$ $$\frac{dP}{-\left(\frac{dV}{V}\right)} = \frac{P}{n}$$

or $$B = \frac{P}{n}.$$ *Ans.*

11. **(b)** $\gamma=1+\frac{2}{f}=1+\frac{2}{6}=\frac{2}{3}$

$$\frac{\Delta W}{\Delta Q}=\left(\frac{\Delta Q-\Delta U}{\Delta Q}\right)=1-\frac{1}{\gamma}=1-\frac{3}{4}=\frac{1}{4}$$

$\therefore \Delta Q = 4(\Delta W)=4\times 25=100$ J. *Ans.*

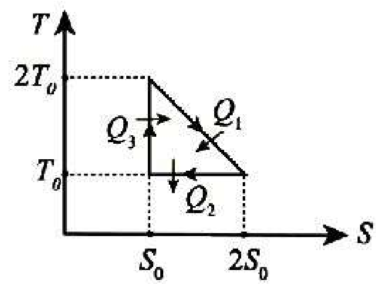
12. **(b)** Using, $P^{1-\gamma}T^{\gamma}$ = constant

or $P \propto T^{\left(\frac{\gamma}{\gamma-1}\right)}$

Here $P \propto T^3$

or $\frac{\gamma}{\gamma-1} = 3$ or $\gamma=\frac{3}{2}$.

13. **(a)** $$Q_1=\frac{2T_0+T_0}{2}\times S_0=\frac{3}{2}T_0S_0$$



$Q_2=T_0S_0$ and $Q_3=0$

$\therefore$ $$\eta=\frac{Q_1-Q_2}{Q_1}=1-\frac{2}{3}=\frac{1}{3}.$$

Ans.

14. **(c)** The change in internal energy depends only on initial and final state, so

$$\Delta U_1=\Delta U_2$$

or $Q_1-W_1=Q_2-W_2$. *Ans.*

15. **(b)** Let, Q_H : Heat input to I[st] engine

Q_L : Heat rejected from I[st] engine

Q_L : Heat rejected from II[nd] engine

Work done by I[st] engine = work done by II[nd] engine

$$Q_H-Q_L = Q_L-Q_L$$

$$2Q_L = Q_H+Q_L$$

$$2 = \frac{T_1}{T}+\frac{T_2}{T}$$

$$T = \frac{T_1+T_2}{2}$$ *Ans.*

Oscillations 13

TOPIC 13.1: *Linear and Angular SHM, Velocity and Acceleration in SHM, Time Period of Pendulum, Coupled Oscillator, Springs in Series and Parallel.*

Review of Formulae

1. **Differential equation**

   (a) Linear SHM :

$$F = -kx$$

$$\frac{d^2\vec{x}}{dt^2} + \frac{k\vec{x}}{m} = 0, \text{ here } k = \omega^2 m.$$

   (b) Angular SHM :

$$\tau = -c\theta$$

$$\frac{d^2\vec{\theta}}{dt^2} + \frac{c}{I}\vec{\theta} = 0, \text{ here } c = \omega^2 I$$

2. Time taken to travel from mean position to $\frac{A}{2}$ is $\frac{T}{12}$ and from $\frac{A}{2}$ to A will be $\frac{T}{6}$.

3. **Velocity of the particle :**

$$v = \omega A\cos(\omega t + \phi_0) = \omega\sqrt{A^2 - x^2}$$

   $v_{max} = \omega A$, at mean position.

   $v_{max} = 0$, at extreme positions.

4. **Acceleration of the particle :**

$$a = -\omega^2 A\sin(\omega t + \phi_0) = -\omega^2 x$$

   $|a_{max}| = \omega^2 A$, at extreme positions

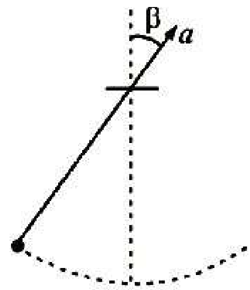
   $|a_{min}| = 0$, at mean position

5. Average K.E. of the period = Average P.E. = $\frac{1}{4}m\omega^2 A^2$.

6. Time period of simple pendulum

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

7. When point of suspension is accelerating, then



$$T = 2\pi\sqrt{\frac{\ell}{a_{net}}}$$

where $a_{net} = \sqrt{a^2 + g^2 + 2ag\cos\beta}$

8. Time period of a infinitely large pendulum

$$T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ minute}$$

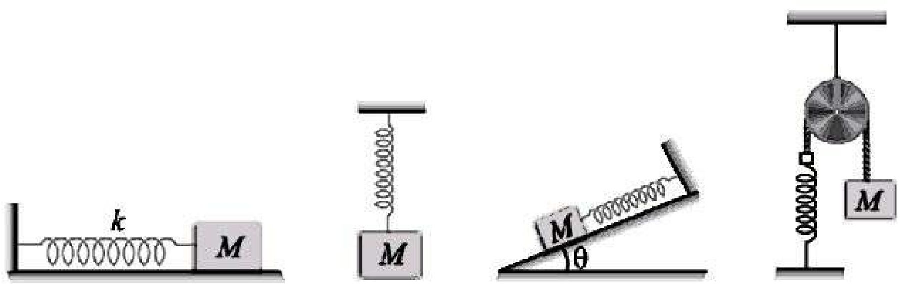
9. Time period of a body oscillating in a tunnel dug along the diameter of the earth

$$T = 2\pi\sqrt{\frac{R}{g}}$$

10. Time period of a physical pendulum

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

11. Time period of mass – springs shown in figures, $T = 2\pi\sqrt{\frac{M}{k}}$



12. Time period of torsional pendulum

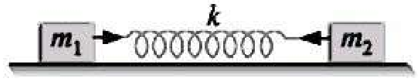
$$T = 2\pi\sqrt{\frac{I}{C}}$$

13. Time period of a floating body

$$T = 2\pi\sqrt{\frac{M}{A\rho_\ell g}}$$

14. Time period of coupled oscillator

$$\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right)$$



$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

15. Springs in series : $k \propto \frac{1}{\ell}$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + ... + \frac{1}{k_n}$$

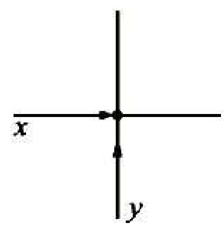
Springs in parallel : $k = k_1 + k_2 + ... + k_n$

16. Composition of two SHM^s of equal frequency in perpendicular directions.

$$x = a \sin \omega t$$

and $y = b \sin(\omega t + \phi)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\phi = \sin^2\phi$$



For $\phi = 0$ and π : straight line.

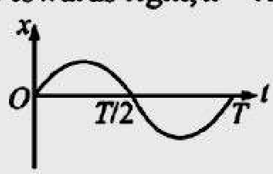
$\phi = \frac{\pi}{2}$: an ellipse

$\phi = \frac{\pi}{2}$, $a = b$: a circle.

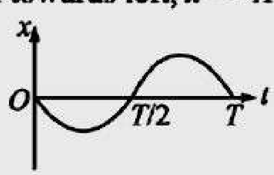
Tips and Tricks for Shortcut Solutions

1. When particle starts oscillating:

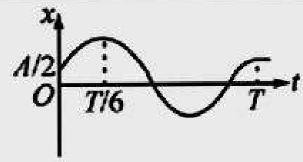
   (i) from mean position towards right, $x = A \sin \omega t$



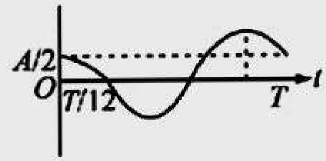
   (ii) from mean position towards left, $x = -A \sin \omega t$



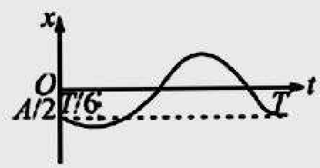
   (iii) from, $x = +\frac{A}{2}$ towards right, $x = A\sin\left(\omega t + \frac{\pi}{6}\right)$



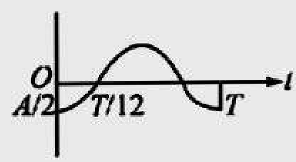
(iv) from, $x = +\frac{A}{2}$ towards left, $x = A\sin\left(\omega t + \frac{5\pi}{6}\right)$



(v) from, $x = -\frac{A}{2}$ towards left, $x = A\sin\left(\omega t - \frac{5\pi}{6}\right)$



(vi) from, $x = -\frac{A}{2}$ towards right, $x = A\sin\left(\omega t - \frac{\pi}{6}\right)$



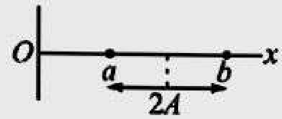
2. If U_0 is the PE of the oscillating particle at mean position, then its total mechanical energy, $E = \frac{1}{2} m\omega^2 A^2 + U_0$. Then $U_{max} \neq K_{max}$. If $U_0 = 0$, then $U_{max} = K_{max} = \frac{1}{2} m\omega^2 A^2$.

3. The position at which KE and PE of the oscillating particle are equal is, $x = \pm\frac{A}{\sqrt{2}}$ and at $t = \frac{T}{8}, \frac{3T}{8}$. Also we can say that at this position KE and PE, each one is half the maximum value.

4. The frequency of KE or PE is twice the frequency of oscillations.

5. The energy of oscillations, $= K_{max} = \frac{1}{2} m\omega^2 A^2$.

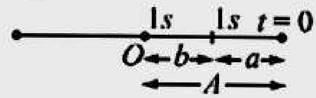
6. If a particle oscillates between $x = +a$ to $x = +b$, its amplitude of motion, $A = \left(\frac{b-a}{2}\right)$.



If particle oscillates between $x = +a$ and $x = -b$, then $A = \left(\frac{a+b}{2}\right)$.

7. If a particle oscillates in a straight line, and travels distances a and b in successive seconds, starting from rest, then its amplitude of motion.

$$A = \left(\frac{2a^2}{3a-b}\right).$$



8. If particle has velocities v_1 and v_2 at x_1 and x_2 from mean position, then its time period $T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$.

9. The amplitude of motion of $x = A + B \sin \omega t$ will be B.

10. The amplitude of motion of $x = A \sin^2 \omega t$ will be $\frac{A}{2}$.

11. The amplitude of, $x = A \sin \omega t + B \sin(\omega t + \theta)$, will be $\sqrt{A^2 + B^2 + 2AB\cos\theta}$.

12. The equation, $x = A \sin \omega t + B \cos \omega t = \sqrt{A^2 + B^2} \sin\left(\omega t + \tan^{-1}\frac{B}{A}\right)$.

Illustration 1

A particle is moving such that its displacement along x-axis as a function of time is given by $x(x-6) = 1 - 10 \cos \omega t$. Find mean position, amplitude of motion and time period.

Short-cut solution :

Given, $x(x-6) = 1 - 10 \cos \omega t$

or $x^2 - 6x + 9 = 10 - 10 \cos \omega t$

or $(x-3)^2 = 10(1 - \cos \omega t)$

or $(x-3)^2 = 5 \sin^2 \frac{\omega t}{2}$

$\therefore$ $x = 3 + \sqrt{5} \sin \frac{\omega t}{2}$.

On comparing with $x = x_0 + b \sin \omega t$, we have

$x_0 = 3$, and amplitude $b = \sqrt{5}$.

Time period, $T = \dfrac{2\pi}{\frac{\omega}{2}} = \dfrac{4\pi}{\omega}$. ***Ans.***

Illustration 2

Find amplitude of motion of the following motions.

(i) $x = 3 + 4 \sin \omega t$

(ii) $x = \cos^2 \omega t - \sin^2 \omega t$

(iii) $x = \sin^2 \omega t$

(iv) $x = a \sin \omega t + b \cos \left(\omega t + \frac{\pi}{2}\right)$.

(v) $x = A \sin \omega t + B \sin \left(\omega t + \frac{\pi}{3}\right)$

Short-cut solution :

(i) In this motion (SHM), mean position $x_0 = 3$ and amplitude is $A = 4$ unit.

(ii) $x = \cos^2 \omega t - \sin^2 \omega t$

or $x = \cos 2\omega t$. It represents SHM of amplitude $A = 1$, and $T = \dfrac{2\pi}{2\omega} = \dfrac{\pi}{\omega}$.

(iii) $x = \sin^2 \omega t = \dfrac{1 - \cos 2\omega t}{2} = \dfrac{1}{2} - \dfrac{1}{2} \cos 2\omega t$. It is SHM with amplitude $A = \dfrac{1}{2}$

and $T = \dfrac{2\pi}{2\omega} = \dfrac{\pi}{\omega}$.

(iv) $x = a \sin \omega t + b \cos \left(\omega t + \frac{\pi}{2}\right)$

$= a \sin \omega t - b \sin \omega t$

$= (a - b) \sin \omega t$

$\therefore \quad A = (a - b)$ and $T = \dfrac{2\pi}{\omega}$.

(v) $x = A \sin \omega t + B \sin \left(\omega t + \frac{\pi}{3}\right)$

Amplitude,
$$a = \sqrt{A^2 + B^2 + 2AB \cos \frac{\pi}{3}}$$
$$= \sqrt{A^2 + B^2 + 2AB \times \frac{1}{2}}$$
$$= \sqrt{A^2 + B^2 + AB}.$$
Ans.

Illustration 3

The displacement of a particle is given at time t as $x = A\sin(-2\omega t) + B\sin^2\omega t$. Find its amplitude of motion.

Short-cut solution :

We know that $\sin^2\omega t = \frac{1}{2}(1-\cos 2\omega t)$, so

$$x = A\sin(-2\omega t) + \frac{B}{2}(1-\cos 2\omega t)$$

$$= -(A\sin 2\omega t + \frac{B}{2}\cos 2\omega t) + \frac{B}{2}$$

The motion of the particle is SHM, with an amplitude $\sqrt{A^2+\left(\frac{B}{2}\right)^2}$. ***Ans.***

Illustration 4

Which of the following is not SHM?

(i) $y = a\sin 2\omega t + b\cos 2\omega t$

(ii) $y = a\sin 2\omega t + b\cos\omega t$

(iii) $y = \sqrt{a^2+b^2}\sin\omega t\cos\omega t$

(iv) $y = 1 - 2\sin^2\omega t$.

Short-cut solution :

(i) $y = a\sin 2\omega t + b\cos 2\omega t$

The above equation can be written in the form

$y = A\sin(\omega' t + \phi)$.

$= A\sin\omega' t\cos\phi + A\cos\omega' t\sin\phi$

On comparing, $A\cos\phi = a$ and $A\sin\phi = b$, $\omega' = 2\omega$

$\therefore \quad A = \sqrt{a^2+b^2}$ and $\tan\phi = \frac{b}{a}$.

or $\quad y = \sqrt{a^2+b^2}\sin(2\omega t + \tan^{-1}(b/a))$.

It represent SHM.

(ii) $y = a\sin 2\omega t + b\cos\omega t$. This cannot be represent in the form $y = A\sin(\omega t + \phi)$, so not SHM.

(iii) $y = \sqrt{a^2+b^2}\sin\omega t\cos\omega t$

$= \sqrt{\frac{a^2+b^2}{2}}\sin^2\omega t$. It represents SHM.

(iv) $y = 1 - 2\sin^2\omega t$

$= \cos 2\omega t$.

It represents SHM. ***Ans.***

Illustration 5

Two SHM's, $x_1 = A \sin \omega t$ and $x_2 = A \cos \omega t$ are superposed on a particle of mass 'm'. Find total mechanical energy of the particle.

Short-cut solution :

$$x = x_1 + x_2 = A \sin \omega t + A \cos \omega t$$

$$= \sqrt{2}A \sin\left(\omega t + \frac{\pi}{4}\right)$$

Total ME, $$E = \frac{1}{2} m\omega^2 \left(\sqrt{2}A\right)^2 = m\omega^2 A^2.$$ ***Ans.***

Illustration 6

A particle of mass m is executing oscillations about the origin on the x-axis. Its PE varies with position as, $U(x) = k|x|^3$, here k is a constant. The amplitude of oscillation is 'a', then how does its time period T vary with amplitude?

Short-cut solution :

Given, $$U = k|x|^3, \text{ but } U = \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t$$

Putting $|x| = a$ and $A = a$, we get

$$ka^3 \propto m\omega^2 a^2$$

or $$\omega \propto \sqrt{a}$$

and $$T \propto \frac{1}{\sqrt{a}}.$$ ***Ans.***

Illustration 7

A particle moves simple harmonically in a straight line starting from rest. In first t second it travels a distance 'a' and in next t second it travels a distance '2a' in the same direction, find its time period.

Short-cut solution :

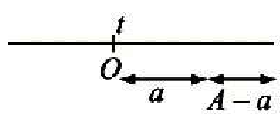
Using, amplitude $$A = \left[\frac{2a^2}{3a - b}\right]$$

$$= \frac{2a^2}{3a - 2a} = 2a.$$

As particle starts from rest, so

$$x = A \cos \omega t$$

$$= 2a \cos \omega t$$



or $$A - a = 2a \cos\left(\frac{2\pi t}{T}\right)$$

or $$2a - a = 2a \cos \frac{2\pi t}{T}.$$

or $$\cos\left(\frac{2\pi t}{T}\right) = \frac{1}{2}$$

or $$\frac{2\pi t}{T} = \frac{\pi}{3}$$

$\therefore$ $$T = 6t.$$ ***Ans.***

Illustration 8

The displacement of a particle executing periodic motion is given by, $y = 4\cos^2\left(\frac{t}{2}\right)\sin(1000\,t)$. Find the independent constituent of SHM.

Short-cut solution :

$$y = 4\cos^2\left(\frac{t}{2}\right)\sin(1000\,t)$$

$$= 2\cos^2\frac{t}{2} \times 2\sin 1000\,t$$

$$= 2(1 + \cos t)\sin(1000\,t)$$

$$= 2\sin 1000\,t + (2\sin 1000\,t\cos t)$$

$$= 2\sin 1000\,t + \sin 1001\,t + \sin 999\,t$$

So it is the combinations of three SHM^s. ***Ans.***

Illustration 9

A particle of mass 0.1 kg executes SHM under a force $F = -10x$ (N). Speed of the particle at mean position is 6 m/s. Find its amplitude.

Short-cut solution :

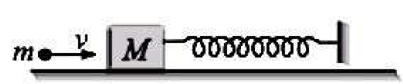
On comparing with $F = -kx$, we have $k = 10$ Nm

If A is the amplitude then,

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2$$

or $$A = v\sqrt{\frac{m}{k}} = 6\sqrt{\frac{0.1}{10}} = 0.6 \text{ m}.$$ ***Ans.***

Illustration 10



A block of mass M is connected to a massless spring of constant k, whose other end is connected to the wall as shown in figure. A bullet of mass m strikes to the block will velocity v and embedded is it, find amplitude and time period of resulting motion

Solution :

Using, conservation of momentum

$$mv + 0 = (m + M)v'$$

or

$$v' = \left(\frac{mv}{m+M}\right)$$

If A is the amplitude of motion, then

$$\frac{1}{2}kA^2 = \frac{1}{2}(m+M)v'^2$$

or

$$A = \sqrt{\frac{m+M}{k}}\,v'$$

$$= \sqrt{\frac{m+M}{k}} \times \left(\frac{mv}{m+M}\right)$$

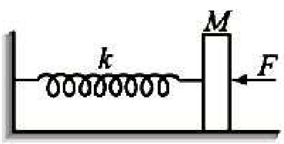
$$= \sqrt{\frac{m^2v^2}{k(m+M)}}. \qquad \textit{Ans.}$$

Time period,

$$T = 2\pi\sqrt{\frac{(m+M)}{k}}. \qquad \textit{Ans.}$$

Illustration 11

In figure a sharp blow by some external agent imparts a speed of 2 m/s to the block towards left when it is in equilibrium position. The PE of the spring when the block is at the right extreme is (k = 100 N/m, M = 1 kg and F = 10 N)



(a) ***4.5 J*** *(b)* ***4 J*** *(c)* ***0.5 J*** *(d)* ***2.5 J***

Short-cut solution :

If A is the amplitude of motion, then

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2$$

or

$$A = \sqrt{\frac{mv^2}{k}} = \sqrt{\frac{1\times 2^2}{100}} = 0.2 \text{ m.}$$

Compression of the spring due to F = 10 N.

$$x_0 = \frac{F}{k} = \frac{10}{100} = 0.1 \text{ N.}$$

Energy stored,

$$= \frac{1}{2}\times 100\times(0.2-0.1)^2 = 0.5 \text{ J.} \qquad \textit{Ans. (c)}$$

Illustration 12

A linear harmonic oscillator of force constant 2×10^6 N/m and amplitude 0.01 m has a total mechanical energy 160 J. Find maximum and minimum values of PE and kE.

Short-cut solution :

$$K_{max} = \frac{1}{2}kA^2 = \frac{1}{2}\times(2\times10^6)\times(0.01)^2$$

$$= 100 \text{ J}$$

Using, $K_{max} + U_{min} = 160$

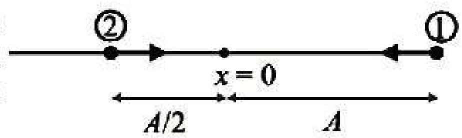
or $100 + U_{min} = 160$, $\therefore$ $U_{min} = 60$ J.

Maximum PE = total mechanical energy

$= 160$ J. ***Ans.***

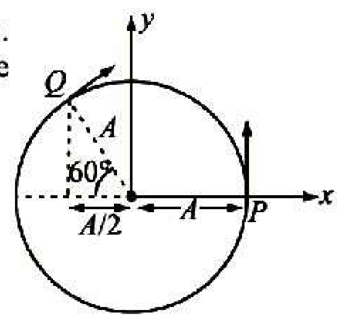
Illustration 13

Two particles are performing SHM in a straight line about the same equilibrium point. The amplitude and time period for both particles are same and equal to A and T respectively. At t = 0, one particle has displacement A while the other one has displacement –A/2, and they are moving towards each other. Find the time when they cross, each other.



Short-cut solution :

This problem can be solved by using a phasor diagram. The particles are located at P and Q as shown in the diagram.



The angle to be covered by the particles,

$$\theta = 120° = \frac{2\pi}{3} rad$$

Their relative angular velocity = 2ω.

So time to cross each other, $t = \frac{\theta}{2\omega} = \frac{2\pi/3}{2\times\left(\frac{2\pi}{T}\right)} = \frac{T}{6}$. ***Ans.***

Illustration 14

Two particles are executing SHM with amplitude A and frequency ω along the x-axis. Their mean positions are separated by distance $x_0 (x_0 > A)$. If maximum separation between them is $(x_0 + A)$. The phase difference between their motion is:

(JEE Main 2011)

Short-cut solution :

$$x_1 = A \sin \omega t \text{ and } x_2 = x_0 + A \sin(\omega t + \phi)$$

$$\therefore \quad x_2 - x_1 = x_0 + [A \sin (\omega t + \phi) - A \sin \omega t]$$

$$= x_0 + 2A \sin \frac{\phi}{2} \cdot \cos\left(\omega t + \frac{\phi}{2}\right)$$

The distance between then is changing harmonically with an amplitude $x_0 + 2A \sin \frac{\phi}{2}$.

Given, $$x_0 + 2A \sin \frac{\phi}{2} = x_0 + A$$

or $$\sin \frac{\phi}{2} = \frac{1}{2}$$

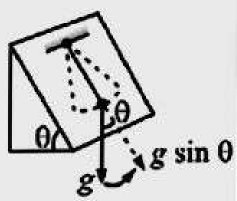
or $$\phi = \frac{\pi}{3}.$$ ***Ans.***

TIPS & TRICKS Trips and Tricks in Simple Pendulum and Physical Pendulum

1. Time period of a pendulum with point mass bob $T = 2\pi\sqrt{\frac{\ell}{a_{net}}}$ and time period of physical pendulum $T = 2\pi\sqrt{\frac{I}{mdg}}$. Here '$d$' is distance of CG from point of suspension (ℓ also the same).

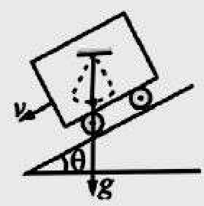
2. The value of a_{net} in different cases as follows:

   (i) Pendulum oscillating of smooth inclined plane

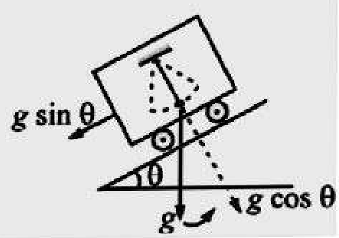
   $$T = 2\pi\sqrt{\frac{\ell}{g \sin\theta}}.$$

   (ii) Pendulum in a cart moving on an inclined with constant velocity

   $$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

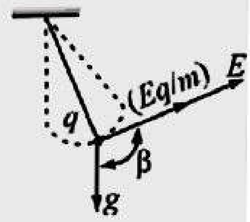
   (iii) Pendulum in a cart moving freely with acceleration $g \sin \theta$

   $$T = 2\pi\sqrt{\frac{\ell}{g \cos\theta}}.$$

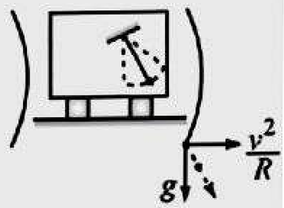
(iv) Oscillations of charged bob in uniform $\vec{E}$-field

$$T = 2\pi\sqrt{\frac{\ell}{\left[g^2 + \left(\frac{Eq}{m}\right)^2 + 2g\left(\frac{Eq}{m}\right)\cos\theta\right]}}$$



(v) Pendulum in a cart moving on a circular path of large radius,

$$T = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2 + \left(\frac{v^2}{R}\right)^2}}}.$$



(vi) Pendulum oscillates in a lift

$$T = 2\pi\sqrt{\frac{\ell}{(g \pm a)}}$$; Use $+a$ for acceleration upwards.

Time period of simple pendulum inside satellite will be infinite.

3. Pendulum oscillates in non-viscous liquid of density $\rho_\ell = \frac{\text{bob}}{\ }$ $(\eta > 1)$:

$$T = 2\pi\sqrt{\frac{\ell}{g\left(1 - \frac{1}{\eta}\right)}}.$$

4. If α is coefficient of linear expansion of the string of the pendulum, then due to change in temperature Δt, the change in time period

$$\frac{\Delta T}{T} = \frac{\alpha(\Delta t)}{2}.$$

If temperature increases, there is loss in time and if temperature decreases, there is gain in time.

5. If two pendulums of different lengths start from same position, then they will start again from same position when

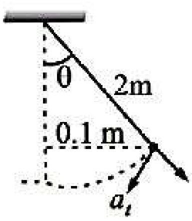
$$t = nT_1 = mT_2.$$

Here T_1 and T_2 are there time periods and n and m are their number of oscillations.

Illustration 15

A simple pendulum 2m long swings with an amplitude of 0.1 m. What is its acceleration at the extreme position? ($g = 10\ m/s^2$)

Short-cut solution :



At extreme position is only tangential acceleration, so

$$a_t = g\sin\theta = 10\times\frac{0.1}{2} = 0.5\ m/s^2.$$

Ans.

Illustration 16

Two pendulums of lengths 1 m and 1.44 m start swinging together. After how many oscillations will they again start swinging together?

Short-cut solution :

Using, $nT_1 = mT_2$

or $n\times 2\pi\sqrt{\frac{1}{g}} = m\times 2\pi\sqrt{\frac{1.44}{g}}$

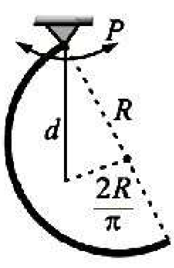
$\therefore$ $\frac{n}{m} = \frac{12}{10}$

So they swing together after shorter pendulum completes 12 oscillations and longer one 10. Thereafter 24 and 20, and so on. ***Ans.***

Illustration 17

A semicircular rigid wire of radius is smoothly pivoted at P. Find its time period of motion.

Short-cut solution :



Here $d = \sqrt{R^2+\left(\frac{2R}{\pi}\right)^2} = \frac{R}{\pi}\sqrt{\pi^2+4}$

$I = 2mR^2$ (MI of ring about tangent)

So, $T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{2mR^2}{mg\frac{R}{\pi}(\pi^2+4)}}$

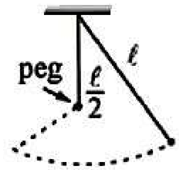
$$= 2\pi\sqrt{\frac{g\sqrt{\pi^2+4}}{2\pi R}}.$$

Ans.

Illustration 18

A simple pendulum is of length ℓ, but there is a peg at a distance $\frac{\ell}{2}$ below the point of suspension. Find time period of such a pendulum.

Short-cut solution :



$$T = \frac{T_1}{2} + \frac{T_2}{2}$$

$$= \left(\frac{2\pi\sqrt{\frac{\ell}{g}}}{2}\right) + \frac{\left(2\pi\sqrt{\frac{\left(\frac{\ell}{2}\right)}{g}}\right)}{2}$$

$$= \pi\left[\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{\sqrt{2}}\right)\right].$$ ***Ans.***

Illustration 19

The length of a pendulum is equal to the half the radius of earth. Find its time period.

Short-cut solution :

Using,

$$T = 2\pi\sqrt{\frac{R}{g\left(1+\frac{R}{\ell}\right)}}$$

$$= 2\pi\sqrt{\frac{R}{g\left(1+\frac{R}{\frac{R}{2}}\right)}}$$

$$= \frac{\left[2\pi\sqrt{\frac{R}{g}}\right]}{\sqrt{3}} = 49 \text{ min.}$$ ***Ans.***

TIPS & TRICKS Tips and Tricks in Spring and Other Devices

1. The time period of block-spring system is, $T = 2\pi\sqrt{\frac{M}{k}}$, which is independent on value of 'g', so it remains same on moon, inside satellite and inclined plane etc.

2. In the spring device, if body has translation only, then we can use

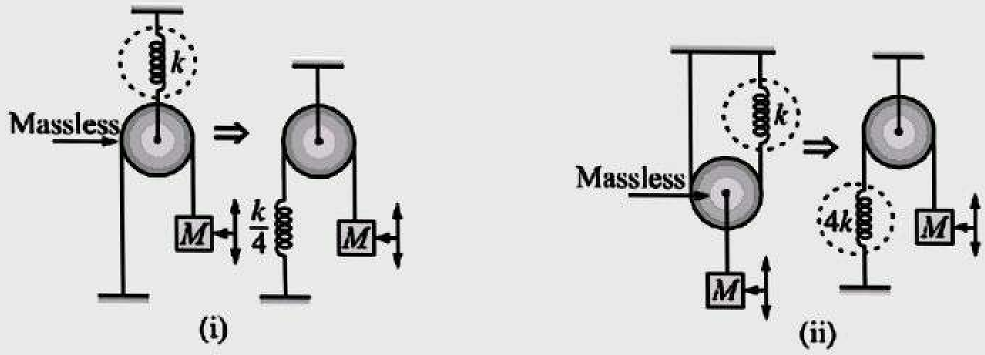
$$m \times a = -kx$$

If body has both translation and rotation, then

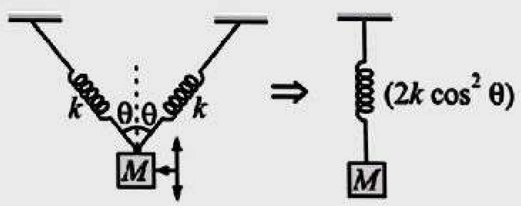
$$\left(m + \frac{I}{R^2}\right) \times a = -kx$$

3. Other simplifications

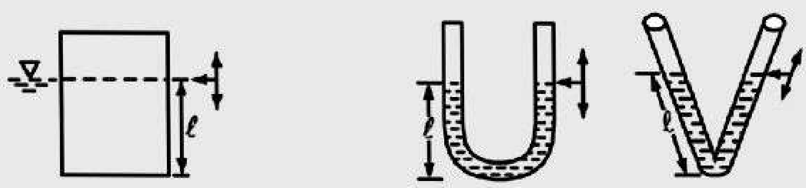
$$T = 2\pi\sqrt{\frac{M}{\left(\frac{k}{4}\right)}} \qquad T = 2\pi\sqrt{\frac{M}{4k}}$$



(i) (ii)

4. 

$$T = 2\pi\sqrt{\frac{M}{2k\cos^2\theta}}$$

5. 

Oscillations of floating body Liquid oscillates in U-tube or V-tube

These devices has, $T = 2\pi\sqrt{\frac{\ell}{g}}$.

6. Body oscillates in tunnel in earth, $T = 2\pi\sqrt{\frac{R}{g}}$.

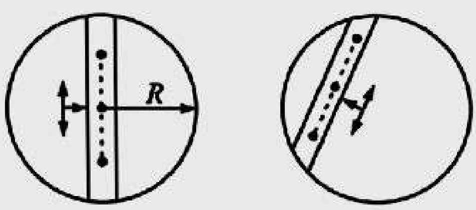
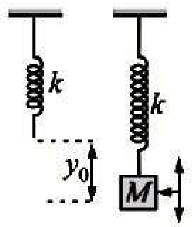


Illustration 20

A block of mass m is attached to the light spring of force constant k and released from its normal length. Find amplitude of subsequent motion.

Short-cut solution :

Using conservation of ME, we have



$$mgy = \frac{1}{2}kg^2$$

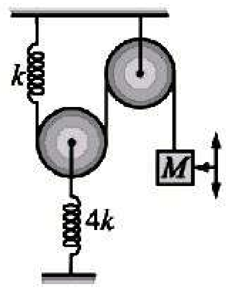
or
$$y = \frac{2mg}{k}$$

The equilibrium position is at $y_0 = \frac{mg}{k}$. So amplitude of motion becomes $\frac{2mg}{k} - \frac{mg}{k}$

$= \frac{mg}{k}$. ***Ans.***

Illustration 21

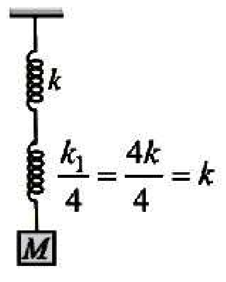
Find time period of oscillations of the vertical motion of the block in the device shown. The mass of the block is M and force constant of springs are shown in the figure. (pulleys are massless).



Short-cut solution :

The given device can be equivalent to the following device.

$$k_e = \frac{k \times k}{k + k} = \frac{k}{2}$$



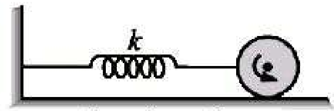
$\therefore$
$$T = 2\pi\sqrt{\frac{M}{\frac{k}{2}}}$$

$$= 2\pi\sqrt{\frac{2M}{k}}.$$ ***Ans.***

Illustration 22

In the device cylinder has mass m and radius R. There is sufficient friction between the surfaces to cause pure rolling of the cylinder. The spring connected is massless and force constant k. Find time period of motion of cylinder.

Short-cut solution :



Using, $\left(m+\frac{I}{R^2}\right)\times a = -kx$

$$\therefore \quad T = 2\pi\sqrt{\frac{\left(m+\frac{I}{R^2}\right)}{k}} = 2\pi\sqrt{\frac{m+\frac{mR^2}{2R^2}}{k}}$$

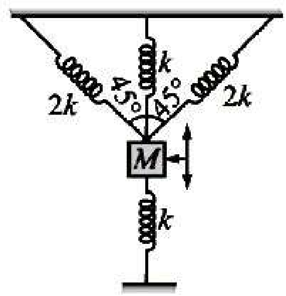
$$= 2\pi\sqrt{\frac{3m}{2k}}. \qquad \textit{Ans.}$$

Illustration 23

In the device shown, find time period of small vertical oscillations of the block. All the springs are massless and block is of mass M.

Short-cut solution :

The effective force constant



$$k_e = \underbrace{(k+k)}_{\text{Vertical springs}} + \underbrace{2(2k)\cos^2 45°}_{\text{Inclined springs}}$$

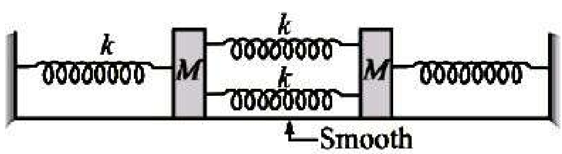
$$= 2k + 2\times 2k\times\frac{1}{2}$$

$$= 4k.$$

Time period, $T = 2\pi\sqrt{\frac{M}{4k}} = \pi\sqrt{\frac{M}{k}}.$ ***Ans.***

Illustration 24

In the device shown, each block is of mass M, and each spring is of force constant k. The blocks are pushed together and released. Find time period of oscillations.



Short-cut solution :

Reduced mass of the system,

$$\mu = \frac{m_1 m_2}{m_1+m_2} = \frac{M\times M}{M+M} = \frac{M}{2}$$

The effective force constant,

$$k_e = k + k + k + k = 4k.$$

Time period,

$$T = 2\pi\sqrt{\frac{\mu}{k_e}}$$

$$= 2\pi\sqrt{\frac{\frac{M}{2}}{4k}}$$

$$= \pi\sqrt{\frac{M}{2k}}. \qquad \textit{Ans.}$$

Video Solution

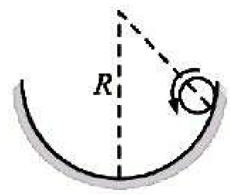
Q. Two solid spheres, each of mass 1 kg are connected to each other with the help of a massless spring of force constant 70 N/m. The spheres can roll without slipping along the horizontal plane. If both the spheres are pulled apart and released they start oscillating. Find time period of their oscillations.

To see the video solution, scan the QR code:

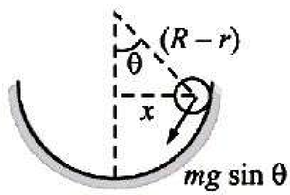
OR Visit https://www.youtube.com/watch?v=LcHEA55Y9S8

Illustration 25

A spherical ball of mass m and radius r rolls without slipping on a rough concave surface of large radius R. It makes small oscillations about the lowest point. Find the time period.



Short-cut solution :



Using,
$$\left(m + \frac{I}{R^2}\right) \times a = -mg \sin\theta$$

or
$$a = \frac{-mg\left(\frac{x}{R-r}\right)}{m + \frac{I}{R^2}}$$

$$= \frac{-mg\left(\frac{x}{R-r}\right)}{\left(m + \frac{2}{5}\frac{mR^2}{R^2}\right)}$$

$$= \frac{5g(-x)}{7(R-r)}$$

On comparing with, $a = -\omega^2 x$, we get

$$\omega = \sqrt{\frac{5g}{7(R-r)}}$$

and $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7(R-r)}{5g}}.$$ ***Ans.***

TOPIC 13.2: *Damped and Forced Oscillations.*

Review of Formulae

1. **Damped oscillator :** Differential equation of damped oscillator

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

where $x = A_0 e^{-bt/2m} \sin(\omega_d t + \phi)$

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

2. Differential equation of forced oscillations

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin \omega t$$

$$A = \frac{F_0/m}{\sqrt{(\omega_0{}^2 - \omega_d{}^2)^2 + \left(\frac{b\omega}{m}\right)^2}}, \text{ where } \omega_0 = \sqrt{\frac{k}{m}}$$

Illustration 26

A mass of 2 kg oscillates on a spring with force constant 50 N/m. By what factor does the frequency of oscillations decrease when a damping force with b = 12 unit is introduced?

Short-cut solution :

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50}{2}} = 5 \text{ rad/s}$$

$$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{6}{2m}\right)^2} = \sqrt{5^2 - \left(\frac{12}{2\times 2}\right)^2}$$

$$= 4 \text{ rad/s}$$

So $$\frac{\omega_d - \omega_0}{\omega_0} \times 100 = \frac{5-4}{5} \times 100$$

$$= 20\%$$ ***Ans.***

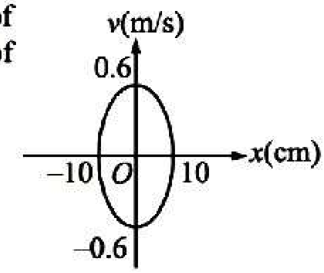
Concept Booster Exercise

1. The displacement of an object attached to a spring and executing SHM is given by, $x = 0.02 \cos \pi t \ m$. The time at which the maximum speed first occurs is:

Numeric/Integer

(a) $0.5\ s$ (b) $1.0\ s$ (c) $1.5\ s$ (d) $2\ s$

2. Figure shows the graph of velocity versus displacement of a particle executing SHM. Find period of oscillations of the particle.



(a) $\pi\ s$ (b) $\frac{\pi}{2} s$

(c) $\frac{\pi}{3} s$ (d) $\frac{\pi}{4} s$

3. The PE of a harmonic oscillator of mass 2 kg at its mean position is 5 J. If its total energy is 9 J and its amplitude is 0.01 m, find its period.

(a) $\frac{\pi}{100} s$ (b) $\frac{\pi}{50} s$ (c) $\frac{\pi}{200} s$ (d) $\pi\ s$

4. An object of mass 4 kg is moving along x-axis and its PE as a function of x varies as $U(x) = 4(1 - \cos 2x)\ J$, the time period of small oscillations is:

(a) $\pi\ s$ (b) $2\pi\ s$ (c) $4\pi\ s$ (d) $2\ s$

5. A particle moves in the xy-plane according to the equation $\vec{r} = (\hat{i} + \hat{j})\ [A \sin \omega t + B \cos \omega t]$. Motion of the particle is:

(a) periodic (b) along a straight line

(c) SHM (d) all of these

6. A simple pendulum has a time period T when on earth surface and T' when on moon, then $\frac{T'}{T}$

Numeric/Integer

(a) 1 (b) 2 (c) $\sqrt{6}$ (d) $\sqrt{3}$

7. A particle moves such that its acceleration is given by $a = -\beta(x - 2)$, here β is positive constant and x is the position from origin. Time period of oscillations is:

(a) $2\pi\sqrt{\beta}$ (b) $\frac{2\pi}{\sqrt{\beta}}$ (c) $2\pi\sqrt{\beta + 2}$ (d) $\pi\sqrt{\frac{1}{2+\beta}}$

8. A particle of mass 4 kg is in a gravitational potential $V = (800\ x^2 + 150)$ J/kg. Its frequency of oscillations is:

(a) $\frac{5}{\pi}$ Hz (b) $\frac{20}{\pi}$ Hz (c) 50π Hz (d) $\frac{50}{\pi}$ Hz

9. A particle free to more along the x-axis has potential energy given by $U(x) = k[1 - e^{-x^2}]$ for $-\infty \le x \le +\infty$, where k is a positive constant of appropriate dimensions. Then

(a) at away from the origin, the particle is in unstable equilibrium

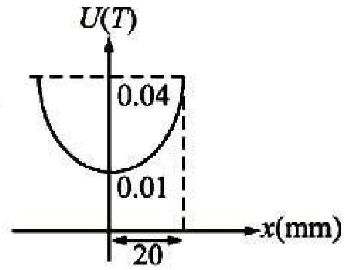
(b) for any finite non-zero value of x, there is a force directed away from the origin

(c) if its total ME is $\frac{k}{2}$, it has its minimum kinetic energy at the origin

(d) for small displacement from $x = 0$, the motion is SHM

10. The variation of PE (U) of a SHO is as shown. Then force constant of the system (PE is in J and displacement x in mm)

Numeric/Integer



(a) 100 N/m (b) 150 N/m

(c) $\frac{200}{3}$ N/m (d) 300 N/m

11. For a body in SHM the velocity is given by the relation $v = \sqrt{144 - 16x^2}$ m/s. The maximum acceleration is:

Numeric/Integer

(a) 12 m/s^2 (b) 16 m/s^2

(c) 32 m/s^2 (d) 48 m/s^2

12. The average KE of a simple harmonic oscillator is 2 J and its total energy is 5 J. Its minimum PE is:

Numeric/Integer

(a) 1 J (b) 1.5 J (c) 2 J (d) 3 J

Solutions

1. **(d)**

$$x = 0.02 \cos \pi t$$

$$\therefore \quad \omega = \pi \text{ and } T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2s$$

The maximum speed first will occur after

$$t = \frac{T}{4} = \frac{2}{4} = 0.5\, s.$$

2. **(c)**

$$v_{max} = \omega A$$

or $$0.6 = \frac{2\pi}{T} \times 0.10 \Rightarrow T = \frac{\pi}{3} s.$$

3. **(a)**

$$\text{Maximum } KE = TE - PE_{min}$$

or $$\frac{1}{2}kA^2 = 9 - 5$$

$$\therefore \quad k = \frac{4 \times 2}{0.01^2} = 8 \times 10^4 \text{ N/m}$$

Time period, $$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{8\times 10^4}} = \frac{\pi}{100} s.$$

4. (a) $$F = -\frac{dU}{dx} = -\frac{d}{dx}[4(1-\cos 2x)] = -8 \sin 2x$$

For small x, $\sin 2x = 2x$

$$F = -8 \times 2x$$

or $$a = \frac{F}{m} = -\frac{8\times 2x}{4} = -4x$$

On comparing with $a = -\omega^2 x$, we have

$$\omega = \sqrt{4} = 2 \text{ rad/s} \quad \text{and } T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi\, s.$$

5. (d) $$\vec{r} = (\hat{i}+\hat{j})\ (A \sin \omega t + B \cos \omega t)$$

$$= \sqrt{2}\sqrt{A^2+B^2}\ \sin\left(\omega t + \tan^{-1}\left(\frac{B}{A}\right)\right).$$

6. (c) $$T = 2\pi\sqrt{\frac{\ell}{g}} \text{ and } T' = 2\pi\sqrt{\frac{\ell}{\left(\frac{g}{6}\right)}} = \sqrt{6}\ T.$$

7. (b) $$a = -\beta x + 2$$

On comparing with, $a = -\omega^2 x$, we have

$$\omega = \sqrt{\beta}$$

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\beta}}.$$

8. (b) *PE*, $$U = mV = 4(800x^2 + 150)$$

$$F = -\frac{dU}{dx} = -4 \times 800 \times 2x = -6400x$$

Acceleration, $$a = \frac{F}{m} = \frac{-6400x}{4} = -1600x$$

On comparing with $a = -\omega^2 x$, we have

$$\omega = \sqrt{1600} = 40 \text{ rad/s}$$

Now, $$f = \frac{\omega}{2\pi} = \frac{40}{2\pi} = \frac{20}{\pi} \text{Hz}.$$

9. (d) $$F = \frac{-dU}{dx} = \frac{-d}{dx}[k(1-e^{-x^2})] = \frac{-2kx}{e^{x^2}}$$

For small value of x, $e^{x^2} \to 1$, so $F \propto (-x)$

It represents SHM.

10. (b) $$U = \frac{1}{2}kA^2$$

$$\therefore \quad k = \frac{2U}{A^2} = \frac{2\times(0.04-0.01)}{(0.02)^2} = 150 \text{ N/m.}$$

11. (d) $$v = \sqrt{144-16x^2} = 4\sqrt{9-x^2} = 4\sqrt{3^2-x^2}$$

On comparing with $v = \omega\sqrt{A^2-x^2}$, we have $\omega = 4$ and $A = 3$.

$$\therefore \quad a_{max} = \omega^2 A = 4^2 \times 3 = 48 \text{ m/s}^2.$$

12. (a) $$K_{max} = 2K_{av} = 2\times 2 = 4\text{J}$$

Now $K_{max} + U_{min} = 5 \Rightarrow U_{min} = 5 - K_{max} = 5 - 4 = 1$ J.

Wave Motion 14

TOPIC 14.1: Equations of Wave Motion, Sound Waves, Speed of Sound Waves in Air, Speed of Transverse Wave is Stretched String, Wave Intenstiy, Loudness of Sound and Doppler Effect.

Review of Formulae

1. **Other equations of wave**

   (i) $y = A\sin\frac{2\pi}{\lambda}(vt - x)$

   (ii) $y = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)$

   Differential equation of a wave : $\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$

2. **Phase difference, $\Delta\phi$**

   (i) Between two particles at any time

   $$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

   (ii) Between two times of a particle

   $$\Delta\phi = \frac{2\pi}{T}\Delta t$$

3. **Sound waves or pressure waves**

   $$\Delta P = \Delta P_m \sin(kx - \omega t),\ \Delta P_m = ABk$$

4. **Speed of sound waves in air**

   $$v = \sqrt{\frac{\gamma P}{\rho}},\ \text{for air } \gamma = 1.4.$$

   Also $$v = \sqrt{\frac{\gamma RT}{M}}$$

5. **Speed of transverse wave in stretched string**

   $$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{\text{stress}}{Y}}$$

6. Power transmitted

$$P_{av} = \frac{1}{2}\rho v S\omega^2 A^2$$

7. Intensity of wave

$$I = \frac{P}{S} = \frac{1}{2}\rho v\omega^2 A^2 = 2\pi^2 f^2 A^2 \rho v$$

8. Variation of intensity with distance

(i) For point source, $I \propto \frac{1}{r^2}$

(ii) For line source, $I \propto \frac{1}{r}$

9. Sound level : The decibel scale

$$\beta = 10\log\left(\frac{I}{I_0}\right) \text{ decibel}$$

and
$$\beta_2 - \beta_1 = 10\log\left(\frac{I_2}{I_1}\right)$$

10. Doppler effect in sound

(i) When source moves towards stationary observer

$$f' = f\left(\frac{v}{v - v_s}\right)$$

(ii) When observer moves towards stationary source

$$f' = f\left(\frac{v + v_0}{v}\right), \lambda' = \lambda$$

In general,

$$f' = f\left(\frac{v - v_0}{v - v_s}\right)$$

11. Doppler effect in light

$$\frac{\Delta\lambda}{\lambda} = \frac{-\Delta f}{f} = \frac{v}{c}$$

where $v \rightarrow$ relative speed between source and observer.

$c \rightarrow$ speed of light.

Frequency of sound as received by the observer himself after reflection from wall,

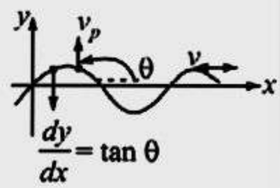
$$f' = f\left(\frac{v+u}{v-u}\right).$$

Here $$v_s = v_0 = u.$$

Tips and Tricks for Shortcut Solutions

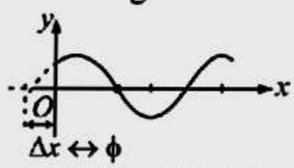
1. In wave equation, three quantities x, v and t must appear in combination of $(x + vt)$ or $(x - vt)$. Thus $y = x^2 + u^2t^2$, $y = (x + vt)^2$, $y = (x - vt)^2$ and $y = \sqrt{x - vt}$ represent one dimensional wave, but $y = x^2 - v^2t^2$ and $y = \sqrt{x} - \sqrt{vt}$ are not.

2. For any periodic function to represent a travelling wave it must satisfy the differential equation $\dfrac{\partial^2 y}{\partial t^2} = v^2 \dfrac{\partial^2 y}{\partial x^2}$.

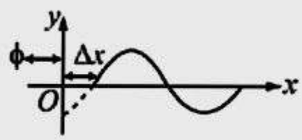


3. Particle velocity, $v_p = -v\left(\dfrac{dy}{dx}\right)$.

4. Positive and Negative initial phase



Positive initial phase
$y = A\sin(kx - \omega t + \phi)$



Negative initial phase
$y = A\sin(kx - \omega t - \phi)$

Illustration 1

Does the function $y = A\cos^2\left(\omega_0 t - \dfrac{2\pi x}{\lambda}\right)$ represent a wave? If yes, then find its amplitude and frequency.

Short-cut solution :

The given equation can be written in single power of 'cosine' *i.e.*,

$$y = A\cos^2\left(\omega_0 t - \frac{2\pi x}{\lambda}\right)$$

$$= \frac{A_0}{2}\left[1 + \cos 2\left(\omega_0 t - \frac{2\pi x}{\lambda}\right)\right]$$

$$= \frac{A_0}{2} + \frac{A_0}{2}\cos\left(2\omega_0 t - \frac{4\pi x}{A}\right).$$

Clearly the amplitude of the waves $\frac{A_0}{2}$ and angular frequency $2\omega_0$. ***Ans.***

Illustration 2

A progressive wave of frequency 500 Hz is travelling with a velocity of 360 m/s. Two points A and B are separated by a distance of 12 cm, find the phase difference between the two points.

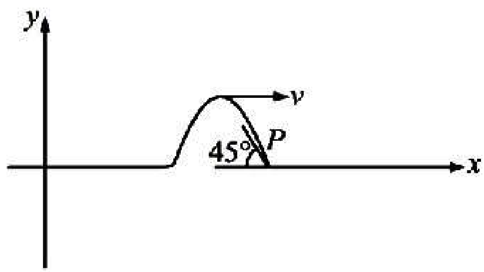
Short-cut solution :

$$\lambda = \frac{v}{f} = \frac{360}{500} = 0.72 \text{ m}$$

Phase difference, $$\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{0.72}\times 0.12 = \frac{\pi}{3} \text{ rad.}$$ ***Ans.***

Illustration 3

The wave pulse on a string shown in figure is moving to the right without changing the shape, with a velocity 2 m/s. Find velocity of a particle of the string at which slope of the y-x of the pulse is from 45° –ve x-axis.



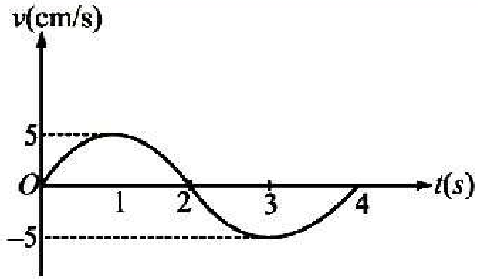
Short-cut solution :

The particle velocity

$$v_P = -v\left(\frac{dy}{dx}\right)$$
$$= -2\tan(135°)$$
$$= -2 \times (-1)$$
$$= 2 \text{ m/s.}$$ ***Ans.***

Illustration 4

A certain sinusoidal wave of wavelength 20 cm is moving in the positive x-direction. The velocity of the particle at x = 0 as a function of time is shown. Find the amplitude of motion.



Short-cut solution :

Time period of the particle, $T = 4$s, and so frequency,

$f = \frac{1}{T} = \frac{1}{4} Hz$, and $\omega = 2\pi f = \frac{2\pi}{4} = \frac{\pi}{2}$

We know that, maximum particle velocity $v_0 = \omega A$

or $$A = \frac{v_0}{\omega} = \frac{5}{\frac{\pi}{2}} = \frac{10}{\pi} \text{ cm.}$$ ***Ans.***

Illustration 5

The wave function of a pulse is given by

$$y = \frac{3}{(2x+3t)^2}$$

where x and y are in metre and t is in second. Determine the wave velocity of the pulse.

Short-cut solution :

Putting, $2x + 3t = \text{constant}$

or $$\frac{d}{dt}(2x+3t) = 0$$

or $$2\frac{dx}{dt}+3 = 0$$

$\therefore$ $$\frac{dx}{dt} = -\frac{3}{2} \text{ m/s.}$$ ***Ans.***

Illustration 6

A tuning fork of frequency 220 Hz produces sound waves of wavelength 1.5 m in air at STP. Calculate the increase in wavelength, when temperature of air is 27°C.

Solution :

Given $f = 220$ Hz, $\lambda_0 = 1.5$ m at $T_0 = 273$ K

Speed of sound at STP, $v_0 = f\lambda_0 = 220 \times 1.5 = 330$ m/s.

Final temperature, $T = 273 + 27 = 300$ K

Let v the speed of sound at this temperature, then

$$\frac{v}{v_0} = \sqrt{\frac{T}{T_0}}$$

$$\therefore \quad v = v_0\sqrt{\frac{T}{T_0}}$$

$$= 330\sqrt{\frac{300}{273}} = 346.1 \text{ m/s}$$

Final wavelength, $\lambda = \frac{v}{f} = \frac{346.1}{220} = 1.57$ m

The increase in wavelength $= \lambda - \lambda_0 = 1.57 - 1.50 = 0.07$ m. ***Ans.***

Illustration 7

The maximum pressure amplitude ΔP_m that the human ear can tolerate in loud sounds is about 28 Pa. What is the displacement amplitude A for such a sound in air of density $\rho = 1.21$ kg/m^3, at a frequency of 1000 Hz and a speed of 343 m/s?

Short-cut solution :

The pressure amplitude is related to displacement amplitude as

$$\Delta P_m = ABk$$

$$= Av^2\rho \times \frac{\omega}{v}$$

$$= Av\rho \times 2\pi f$$

$\therefore$ Displacement amplitude

$$A = \frac{\Delta P_m}{v\rho(2\pi f)}$$

$$= \frac{28}{343 \times 1.21 \times 2\pi \times 1000}$$

$$= 1.1 \times 10^{-5} \text{ m.}$$ ***Ans.***

Video Solution

Q. A sample of oxygen at NTP has volume V and a sample of hydrogen at NTP has volume 4V. Both the gases are mixed and the mixture is maintained at NTP. If the speed of sound in hydrogen at NTP is 1270 m/s, calculate the speed of sound in the mixture.

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=pkHDvwSebaI

Illustration 8

A line source of sound of length 10 m, emitting a pulse of sound that travels radially outward from the source. The power of the source is $P = 1.0 \times 10^4$ W. What is the intensity I of the sound when it reaches a distance of 10m from the source?

Short-cut solution :

The intensity at a distance r from a line source is given by

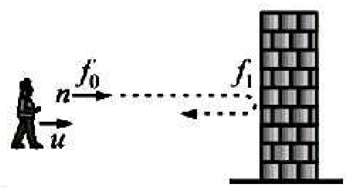
$$I = \frac{P}{2\pi r L}$$

$$= \frac{1.0\times10^4}{2\pi\times10\times10} = 15.92 \text{ W/m}^2. \qquad \textit{Ans.}$$

Illustration 9

A person sounds a whistle of frequency f_0 moves with a velocity 'u' towards a wall. What is the frequency detected (after reflection from wall). by the person himself speed of sound is v

Short-cut solution :



Frequency as received at wall $f_1 = f_0 \frac{v}{v-u}$

The frequency of reflected sound as detected by person

$$f_2 = f_1 \frac{v+u}{v} = f_0 \frac{v+u}{v-u}. \qquad \textit{Ans.}$$

Illustration 10

The sound emitted by a point source reaches a particular position with an intensity I. What is the change in intensity level at that position if N such sources are placed together. (Assuming that the sources do not produce interference).

Short-cut solution :

Using, $\qquad \beta_i = 10\log\frac{I}{I_0}$

and $\qquad \beta_f = 10\log\frac{NI}{I_0}.$

$\therefore \qquad \Delta\beta = 10\left[\log\frac{NI}{I_0} - \log\frac{I}{I_0}\right] = 10 \log N. \qquad$ *Ans.*

Illustration 11

The source is moving towards a stationary observer, plot variation $\frac{f'}{f}$ to $\frac{v_s}{v}$.

Short-cut solution :

Using, $$f' = f\frac{v}{v - v_s}$$

or $$\frac{f'}{f} = \frac{1}{\left(1 - \frac{v_s}{v}\right)}.$$ ***Ans.***

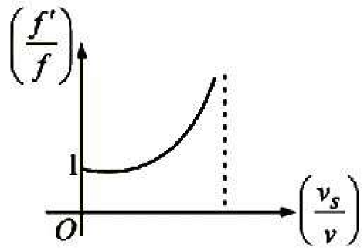


Illustration 12

In a mixture of gases, the average number of degrees of freedom per molecule is 6. The rms speed of the molecules of the gas is c. The velocity of sound in the gas is:

(a) $\frac{c}{\sqrt{2}}$ *(b)* $\frac{3c}{4}$ *(c)* $\frac{2c}{3}$ *(d)* $\frac{c}{\sqrt{3}}$

Short-cut solution :

Using, $$\gamma = 1 + \frac{2}{f} = 1 + \frac{2}{6} = \frac{4}{3}$$

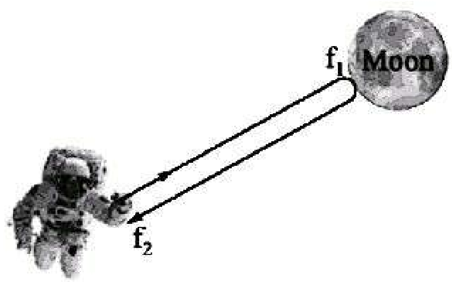
As $$C = \sqrt{\frac{3P}{\rho}} \text{ and } v = \sqrt{\frac{\gamma P}{\rho}}$$

$\therefore$ $$v = c\sqrt{\frac{\gamma}{3}} = c\sqrt{\frac{4/3}{3}} = \frac{2c}{3}.$$ ***Ans. (c)***

Illustration 13

An astronaut is approaching the moon. He sends a radio signal of frequency 5×10^9 Hz and finds that the frequency shift in echo received is 10^3Hz. Find his speed of approach.

Short-cut solution :



The frequency shift as observed on moon

$$\frac{\Delta f}{f} = \frac{v}{c} \Rightarrow \Delta f = \frac{v}{c}f$$

Now moon becomes source of frequency $f_1 = (f + \Delta f)$ the shift in frequency in reflected light is observed,

$$\Delta f = \frac{vf}{c}$$

Therefore total shift observed $= 2\Delta f = 2f\frac{v}{c}$

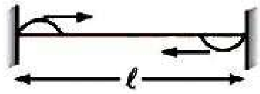
$\therefore$ $$10^3 = 2 \times 5 \times 10^9 \times \frac{v}{3 \times 10^8}$$

$\Rightarrow$ $$v = 30 \text{ m/s.}$$ ***Ans.***

Illustration 14

A string of length 0.4 m and mass 0.01 kg is clamped at its ends. The tension in the string is 1.6 N. When a pulse travels along the string, the shape of the string found to be same after a time t. Find value of t.

Short-cut solution :



Speed of pulse, $v = \sqrt{\frac{F}{\mu}}$

$$= \sqrt{\frac{1.6}{\left(\frac{0.01}{0.4}\right)}} = 8 \text{ m/s}$$

Required time $= \frac{2\ell}{v} = \frac{2\times0.4}{8} = 0.1s$ ***Ans.***

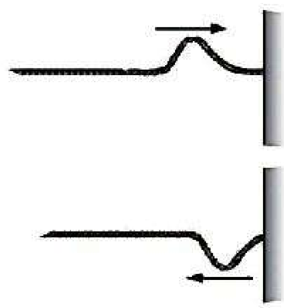
TOPIC 14.2: *Boundary Effects, Principle of Superposition, Interference, Beats, Stationary Waves, Resonance Tube, Vibration of Stretched Strings and Organ Pipes.*

Review of Formulae

1. Boundary effects :

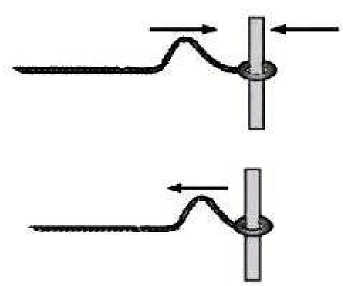
(i) When wave is reflected from rigid boundary, the reflected wave will suffer a phase change of π radian. Thus if

$$y_i = \sin(kx - \omega t), \text{ then}$$



$$y_r = A\sin[k(-x) - \omega t + \pi] = A\sin(kx + \omega t)$$

(ii) When wave is reflected from free boundary, the reflected wave suffers no phase change. Thus if



$$y_i = \sin(kx - \omega t), \text{ then}$$

$$y_r = A\sin[-kx - \omega t] = A\sin(kx + \omega t)$$

2. **Principle of superposition :** If $y_1, y_2, \ldots.. y_n$ are the displacements produced by waves acting separately, then the resultant displacement

$$y = y_1 + y_2 + \ldots\ldots + y_n.$$

There are three types of superpositions.

3. **Interference :** When two or more waves of same frequency travel simultaneously in the same direction or nearly along the same direction in a medium, they superpose on each other and give rise new disturbance is called interference.

If A_1 and A_2 are the amplitudes of the interfering waves, then resultant amplitude

$$R = A_1^2 + A_2^2 + 2A_1A_2 \cos\phi$$

(a) **Constructive interference :**

$$\phi = 2\pi n,\ n = 0, 1, 2, \ldots.$$

or

$$\Delta x = n\lambda$$

$$R_{max} = (A_1 + A_2)$$

(b) **Destructive interference :**

$$\phi = (2n-1)\pi,\ n = 1, 2, \ldots..$$

or

$$\Delta x = (2n-1)\frac{\lambda}{2}$$

$$R_{min} = A_1 - A_2$$

The ratio of maximum and minimum intensities

$$\frac{I_{max}}{I_{min}} = \frac{(A_1 + A_2)^2}{(A_1 - A_2)^2}$$

4. In Quinke's tube, each x cm slide of tube will cause a path difference 2x. Thus for a maxima and next minima

$$\frac{\lambda}{2} = x \quad \text{or} \quad \lambda = 4x.$$

5. **Beats :** In the superposition of two waves of slightly different frequencies

$$y_1 = A_1 \sin 2\pi f_1 t$$

and

$$y_2 = A_1 \sin 2\pi f_2 t$$

number of maximum per second are $f_1 \sim f_2$, and number of minimum per second are $f_1 \sim f_2$.

Beats frequency, $\Delta f = f_1 \sim f_2$.

6. **Stationary waves :**

(a) Consider two waves travelling from opposite directions

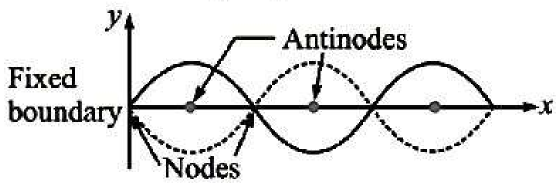
$$y_1 = A \sin(kx - \omega t)$$

and

$$y_2 = A \sin(kx + \omega t)$$

The resultant wave

$$y = y_1 + y_2 = [2A \sin kx] \cos \omega t$$



Positions of nodes : $x = 0, \frac{\lambda}{2}, \lambda, ...$

Positions of antinodes : $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, ...$

(b) When wave is reflected from free boundary, then

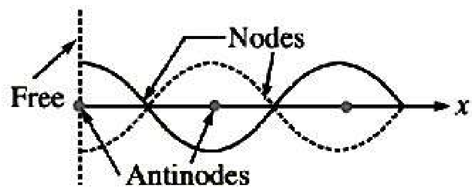
$$y_1 = A \sin(kx - \omega t)$$

and $$y_2 = A \sin(-kx - \omega t) = -A \sin(kx + \omega t)$$

The resultant wave

$$y = [2A \cos kx] \sin \omega t$$

Positions of nodes : $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, ...$



7. **Resonance tube :** It is used to determine the speed of sound in air.

For the two consecutive resonances

$$L_1 + e = \frac{\lambda}{4} \quad \text{and} \quad L_2 + e = \frac{3\lambda}{4}$$

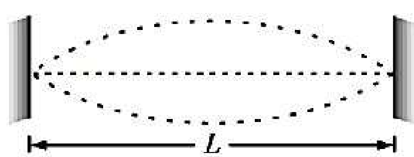
$$\therefore \qquad \lambda = 2(L_2 - L_1) \quad \text{and} \quad e = \frac{(L_2 - 3L_1)}{2}$$

Speed of sound $v = f\lambda = 2f(L_2 - L_1)$

8. **Vibrations of stretched string :**

(a) Frequency of fundamental note

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$



(b) If string vibrates in P loops, then

$$f_p = \frac{P}{2L}\sqrt{\frac{F}{\mu}}$$

9. Organ pipes

(a) Open organ pipe : The frequency of fundamental note

$$f = \frac{v}{2L}$$

Harmonics of frequencies ratio 1 : 2 : 3 : ..... are possible.

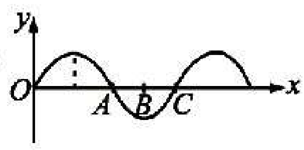
(b) Close organ pipe : The frequency of fundamental note

$$f = \frac{v}{4L}$$

Harmonics of frequencies ratio 1 : 3 : 5 : ..... are possible.

Illustration 15

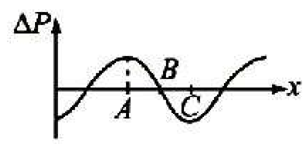
Figure shows an instantaneous displacement position graph of a sound wave travelling along the positive x-axis. Identify the points of maximum and minimum pressure.



Short-cut solution :

The change in pressure is given by

$$\Delta P = -B\left(\frac{dy}{dx}\right).$$



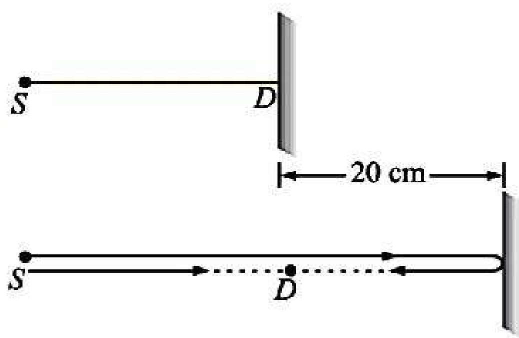
The slope $\frac{dy}{dx}$ is negative at A and positive at C so,

$\Delta P = +ve$ at A and '$-ve$' at C. Point B is the point of normal pressure. The variation of pressure change (ΔP) is shown in figure. ***Ans.***

Illustration 16

A source of sound S and a detector D are placed at some distance from one another. A big cardboard is placed near the detector and perpendicular to the line SD as shown in the Figure. It is gradually moved away and it is found that the intensity changes from maximum to minimum as the board is moved through a distance of 20 cm. Find the frequency of the sound emitted. The velocity of sound in air is 336 m/s.

Short-cut solution :



The distance between maximum to next minimum is λ/2. When cardboard is displaced by 20 cm the path difference produces by 40 cm.

$\therefore \quad \lambda/2 = 40$ cm

or $\quad \lambda = 80$ cm

Frequency of sound $f = \frac{v}{\lambda} = \frac{336}{0.80}$

$= 420$ Hz. *Ans.*

Illustration 17

A tuning fork of unknown frequency gives 4 beats with a tuning fork of frequency 310 Hz. It gives the same number of beats on filing. Find the unknown frequency.

Solution :

The unknown frequency of the tuning fork can be

$= 310 \pm 4$

or $f = 314$ or 306 Hz

Suppose $f = 314$ Hz

On filing, let it becomes $= 318$ Hz.

When it sounded together with a fork of frequency 310 Hz, beats frequency will be more than 4 per second. Therefore unknown frequency can not be 314 Hz.

Now suppose $f = 306$ Hz.

On filing, let it becomes $= 314$ Hz.

When it sounded again with a fork of frequency 310 Hz it gives 4 beats per second. So unknown frequency must be 306 Hz. *Ans.*

Illustration 18

A string under a tension of 129.6 N produces 10 beats/s when it is vibrated along with a tuning fork. When the tension in the string is increased to 160 N, it sounds in unision with the same tuning fork. Calculate the fundamental frequency of the tuning fork.

Short-cut solution :

Suppose f be the frequency of the tuning fork. The frequency of the string will be either $(f - 10)$ or $(f + 10)$. With the increase in tension its frequency becomes f, so initial frequency of the string is $(f - 10)$. Thus

For $F = 129.6\ N$, $(f - 10) = \frac{1}{2L}\sqrt{\frac{129.6}{\mu}}$ ... (i)

and for $f = 160\ N$, $f = \frac{1}{2L}\sqrt{\frac{160}{\mu}}$ ... (ii)

After solving equations (i) and (ii), we get

$f = 100$ Hz. *Ans.*

Illustration 19

The first overtone of an organ pipe beats with the first overtone of a close organ pipe with a beat frequency of 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz. Find the lengths of the pipes. Velocity of sound in air = 330 m/s.

Short-cut solution :

Suppose L_o and L_c are the lengths of open and close pipes respectively.

Frequency of first overtone of open organ pipe,

$$f_o = \frac{2v}{2L_o} = \frac{v}{L_o}$$

Frequency of first overtone of close organ pipe

$$f_c = \frac{3v}{4L_c}$$

Given $$f_0 - f_c = 2.2 \text{ Hz}$$

$$\therefore \quad \frac{v}{L_o} - \frac{3v}{4L_c} = 2.2$$

As $\frac{v}{4L_c} = 110$ Hz and $v = 330$ m/s

$$\therefore \quad \frac{330}{L_o} - 3 \times 110 = 2.2$$

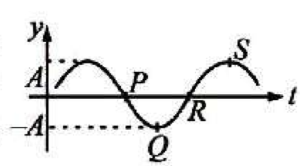
or $$L_o = 0.99\text{m} = L_c = 1.01 \text{ m}.$$ ***Ans.***

Concept Booster Exercise

1. An open organ pipe of length 1m contains an ideal gas whose density is twice the density of atmosphere at STP. Find the difference between fundamental and second harmonic frequencies if speed of sound in atmosphere is 300 m/s. ***[JEE Main 2020]***

Numeric/Integer

2. A wave motion has the function $y = A\sin(\omega t - kx)$. The graph shows how the displacement y at a fixed point varies with time t, which one of the labelled points shows a displacement equal to that at the position $x = \frac{\pi}{2}k$ at time $t = 0$?



(a) P (b) Q (c) R (d) S

3. A sine wave has an amplitude A and wavelength λ. Let v is the wave velocity and v_0 is the maximum part velocity in the medium:

(a) v cannot be equal to v_0
(b) $v = v_0$, if $A = \frac{\lambda}{2\pi}$
(c) $v = v_0$, if $A = 2\pi\lambda$
(d) $v = v_0$, if $\lambda = \frac{A}{\pi}$

4. A travelling wave pulse is given by $y = \frac{6}{2+(x+3t)^2}$ where symbols have their usual meanings x and y are in m and t is in s

(a) The amplitude of the wave pulse is 6 m
(b) The amplitude of wave pulse is 3m
(c) The pulse is travelling along negative x-axis with velocity 3 m/s
(d) The pulse is travelling along positive x-axis with 2 m/s

5. A metal string is fixed between rigid supports. It is initially at negligible tension. Its Young's modulus is Y, density ρ and coefficient of thermal expansion is α. If it is now cooled through a temperature t, transverse waves will move along it with speed.

(a) $\sqrt{\frac{Y\alpha t}{\rho}}$
(b) $Y\sqrt{\frac{\alpha t}{\rho}}$
(c) $\alpha t\sqrt{\frac{Y}{\rho}}$
(d) $t\sqrt{\frac{Y\alpha}{\rho}}$

6. A bus is moving with a velocity of 5m/s towards a large wall. The driver sounds a horn of frequency 165 Hz. If the speed of sound in air = 335 m/s, the number of beats heard per second by a passenger on the bus will be **Numeric/Integer**

(a) 3
(b) 4
(c) 5
(d) 6

7. Two identical sounds A and B reach a point in the same phase. The resultant sound is C. The loudness of C is n dB higher than the loudness of A. The value of n is: **Numeric/Integer**

(a) 2
(b) 3
(c) 5
(d) 6

8. A pipe of length 1m is closed at one end. The velocity of sound in air is 300 m/s. The air column in the pipe will not resonate for sound of frequency **Numeric/Integer**

(a) 75 Hz
(b) 225 Hz
(c) 300 Hz
(d) 375 Hz

9. Two sources A and B are sounding notes of frequency 680 Hz. An observer 'O' moves from A to B with a constant velocity v_0. If the speed of sound is 340 m/s, what must be the value of u so that he identifies 10 beats per second? **Numeric/Integer**

(a) 2.0 m/s
(b) 2.5 m/s
(c) 3.0 m/s
(d) 4.0 m/s

10. Sound of wavelength λ passes through a Quincke's tube, which is adjusted to give a maximum intensity I_0. Through what distance should the sliding tube be moved to give an intensity $\frac{I_0}{2}$?

(a) $\frac{\lambda}{8}$
(b) $\frac{\lambda}{4}$
(c) $\frac{\lambda}{2}$
(d) λ

11. A string of length L is stretched along the x-axis and is rigidly clamped at its two ends. It undergoes transverse vibrations. If n is an integer, which of the following may represent the shape of the string at anytime?

(a) $y = A\sin\left(\frac{n\pi x}{L}\right)\cos\omega t$ (b) $y = A\cos\left(\frac{n\pi x}{L}\right)\sin\omega t$

(c) $y = A\sin\left(\frac{n\pi x}{L}\right)\sin\omega t$ (d) $y = A\cos\left(\frac{n\pi x}{L}\right)\cos\omega t$

12. A standing sound wave in a pipe has five displacement nodes and five antinodes. The harmonic number for the standing wave is **Numeric/Integer**

(a) 3 (b) 4 (c) 5 (d) 9

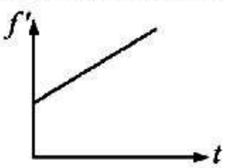
13. A source of sound placed at the open end of a resonance column sends an acoustic wave of pressure amplitude P_0 inside the tube. If the atmospheric pressure is P_A, then the ratio of maximum and minimum pressure at the closed end of the tube will be

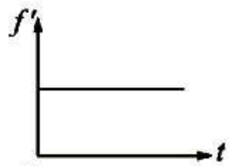
(a) $\frac{(P_A+P_0)}{(P_A-P_0)}$ (b) $\frac{(P_A+2P_0)}{(P_A-2P_0)}$ (c) $\frac{P_A}{P_A}$ (d) $\frac{\left(P_A+\frac{1}{2}P_0\right)}{\left(P_A-\frac{1}{2}P_0\right)}$

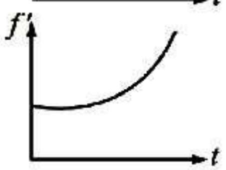
14. A string is rigidly tied at two ends and its equation of vibration is given by $y = \cos 2\pi t \sin 2\pi x$. Then minimum length of string is **Numeric/Integer**

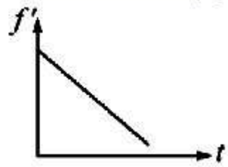
(a) 1 m (b) $\frac{5}{2}$ m (c) 5 m (d) 2π m

15. A source of frequency f is stationary and an observer starts moving towards it at $t = 0$ with constant small acceleration. Then the variation of observed frequency f' registered by the observer with time is:

(a)


(b)


(c)


(d)


Solutions

1. **(105.75)**

$$V = \sqrt{\frac{B}{\rho}}$$

$$\frac{V_{pipe}}{V_{air}} = \frac{\sqrt{\frac{B}{2\rho}}}{\sqrt{\frac{B}{\rho}}} = \frac{1}{\sqrt{2}}$$

$$V_{pipe} = \frac{V_{air}}{\sqrt{2}}$$

$$f_n = \frac{(n+1)V_{pipe}}{2\ell}$$

$$f_1 - f_0 = \frac{V_{pipe}}{2\ell} = \frac{300}{2\sqrt{2}} = 105.75 \text{ Hz (If } \sqrt{2} = 1.41)$$

$$= 106.05 \text{ Hz (If } \sqrt{2} = 1.414).$$ ***Ans.***

2. **(b)** $y = A\sin(\omega t - kx)$

$$= A\sin\left(0 - k \times \frac{\pi}{2k}\right) = A\sin\left(\frac{-\pi}{2}\right) = -A.$$

$\longrightarrow$ Point Q.

3. **(b)** We know that, $v = f\lambda$ and $v_0 = \omega A = 2\pi f A$.

If we put $A = \frac{\lambda}{2\pi}$, we find $v = v_0$.

4. **(b, c)** $y = \frac{6}{2 + (x+3t)^2}$. For maximum value of y, $x + 3t = 0$ or $y_{max} = A = - = 3$m.

On comparing $(x + 3t)$ with $(x - vt)$, we find $v = -3$ m/s.

5. **(a)** $v = \sqrt{\frac{f}{\rho}} = \sqrt{\frac{Y\alpha t}{\rho}}$.

6. **(c)** $f' = f\left(\frac{v + v_0}{v - v_s}\right) = 165\left(\frac{335+5}{335-5}\right) = 170$ Hz

So number of beats, $f_b = 170 - 165 = 5$ Hz.

7. **(d)** $\Delta\beta = 10\log_{10}\frac{4I}{I} = 10\log_{10}{}^4 = 10\log_{10}2^2 = 10 \times 2 \times \log_{10}{}^2$

$$= 20 \times 0.3 = 6.$$

8. **(c)** $f = \frac{v}{4\ell} = \frac{300}{4 \times 1} = 75$ Hz.

The resonance frequencies are in the ratio 1 : 3 : 5.

9. **(b)** Using, $\Delta f = f\left[\frac{v + v_0}{v} - \frac{v - v_0}{v}\right]$

or $10 = \frac{2fv_0}{v}$

or $10 = \frac{2 \times 680 \times v_0}{340} \Rightarrow v_0 = 2.5$ m/s.

10. **(a)** If I is the intensity of sound, then

$$I_0 = 4I, \text{ when } \phi = 0$$

and $$\frac{I_0}{2} = I + I + \sqrt{II}\cos\phi$$

or $$2I = 2I + I\cos\phi \Rightarrow \cos\phi = 0 \text{ or } \phi = \frac{\pi}{2}$$

$\therefore$ $$\Delta x = \frac{\pi}{4}.$$

To cause path difference of $\frac{\pi}{4}$ the tube must be slide by half of Δx, so $\frac{\pi}{8}$.

11. **(a, c)** At rigid supports $x = 0, y = 0$, so $\sin\frac{n\pi x}{L}$ satisfies the given conditions.

12. **(d)** Equal number of nodes and antinodes occur in close pipe. It is corresponding to ninth harmonic.

13. **(a)** The maximum pressure is = $P_0 + P_A$, at the closed end and minimum pressure is $P_A - P_0$, at the open end.

14. **(b)** On comparing with standard equation, we get

$$\omega = 2\pi \text{ and } k = 2\pi$$

or $$\frac{2\pi}{\lambda} = 2\pi, \therefore \lambda = 1 \text{ m.}$$

Thus $$\ell = \frac{\lambda}{2} = 0.5 \text{ m.}$$

So the length of the string will be in integral multiple of 0.5, which is 2.5 m.

15. **(a)** Using, $$f' = f\frac{v + v_0}{v} = f\left(\frac{v + at}{v}\right).$$

It represents a straight line with positive slope.

Electrostatics 15

TOPIC 15.1: *Coulomb's Law, Electrostatic Field and Potential, Gauss's Law, Electric Flux, Energy Density in Electric Field and Electric Dipole.*

Review of Formulae

1. **Coulomb's law in vector form :** If charges q_1 and q_2 are placed at the position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively, then force between them

$$\vec{F}_{21} = \frac{1}{4\pi \epsilon_0} \frac{|q_1||q_2|(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}.$$

2. If the space of thickness t between the charges is filled with a dielectric of dielectric constant k, then

$$F = \frac{1}{4\pi \epsilon_0} \frac{|q_1||q_2|}{\left[(r-t)+t\sqrt{k}\right]^2}$$

3. **Relationship between $\vec{E}$ and V :**

$$V_A - V_B = -\int_B^A \vec{E} \cdot d\vec{r}$$

$$= \int_A^B \vec{E} \cdot d\vec{r}$$

Also $\quad E = \dfrac{-dV}{dr}$

For constant field

$$V_A - V_B = Ed$$

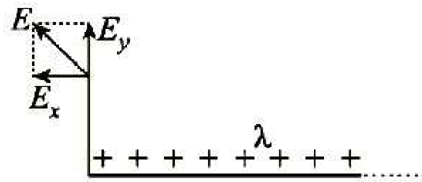
4. **Work done from A to B :**

$$W_A^B = (V_B - V_A)q$$

5. **Gauss's law :** According to it, the total electric flux through a Gaussian surface is equal to $\left(\frac{1}{\epsilon_0}\right)$ times the charge inside the close surface. Thus

$$\oint \vec{E} \cdot d\vec{A} = \left(\frac{1}{\epsilon_0}\right) q_{in}.$$

6. Line charge with linear charge density λ



$$E = \frac{\lambda}{2\pi \epsilon_0 r},$$

at the middle of the line charge

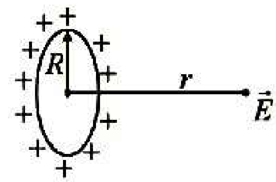
E along the end of the line charge

$$E_x = E_y = \frac{\lambda}{4\pi \epsilon_0 r}$$

$$E = \sqrt{E_x^2 + E_y^2}$$

$$= \frac{\lambda}{2\sqrt{2}\pi \epsilon_0 r}$$

7. Ring of radius R with a uniformly distributed charge q :



$$E = \frac{1}{4\pi \epsilon_0} \frac{qr}{\left(R^2 + r^2\right)^{3/2}}$$

$$E = 0, \quad (r = 0)$$

$$E_{\max} = \frac{q}{6\sqrt{3}\, \pi \epsilon_0 R^2} \qquad \left(r = \frac{R}{\sqrt{2}}\right)$$

$$V = \frac{1}{4\pi \epsilon_0} \frac{q}{\sqrt{R^2 + r^2}} \qquad (r > 0)$$

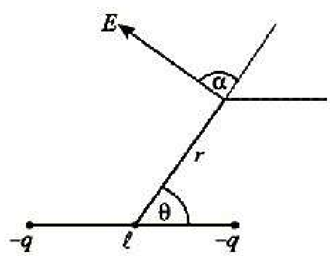
$$V = \frac{1}{4\pi \epsilon_0} \frac{q}{R} \qquad (r = 0)$$

8. Energy density in electric field :

$$u = \frac{1}{2} \epsilon_0 E^2 \qquad \text{(in free space)}$$

Electric Dipole : Dipole moment $\vec{P} = q\vec{\ell}$

Electric field at any position (r, θ) :

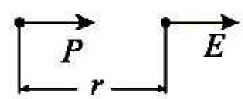


$$E = \frac{1}{4\pi \epsilon_0} \frac{P\sqrt{3\cos^2\theta + 1}}{r^3}$$

$$V = \frac{1}{4\pi \epsilon_0} \frac{P\cos\theta}{r^2}$$

$$\tan\alpha = \frac{\tan\theta}{2}$$

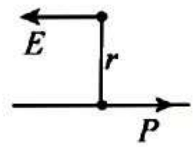
(i) Along the axis of the dipole, $\theta = 0$



$$E = \frac{1}{4\pi \epsilon_0} \frac{2P}{r^3}$$

$$V = \frac{1}{4\pi \epsilon_0} \frac{P}{r^2}.$$

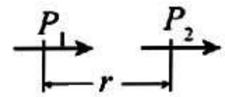
(ii) Along the equator of the dipole, $\theta = 90°$



$$E = \frac{1}{4\pi \epsilon_0} \frac{P}{r^3},$$

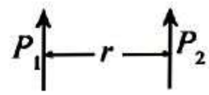
$$V = 0.$$

(iii) Force between two short dipoles placed on the same axis



$$F = \frac{1}{4\pi \epsilon_0} \frac{6P_1P_2}{r^4}.$$

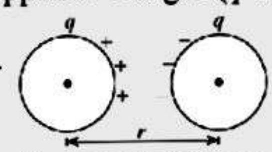
Force between two short dipoles placed parallel on different axes



$$F = \frac{1}{4\pi \epsilon_0} \frac{3P_1P_2}{r^4}$$

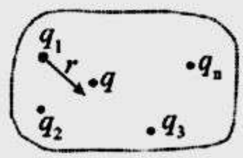
TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. If a metal sheet is placed between the charges, the force between the charges becomes zero, but there is net force on each charge because of induced charge on the sheet.
2. The force between two charged spheres with opposite charges (q and $-q$, separation r) will be more than, $F > \frac{1}{4\pi \epsilon_0} \frac{qq}{r^2}$.



3. A charged particle q is in equilibrium due to the forces of charges; q_1, q_2, ..., q_n. If now q_1 is removed, then the net force on q due to rest charges is equal to

$$F_{net} = \left(\frac{1}{4\pi \epsilon_0}\right) \frac{qq_1}{r^2}.$$



where r is the separation between the charges q and q_1.

4. The ratio of number of field lines emerge or terminate is equal to the ratio of the magnitude of charges, i.e.,

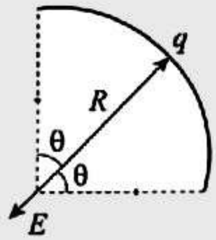
$$\frac{N_1}{N_2} = \frac{|q_1|}{|q_2|}.$$

5. The net electric field inside the metal body due to charges on the body and surrounding will always be zero. If $\vec{E}_1$ is the electric field inside, due to induced charges on the body and $\vec{E}_2$ due to all other charges, then

$$\vec{E}_1 + \vec{E}_2 = 0 \quad \text{or} \quad E_1 = E_2.$$

6. Electric field depends on the spread of charge while electric potential does not

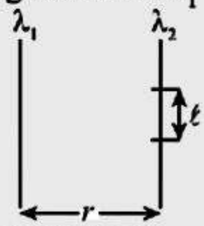
Electric field at the center of arc



$$E = \frac{1}{4\pi \epsilon_0} \frac{q}{R^2} \left(\frac{\sin \theta}{\theta_{rad}} \right)$$

and $$V = \frac{1}{4\pi \epsilon_0} \frac{q}{R}.$$

7. The force between two long charged threads with linear charge densities λ_1 and λ_2



$$F = E_1 q_2$$

$$= \frac{\lambda_1}{2\pi \epsilon_0 r} (\lambda_2 \ell).$$

8. The ratio of energy stored of a uniform charged sphere outside to inside

$$\frac{U_o}{U_i} = 5.$$

9. Electric flux associated by a closed surface

$$\phi = \frac{q_{in}}{\epsilon_0}.$$

or charge enclosed, $q_{in} = \epsilon_0 \phi$.

10. Energy stored by charged spherical conductor is, $U = \frac{1}{8\pi \epsilon_0} \frac{q^2}{R}$ and that of non-conductor, $U = \frac{3}{20\pi \epsilon_0} \frac{q^2}{R}$.

11. Force between two short dipole can be obtained by

$$F_{12} = P_1 \left(\frac{\partial E_2}{\partial r} \right).$$

12. The potential of earthed conductor will be zero, but charge may or may not be zero.

13. 

Flux associated to each of the closed surfaces is $\frac{q}{2 \epsilon_0}$.

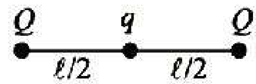
Illustration 1

A charge q is placed at the centre of the line joining two equal charges Q. If the system of charges be in equilibrium, then find minimum value of charge Q from the options given. (e → charge of an electron)

(a) *e* *(b)* *2e* *(c)* *4e* *(d)* *8e*

Short-cut solution :

As charge q is at the centre of the charges Q and Q, so it remains in equilibrium for any value of q. For Q to be in equilibrium,



$$\vec{F}_{Qq} + \vec{F}_{QQ} = 0$$

or $$\vec{F}_{QQ} = -\vec{F}_{Qq}$$

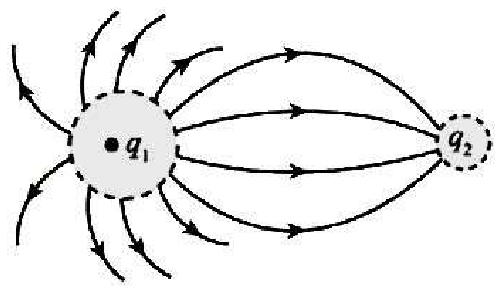
or $$\frac{QQ}{\ell^2} = \frac{Qq}{\left(\frac{\ell}{2}\right)^2}$$

$\therefore$ $$q = \frac{-Q}{4}$$

As minimum value of charge can be $\pm e$, so Q should be $4e$. ***Ans.(c)***

Illustration 2

The field lines of two charges are shown in figure. Determine the ratio $\frac{q_1}{q_2}$.



Short-cut solution :

The field lines emerge from q_1 are 12 while terminate towards q_2 are 4, so

$$\frac{q_1}{q_2} = \frac{12}{4} = 3.$$

The charge q_2 is negative. ***Ans.***

Illustration 3

An electric field is given as $\vec{E} = 2\hat{i} + 3\hat{j}$. ***Find the potential difference between two points A and B*** ($V_A - V_B$) ***whose position vectors are given by*** $\vec{r}_A = \hat{i} + 2\hat{j}$ ***and*** $\vec{r}_B = 2\hat{i} + \hat{j} + 3\hat{k}$.

Short-cut solution :

In uniform field, we may write

$$V_A - V_B = \vec{E}.\vec{d} = \vec{E}.(\vec{r}_A - \vec{r}_B)$$
$$= (2\hat{i}+3\hat{j}).\left[(\hat{i}+2\hat{j})-(2\hat{i}+\hat{j}+3\hat{k})\right]$$
$$= 1\text{ V.} \qquad \textit{Ans.}$$

Illustration 4

The electric field in the region is given by $\vec{E} = E_x\hat{i} + E_y\hat{j}$. If V_0 is the potential at the origin, then find potential at point (x, y).

Short-cut solution :

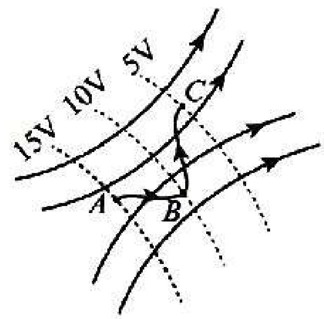
$$V_0 - V_{(x,\,y)} = \vec{E}.\vec{d} = (E_x\hat{i} + E_y\hat{j}).(x\hat{i} + y\hat{j})$$

or

$$V_{(x,\,y)} = V_0 - (E_x x + E_y y). \qquad \textit{Ans.}$$

Illustration 5

Figure shows field lines and equipotential by dashed lines. Calculate work done to move $-2\mu C$ charge from A to C by B.

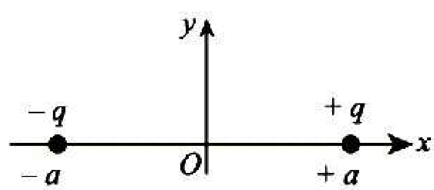


Short-cut solution :

$$W_{AC} = q(V_C - V_A)$$
$$= -2 \times 10^6\,(5 - 15)$$
$$= 2 \times 10^7\text{ J.} \qquad \textit{Ans.}$$

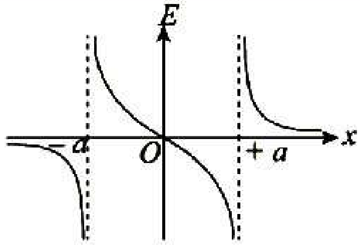
Illustration 6

Two identical positive charges each of q are placed on the x-axis at x = – a and x = + a as shown. Plot the variation of $\vec{E}$ and V along the x-axis.

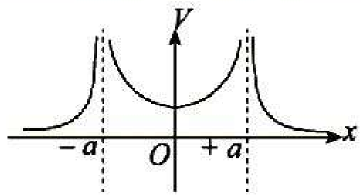


Short-cut solution :

The electric field will be zero at their centre and maximum on both sides of centre. The potential at their centre is minimum but not zero, and maximum on both sides of centre.



Variation of $\vec{E}$ with x



Variation of V with x

Illustration 7

A point charge Q is placed outside a hollow conductor of radius R, at a distance r (r > R) from its centre C. Find electric field at the centre C of the conductor due to charges induced on it.

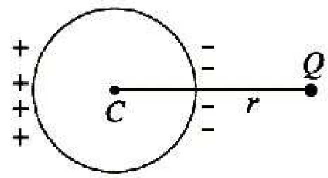
Short-cut solution :

If $\vec{E}_1$ is the electric field of charge Q at C and $\vec{E}_2$ due to induced charge, then

$$\vec{E}_1 + \vec{E}_2 = 0$$

$$\therefore \quad \vec{E}_2 = -\vec{E}_1$$

$$\text{or} \quad E_2 = E_1 = \frac{1}{4\pi \epsilon_0} \frac{Q}{r^2}.$$



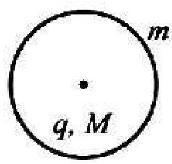
Ans.

Illustration 8

A charged ring has mass m and radius R. A point charge q and mass m is placed at its centre. If total potential energy of the system is zero, then find charge on the ring.

Short-cut solution :

If Q is the charge on the ring, then



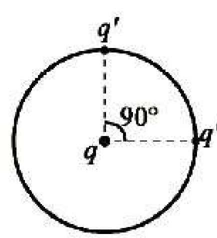
$$\left(\frac{1}{4\pi \epsilon_0}\right)\frac{Qq}{R} - \frac{GM}{R} = 0$$

$$\therefore \quad Q = \frac{4\pi \epsilon_0 Gm}{q}.$$

Ans.

Illustration 9

A point charge q is placed at the centre of a uniformly charge ring of total charge Q. Two small pieces, each with charge q' are removed from the ring as shown in the figure. Find net force on charge q due to the remaining ring.



Short-cut solution :

If $\vec{F}_1$ is force due to remaining ring and $\vec{F}_2$ and $\vec{F}_3$ by the removed charges then

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

or $$\vec{F}_1 = -(\vec{F}_2 + \vec{F}_3)$$

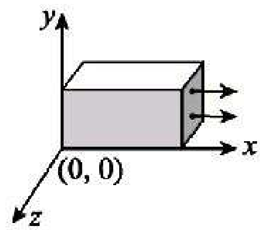
or $$F_1 = \sqrt{F_2^2 + F_3^2} = \sqrt{F^2 + F^2}$$

$$= \sqrt{2} \times F = \sqrt{2} \times \frac{1}{4\pi \in_0} \frac{qq'}{R^2}. \qquad \textit{Ans.}$$

Illustration 10

In a region of space, the electric field is in the x-direction and is given by $\vec{E} = E_0 x\hat{i}$. Consider an imaginary cubical volume of edge 'a' with edges parallel to the axes of coordinates. Find the charge inside the cube.

Short-cut solution :



At $x = 0, E = 0$

and $x = a, E = E_0 a$

Total flux, $\phi = 0 \times a^2 + E \times a^2 = Ea^2$

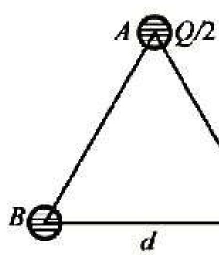
$= E_0 a \times a^2 = E_0 a^3$

Using Gauss's law $\quad q = \in_0 \phi = \in_0 E_0 a^3$. ***Ans.***

Illustration 11

Three identical metallic uncharged spheres A, B and C of radius a are kept on the corners of an equilateral triangle of side d(a << d), A fourth sphere radius 'a' has charge Q touches A and is then removed to a position far away. B is earthed and then the earth connection is removed, C is then earthed. Find the charge on C.

Short-cut solution :



When charge Q touches A, then

$$q_A = \frac{Q}{2}.$$

Now if charge on B is q_B, then its potential

$$0 = \frac{1}{4\pi \in_0} \left[\frac{q_B}{a} + \frac{Q/2}{d} \right]$$

$\therefore$ $$q_B = -\frac{Qa}{2d}.$$

As C is earthed

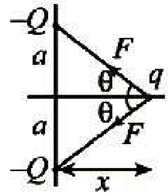
$$V_C = 0 = \frac{1}{4\pi \in_0} \left[\frac{q_C}{a} + \frac{q_B}{d} + \frac{q_A}{d} \right]$$

or $$0 = \left[\frac{q_C}{a} - \frac{Qa}{2d^2} + \frac{Q/2}{d}\right]$$

$\therefore$ $$q_C = -\frac{Qa}{2d}\left(\frac{d-a}{d}\right).$$ ***Ans.***

Illustration 12

Two point charges, each one –Q are fixed at (0, a) and (0, –a). Another point charge q and mass m is placed on x-axis at a small distance x (x << a). Find time period of oscillations of the charge q.



Short-cut solution :

Restoring force,
$$F_R = -2F\cos\theta$$
$$= -2\left(\frac{1}{4\pi\epsilon_0}\right)\frac{Qq}{(a^2+x^2)} \times \frac{x}{\sqrt{a^2+x^2}}$$

Acceleration, $$A = \frac{F_e}{m}$$
$$= \frac{1}{4\pi\epsilon_0}\frac{2Qq}{m(a^2+x^2)^{3/2}}(-x)$$

For $x << a$
$$A = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{2Qq}{ma^3}(-x)$$

On comparing with, $a = -\omega^2 x$
$$\omega = \sqrt{\left(\frac{1}{4\pi\epsilon_0}\right)\frac{2Qq}{ma^3}}$$

$\therefore$ $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4\pi\epsilon_0\, ma^3}{2Qq}}.$$ ***Ans.***

Illustration 13

In a parallel plate capacitor of capacitance C, the plate A carries a positive charge and the plate B carries a negative charge. The potential difference between the plates is V_0. If the plate A is given an additional charge +Q, then find the potential difference between the plates.

Short-cut solution :

When charge Q is given to the plate A, then charge appear on its one face, will be $Q/2$, therefore $V = V_0 + \frac{Q/2}{C} = V_0 + \frac{Q}{2C}$. ***Ans.***

TOPIC 15.2: Capacitance of a Capacitor, Parallel Plate, Spherical and Cylindrical Capacitor; Combination of Capacitors, Kirchhoff Laws of Capacitors and Charging and Discharging of a Capacitor.

Review of Formulae

1. **Capacitor and capacitance :** A capacitor is a device which is used to store electrical energy. A capacitor consists of two isolated conductors (plates) with equal and opposite charges $+q$ and $-q$. Its capacitance C is defined as :

$$C = \frac{q}{V},$$

where V is the potential difference between the plates. The SI unit of capacitance is the farad.

2. **Capacitance with a dielectric :** If the space between the plates of a capacitor is completely filled with a dielectric material, the capacitance C is increased by a factor k. Thus

$$C_{\text{med}} = k\,C_{\text{air}}.$$

In a region that is completely filled by a dielectric, all electrostatic equations containing ϵ_0 must be modified by replacing ϵ_0 with $k\,\epsilon_0$.

3. **Electric potential energy of the capacitor :** The electric potential energy U of a charged capacitor is given by

$$U = \frac{q^2}{2C} = \frac{1}{2}CV^2.$$

This energy is associated with the capacitor's electric field $\vec{E}$ between the plates. The energy density within an electric field of magnitude is given by

$$u = \frac{1}{2}\epsilon_0 E^2 \quad \text{(In vacuum)}$$

4. **Redistribution of charges :** When two charged conductors are connected by a conducting wire, charge flows from the conductor of higher potential towards the conductor at lower potential, until their potentials become equal. If q_1 and q_2 are the charges on the conductors of capacitances C_1 and C_2, then common potential

$$V = \frac{q_1 + q_2}{C_1 + C_2} = \frac{C_1V_1 + C_2V_2}{C_1 + C_2}$$

If q_1' and q_2' are the final charges on them, then

$$q_1' = C_1V \text{ and } q_2' = C_2V$$

$$\text{Loss of energy} = U_i - U_f$$

$$= \left(\frac{1}{2}C_1V_1^2 + \frac{1}{2}C_2V_2^2\right) - \frac{1}{2}(C_1 + C_2)V^2$$

5. Combinations of capacitors :

(i) In series : $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + ...$

(ii) In parallel : $C = C_1 + C_2 + ...$

6. Capacitance of spherical conductor of radius *R*

$$C = 4\pi \epsilon_0 R$$

7. Parallel plate capacitor :

$$C = \frac{\epsilon_0 A}{d} \quad \text{(Vacuum between the plates)}$$

$$C = \frac{k \epsilon_0 A}{d}$$

(When dielectric k is placed between the plates)

If t is the thickness of the dielectric placed between the plates, then

$$C = \frac{\epsilon_0 A}{\left[(d-t) + \frac{t}{k}\right]}$$

Force between the plates of capacitor $F = \frac{q^2}{2\epsilon_0} A.$

8. Spherical capacitor :

$$C = 4\pi \epsilon_0 \frac{ab}{b-a},$$

When outer shell is earthed

$$C = 4\pi \epsilon_0 \frac{b^2}{b-a},$$

When inner shell is earthed.

9. Cylindrical capacitor :

$$C = \frac{2\pi \epsilon_0 \ell}{\ell n b/a}.$$

10. Dielectrics :

(i) Induced charge $q' = q\left(1 - \frac{1}{k}\right)$

(ii) Polarisation $P = \frac{\text{Induced charge}}{\text{area}} = \frac{q'}{A}$

(iii) Electric displacement

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

(iv) Electric susceptibility

$$\alpha = \frac{\vec{P}}{E} = 1 + \frac{\chi}{\epsilon_0}$$

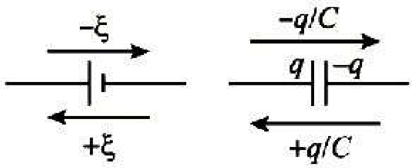
11. Kirchhoff's laws for capacitors :

(i) Junction rule : The algebraic sum of charges at any junction is equal to zero i.e., $\Sigma q = 0$.

(ii) Loop rule : The algebraic sum of the p.d. across all circuit elements is equal to zero i.e.,

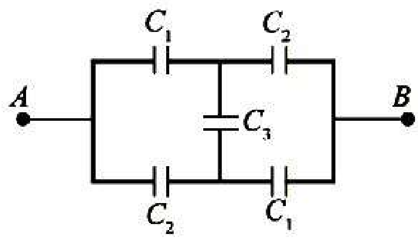
$$\Sigma\xi + \Sigma\frac{q}{C} = 0.$$

(iii) Sign conventions :



12. In unbalanced wheat stone bridge, the equivalent capacitance between *A* and *B*

$$C_{AB} = \left[\frac{2C_1C_2 + C_2C_3 + C_3C_1}{C_1 + C_2 + 2C_3}\right]$$



13. Charging of a capacitor through resistor :

Charge $q = q_0(1 - e^{-t/\tau})$

Current $i = i_0 e^{-t/\tau}$,

where $q_0 = C\xi$ and $\tau = CR$.

Potential $V = \xi(1 - e^{-t/\tau})$

Energy $U = U_0(1 - e^{-t/\tau})^2$

14. Discharging of a capacitor :

Charge $q = q_0 e^{-t/\tau}$,

Current $i = -i_0 e^{-t/\tau}$

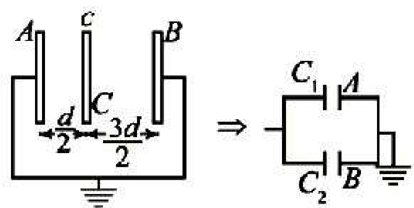
Potential $V = \xi e^{-t/\tau}$

Energy $U = U_0 e^{-2t/\tau}$

Illustration 14

A conducting plate C carrying a charge q is placed between two earthed plates as shown. Find charges on the plates A & B.

Short-cut solution :



If q_1 is the charge on plate A

q_2 on B, then $q_1 + q_2 = q$ ...(i)

and $\frac{q_1}{C_1} = \frac{q_2}{C_2}$

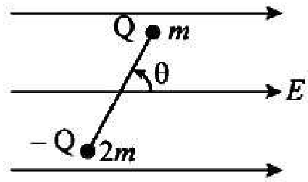
or $\frac{q_1}{\left(\frac{\epsilon_0 A}{d/2}\right)} = \frac{q_2}{\left(\frac{\epsilon_0 A}{3d/2}\right)}$ ...(ii)

On solving above equation, we get $q_1 = \frac{3q}{4}$ and $q_2 = \frac{q}{4}$. ***Ans.***

Illustration 15

A dipole consists of two particles, one with charge Q and mass m, and the other with charge –Q and mass 2m, separated by a distance L. For small oscillations about its equilibrium position, find the angular frequency when placed in a uniform electric field E.

Short-cut solution :



Restoring torque, $\tau = -PE\sin\theta$

or $I_{cm}\alpha = -PE(\theta)$

or $\alpha = \dfrac{PE(-\theta)}{I_{cm}}$

On comparing with $\alpha = -\omega^2\theta$, we have

$$\omega = \sqrt{\frac{PE}{I_{cm}}}$$

Here $P = QL$ and $I_{cm} = m\left(\frac{2L}{3}\right)^2 + 2m\left(\frac{L}{3}\right)^2 = \frac{2}{3}mL^2$

$$\therefore \quad \omega = \sqrt{\frac{3QE}{2mL}}. \qquad \textbf{\textit{Ans.}}$$

Illustration 16

Two isolated metallic solid spheres of radii R and 2R are charged such that both of these have same charge density σ. The spheres are located far away from each other, and connected by a thin conducting wire. Find the new charge density on the bigger sphere.

Short-cut solution :

After connection, the common potential

$$V = \frac{Q_1 + Q_2}{C_1 + C_2}$$

$$= \frac{[4\pi R^2 + 4\pi(2R)^2]\sigma}{4\pi\epsilon_0 R + 4\pi\epsilon_0 (2R)}$$

$$= \left(\frac{20\pi R^2\sigma}{4\pi\epsilon_0 R + 4\pi\epsilon_0 (2R)}\right)$$

$$= \frac{5}{3}\frac{R\sigma}{\epsilon_0}$$

The new charge on bigger sphere

$$Q'_2 = C_2V = 4\pi \epsilon_0 (2R) \times \left(\frac{5R\sigma}{3\epsilon_0}\right)$$

$$= \frac{40}{3}\pi R^2\sigma. \qquad \textbf{\textit{Ans.}}$$

Illustration 17

A conductor is charged from an electrophorous by repeated contacts with a plate which after each contact is recharged with a quantity Q of electricity from the electrophorous. If q_1 is the charge of the conductor after the first operation, calculate the ultimate charge.

Short-cut solution :

If q_1 is the charge on the conductor after first contact, then

$$\frac{q_1}{Q - q_1} = \frac{C_1}{C_2}$$

Let q_{max} be the maximum charge which can be given to the conductor. When potential of conductor becomes equal to the potential of the electrophorous plate, thereafter no charge flows between them, so

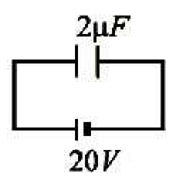
$$V_{Plate} = V_{Conductor}$$

$$\frac{Q}{C_2} = \frac{q_{max}}{C_1}$$

$$\Rightarrow \qquad q_{max} = \frac{C_1}{C_2}Q = \frac{q_1 Q}{Q - q}. \qquad \textbf{\textit{Ans.}}$$

Illustration 18

In figure a capacitor of capacitance 2μF is connected to a cell of emf 20 V. The plates of the capacitor are drawn apart slowly to double the distance between them. Calculate work done by the external agent on the plates.



Solution :

Using, $U_i + W_{battery} + W_{agent} = U_f + \text{loss}$

As the plates are pulled slowly, so loss = 0

$C_f = \frac{C}{2} = \frac{2}{2} = 1\mu F$. Charge flow $Q = Q_f - Q_i = 1 \times 20 - 2 \times 20 = -20\mu C$

$$\therefore \quad \frac{1}{2} \times 2 \times 20^2 - 20 \times 20 + W_{agent} = \frac{1}{2} \times 1 \times 20^2 + 0$$

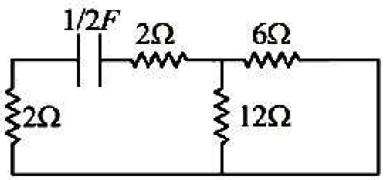
or $\qquad W_{agent} = 200\ \mu J. \qquad$ ***Ans.***

Illustration 19

Find the time constant of the circuits shown in figure.

Short-cut solution :

(a)



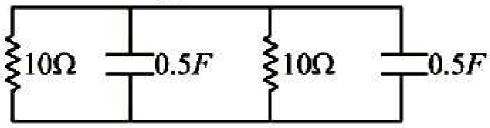
$$R_{net} = \frac{6\times12}{6+12}+2+2=8\,\Omega$$

$$\tau_{net} = R_{net}\,C_{net}$$

$$= 8\times\frac{1}{2}$$

$$= 4 \text{ s.}$$

(b)



$$C_{net} = 0.5 + 0.5 = 1F$$

$$R_{net} = \frac{10\times10}{10+10} = 5\,\Omega$$

$$\tau_{net} = R_{net}\,C_{net}$$

$$= 5 \times 1$$

$$= 5 \text{ s.}$$ ***Ans.***

Illustration 20

A capacitor of capacitance 1 μF can withstand a potential of 4000 V and another capacitor of 2μF can withstand a potential of 3000 V. Find the potential that they can carry when they connected in (i) series (ii) parallel.

Short-cut solution :

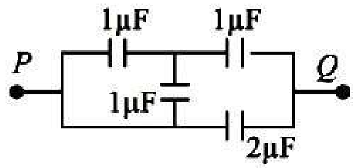
(i) $q_1 = C_1V_1 = 1 \times 4000 = 4000\ \mu C$
and $q_2 = C_2V_2 = 2 \times 3000 = 6000\ \mu C$
When connected in series, the least charge should be taken (because both can carry least charge), so

$$V = \frac{\text{least of } q_1 \text{ and } q_2}{C_{eq}} = \frac{4000}{2/3} = 6000 \text{ V.}$$ ***Ans.***

(ii) When connected in parallel, the least of the voltage should be taken, so $V = 3000$ V. ***Ans.***

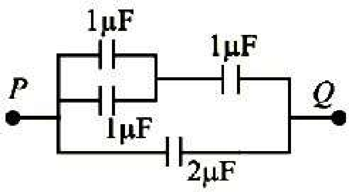
Illustration 21

Find equivalent capacitance between P and Q, in the following circuit.



Short-cut solution :

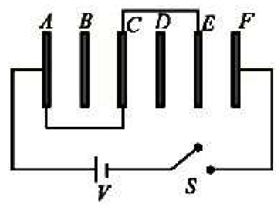
The effective circuit is as shown



$$\therefore \qquad C_{PQ} = \frac{2\times1}{2+1}+2=\frac{8}{3}\mu F. \qquad \textit{Ans.}$$

Illustration 22

There are six identical metal plates arranged as shown in figure. The area of each plate is A and separation between two consecutive plates is d. Find the total energy stored by this system when connected to a cell of emf V,



Short-cut solution :

The effective capacitance is

$$C = \frac{\epsilon_0 A}{d}.$$

So energy stored

$$E = \frac{1}{2}CV^2 = \frac{1}{2}\frac{\epsilon_0 A}{d}V^2. \qquad \textit{Ans.}$$

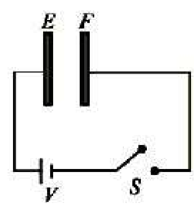
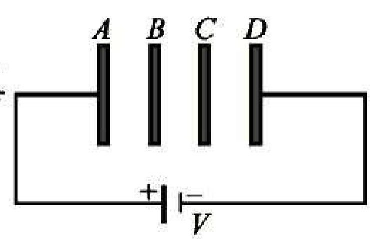


Illustration 23

Four identical metal plates placed parallel with equal separation between consecutive plates. Plates A and D are connected to a cell. What will happen, if plates B and C are connected by a conducting wire?



Short-cut solution :

Potential, $V_A > V_B > V_C > V_D$

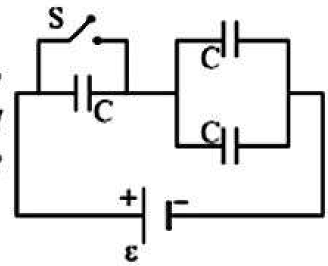
When B is connected to C, $V_B = V_C$, and no electric field remains in between.

Initially there are capacitors (each of C) are in series, so $q_i = \frac{C}{3}V$.

After connecting B and C, there remains two capacitors, so $q_f = \frac{C}{2}V$.

Therefore $$\Delta q = q_f - q_i = \frac{CV}{2} - \frac{CV}{3} = \frac{CV}{6}.$$ **Ans.**

Illustration 24



There are three identical capacitors, each of capacitance C are connected as shown. The emf of the cell connected is ε. Find the charge flows through the switch when it is closed.

Short-cut solution :

Initially, $$q_i = C\varepsilon = \left(\frac{2C \times C}{2C + C}\right)\varepsilon = \frac{2C\varepsilon}{3}.$$

After closing the switch, $$q_f = (2C)\varepsilon = 2C\varepsilon$$

Therefore charge flows through switch

$$= \text{charge flows through cell}$$

$$= 2C\varepsilon - \frac{2C\varepsilon}{3} = \frac{4C\varepsilon}{3}.$$ **Ans.**

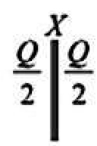
Illustration 25

When a charge Q is given to an isolated metal plate X of surface area A, its charge density becomes σ_1, when an isolated identical plate Y is brought close to X the surface charge density on X becomes σ_2. When Y is earthed, the surface charge density becomes σ_3, then

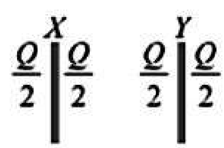
(a) $\sigma_1 = \frac{Q}{A}$ *(b)* $\sigma_1 = \frac{Q}{2A}$ *(c)* $\sigma_1 = \sigma_2$ *(d)* $\sigma_3 = 2\sigma_1$

Short-cut solution :

When there is only plate X, the density, $\sigma_1 = \frac{Q/2}{A}$



When plate Y is brought close to X,



When Y is earthed, $\sigma_3 = \frac{Q}{A}$. **Ans. (b, c, d)**

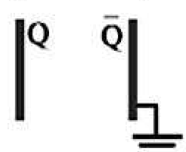
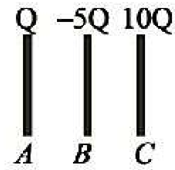


Illustration 26

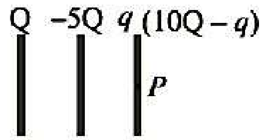
Three large plates are given charges as shown in figure. If the cross-sectional are of each plate is the same, then find charge on both sides of plate C.



Short-cut solution :

Let q is the charge on left side of the plate C. The electric field due to all charges in plate C will be zero.

So, $E_P = 0$



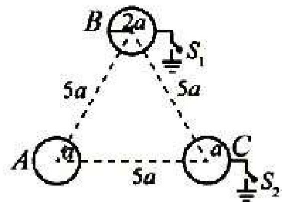
or $$\frac{Q}{\epsilon_0 A} + \frac{-5Q}{\epsilon_0 A} + \frac{q}{\epsilon_0 A} - \left(\frac{10Q - q}{\epsilon_0 A}\right) = 0$$

or $Q - 5Q + q - 10Q + q = 0 \Rightarrow q = 7Q$

and $10Q - q = 10Q - 7Q = 3Q$. ***Ans.***

Illustration 27

The following diagram shows three metal balls. Ball A is charged to Q and B, C are uncharged. Find the charges on balls B and C when the switches S_1 and S_2 are closed.



Short-cut solution :

Let charges on B and C are q_1 and q_2 respectively.

As B is earthed, so $V_B = 0$

or $$\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{5a} + \frac{q_1}{2a} + \frac{q_2}{5a}\right] = 0 \qquad \text{...(i)}$$

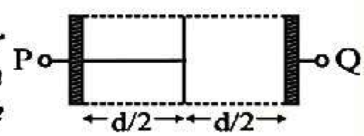
and $V_C = 0$

or $$\frac{1}{4\pi\epsilon_0}\left[\frac{Q}{5a} + \frac{q_1}{5a} + \frac{q_2}{a}\right] = 0 \qquad \text{...(ii)}$$

On solving above equations, we get $q_1 = -\frac{8Q}{23}$ and $q_2 = -\frac{3Q}{23}$. ***Ans.***

Video Solution

Q. *Space between plates of a parallel plate capacitor is filled with three dielectric slabs as shown in figure. Find equivalent dielectric constant for the arrangement shown. Area of each plate is A.*

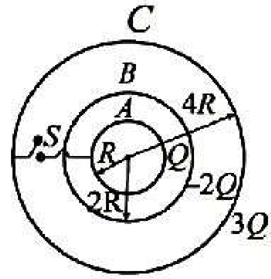


To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=SukN131aFvs

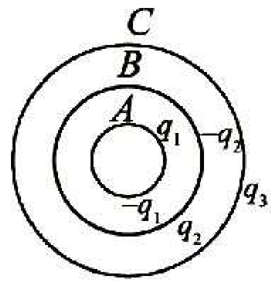
Illustration 28

In the following diagram the conducting shells are concentric. Find the amount of charge that flows through S after closing it.



Short-cut solution :

The charges on the shells are as shown in figure



$$q_2 - q_1 = -2Q \qquad \ldots(i)$$

$$q_1 - q_2 + q_3 = Q + 3Q \qquad \ldots(ii)$$

Also $V_A = V_C$

or $\frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{R} - \frac{q_1}{2R} + \frac{q_2}{2R} - \frac{q_2}{4R} + \frac{q_3}{4R}\right]$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 - q_1 + q_2 - q_2 + q_3}{4R}\right]$$

or $$q_1 - \frac{q_2}{2} + \frac{q_2}{2} - \frac{q_2}{4} + \frac{q_3}{4} = \frac{q_3}{4} \qquad \ldots(iii)$$

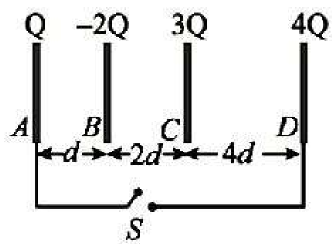
On solving above equations, we get

$$q_1 = — .$$

So the charge flows through S is $Q - \frac{2Q}{3} = \frac{Q}{3}$. ***Ans.***

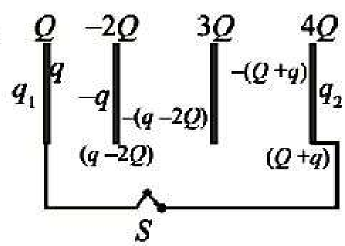
Illustration 29

In the following diagram the plates are conducting and are held parallel. Find the charge flows through the switch when it is closed.



Short-cut solution :

The charges after closing of switch are shown in figure. By conservation of charge, we have



$$q_1 + q - Q - q + q_2 = 5Q$$

or $$q_1 + q_2 = 6Q \qquad \ldots(i)$$

Also $V_{AD} = 0$

or $E_{AB}\, d + E_{BC}\,(2d) + E_{CD}\,(4d) = 0$

or $\frac{q}{\epsilon_0 A} d + \frac{q - 2Q}{\epsilon_0 A}(2d) + \frac{(Q+d)}{\epsilon_0 A}(4d) = 0$

or $$q + 2q - 4Q + 4Q + 4q = 0$$

or $$q = 0 \qquad \ldots(ii)$$

Using, $E_A = 0$

or $\dfrac{q_1}{2\epsilon_0 A} - \dfrac{q_2}{2\epsilon_0 A} = 0 \Rightarrow q_1 = q_2$ ...(iii)

From above equations, we get $q_1 = q_2 = 3Q$

The charge flows through the switch, $\Delta q = 3Q - Q = 2Q$. ***Ans.***

Concept Booster Exercise

1. In a regular polygon of n sides, each corner is at a distance r from the centre. Identical charges of magnitude Q are placed at $(n-1)$ corners. The field at the centre is

(a) $k\dfrac{Q}{r^2}$ (b) $(n-1)k\dfrac{Q}{r^2}$ (c) $\dfrac{n}{(n-1)}k\dfrac{Q}{r^2}$ (d) $\dfrac{(n-1)}{n}k\dfrac{Q}{r^2}$

2. A half ring of radius R has a charge of λ per unit length. The electric potential at the centre of the half ring is

(a) $\dfrac{k\lambda}{R}$ (b) $k\pi\lambda$ (c) $\dfrac{k\lambda}{\pi R}$ (d) $\dfrac{k\pi\lambda}{R}$

3. A solid sphere of radius R is charged uniformly. At what distance from its surface is the potential half of the potential at the centre?

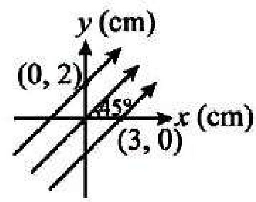
(a) R (b) $R/3$ (c) $R/2$ (d) $R/4$

4. A capacitor of capacitance C and potential V, the flux of the electric field through a closed surface enclosing the capacitor is :

(a) $\dfrac{CV}{2\epsilon_0}$ (b) $\dfrac{2CV}{\epsilon_0}$ (c) $\dfrac{CV}{\epsilon_0}$ (d) Zero

5. A uniform electric field of 400 V/m is directed at 45° above x-axis as shown in figure. The potential difference $V_A - V_B$ is given by

Numeric/Integer

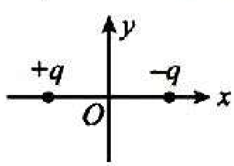


(a) 0 (b) 4 V (c) 6.4 V (d) 2.8 V

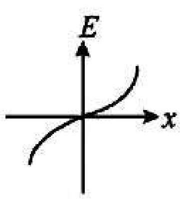
6. A conductor of radius R and carrying charge q is joined to another conductor of radius $2R$ and carrying charge $-2q$. The charge flowing between them will be

(a) $4q/3$ (b) q (c) $2q/3$ (d) $q/3$

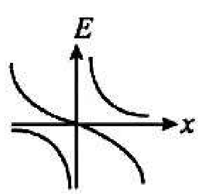
7. A charge $+q$ and a charge $-q$ are placed as shown. The variation of $\vec{E}$ with x-axis is:

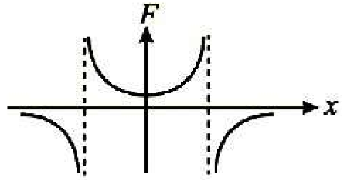


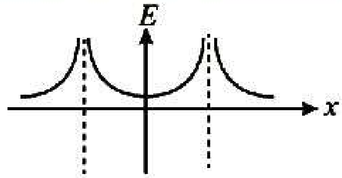
(a)



(b)



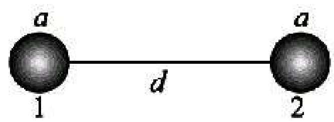
(c) 

(d) 

8. A cubical block of mass m containing a net positive charge Q is placed on a smooth horizontal surface which terminates in a vertical wall. The distance between the wall and the block is 'd'. A horizontal electric field E directed towards the wall is suddenly switched on. Assuming elastic collision (if any) the period of the resulting oscillation is:

(a) $\sqrt{\frac{4md}{QE}}$ (b) $\sqrt{\frac{8md}{QE}}$ (c) $\sqrt{\frac{md}{8QE}}$ (d) Not oscillate

9. There are two uncharged identical metal spheres of radius a, separated by a distance d. A charged metal sphere with charge Q of same radius is brought and touches sphere 1. After some time it moved away to a far off distance. After this, the sphere 2 is earthed. The charge on sphere 2 is:



(a) Q (b) –Q (c) $\frac{-Qa}{2d}$ (d) Zero

10. Four identical pendulums are made by attaching a small ball of mass 100 g on a 20 cm long thread and suspended from the same point. Now each ball is given charge Q so that balls move away from each other with each thread making an angle of 45^0 from the vertical. The value of Q is close to

($\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ in SI units) : ***[KVPY-2017]***

(a) 1 μC (b) 1.5 μC (c) 2 μC (d) 2.5 μC

11. A long string with a charge of λ per unit length passes through an imaginary cube of edge 'a'. The maximum flux of the electric field through the cube will be

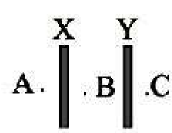
(a) $\lambda a/\epsilon_0$ (b) $\sqrt{2}\lambda a/\epsilon_0$ (c) $6\lambda a^2/\epsilon_0$ (d) $\sqrt{3}\lambda a/\epsilon_0$

12. When two uncharged metal balls of radius 0.09 mm each collide, one electron is transferred between them. The potential difference between them would be :

Numeric/Integer

(a) 16 μV (b) 20 μV (c) 32 μV (d) 40 μV

13. X and Y are large, parallel conducting plates close to each other. Each face has an area A, X is given a charge Q, Y is without any charge. Points A, B and C are as shown in figure :



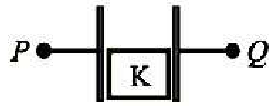
(a) The field at B is $\frac{Q}{2\epsilon_{0A}}$

(b) The field at B is $\frac{Q}{\epsilon_{0A}}$

(c) The field at A, B and C are of the same magnitude

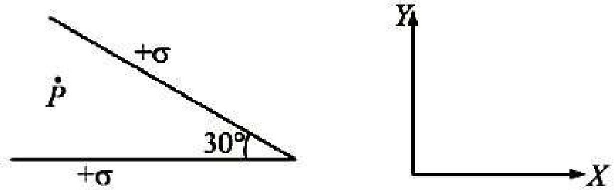
(d) The field at A and C are of the same magnitude, but in opposite direction.

14. A parallel plate capacitor is charged from a cell and then isolated from it. A dielectric slab of dielectric constant K is now introduced in the region between the plates, filling half of it. The intensity of field in the dielectric is E_1 and that in air is E_2, then



(a) $E_1 = \frac{E_2}{k}$ (b) $E_1 = kE_2$ (c) $E_1 = E_2$ (d) $E_1 = \frac{E_2}{(k-1)}$

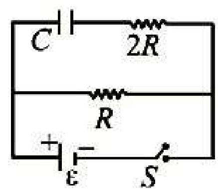
15. There are two infinite plane sheets each having uniform surface charge density $+\sigma$ C/m^2. They are inclined to each other at an angle 30° as shown in the figure. Electric field at arbitrary point P is: *[JEE Main 2020]*



(a) $\frac{\sigma}{2\varepsilon_0}\left[\left(1-\frac{\sqrt{3}}{2}\right)\hat{y}-\frac{1}{2}\hat{x}\right]$ (b) $\frac{\sigma}{2\varepsilon_0}\left[\left(1+\frac{\sqrt{3}}{2}\right)\hat{y}-\frac{1}{2}\hat{x}\right]$

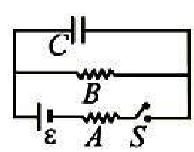
(c) $\frac{\sigma}{2\varepsilon_0}\left[\left(1-\frac{\sqrt{3}}{2}\right)\hat{y}+\frac{1}{2}\hat{x}\right]$ (d) $\frac{\sigma}{2\varepsilon_0}\left[\left(1+\frac{\sqrt{3}}{2}\right)\hat{y}+\frac{1}{2}\hat{x}\right]$

16. In the circuit shown, when switch is closed, the capacitor charge at a time constant xRc. Find the value of x. **Numeric/Integer**



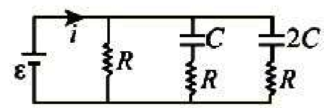
(a) 1 (b) 2
(c) 3 (d) 1.5

17. In the circuit shown, A and B are equal resistances when S is closed, the capacitor C charges from the cell of emf ε and reaches a steady state



(a) Initially current in A is greater than B
(b) At steady state current in A is equal to current in B
(c) In steady state, energy stored in capacitor is $\frac{1}{8}C\varepsilon^2$
(d) Capacitor can not be charged

18. The two capacitors, shown in the circuit are initially uncharged and the cell is ideal. The switch is closed at $t = 0$, which of the following functions represents the current i at any time t through the cell?



(a) Zero (b) $i = \frac{\varepsilon}{R}(1+e^{-t/CR})$

(c) $i = \frac{\varepsilon}{R}(1+e^{-t/CR}+e^{-t/2CR})$ (d) $i = \frac{\varepsilon}{R}(1+e^{-t/2CR})$

Solutions

1. (a) Electric field due to the charges at all 'n' corners will be zero at the centre. Therefore electric field of $(n-1)$ charges is equal to that of single charge. So

$E = k\dfrac{Q}{r^2}$. **Ans.**

2. (b) $V = k\dfrac{q}{R} = k \times \dfrac{\pi R\lambda}{R} = k\pi\lambda$. **Ans.**

3. (b) The required potential will be, $V = \dfrac{1}{2}\left(\dfrac{3}{2}k\dfrac{q}{R}\right) = \dfrac{3}{4}\dfrac{kq}{R}$.

As potential at the surface is $\dfrac{kq}{R}$, so the required point must be outside the sphere, so

$$\frac{kq}{r} = \frac{3kq}{4R} \Rightarrow r = \frac{4R}{3}. \quad \textbf{Ans.}$$

4. (d)

$$\phi = \frac{q_{in}}{\epsilon_0} = \frac{0}{\epsilon_0} = 0. \quad \textbf{Ans.}$$

5. (d)

$$\vec{r}_{AB} = \vec{r}_A - \vec{r}_B = (3\hat{i} - 2\hat{j})\text{ cm}$$

and

$$\vec{E} = 4\cos 45^\circ \hat{i} + 4\sin 45^\circ \hat{j} = \frac{4\hat{i} + 4\hat{j}}{\sqrt{2}}\text{ V/cm.}$$

Now

$$V_A - V_B = \vec{E}.(\vec{r}_A - \vec{r}_B) = \frac{4\hat{i} + 4\hat{j}}{\sqrt{2}}.(3\hat{i} - 2\hat{j})$$

$$= \frac{12 - 8}{\sqrt{2}} = 2\sqrt{2} = 2.8\ V \quad \textbf{Ans.}$$

6. (a)

$$V = \left[\frac{q - 2q}{4\pi\epsilon_0 (R + 2R)}\right] = \frac{-q}{4\pi\epsilon_0 (3R)}$$

Now $q_1 = C_1V = 4\pi\epsilon_0 R \times \left(\dfrac{-q}{4\pi\epsilon_0 \times 3R}\right) = \dfrac{-q}{3}$.

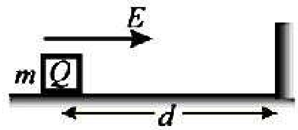
Therefore,

$$\Delta q = -\frac{q}{3} - q = -\frac{4q}{3}. \quad \textbf{Ans.}$$

7. (c) The electric field the centre of charges is minimum and non-zero.

8. (b)

$$d = \frac{1}{2}at^2 = \frac{1}{2}\left(\frac{EQ}{m}\right)t^2$$



$\therefore$

$$t = \sqrt{\frac{2dm}{EQ}}$$

and

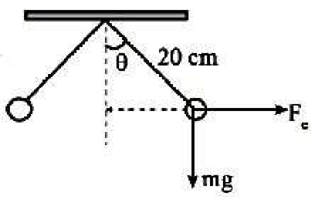
$$T = 2t = 2\sqrt{\frac{2dm}{EQ}} = \sqrt{\frac{8md}{QE}}. \quad \textbf{Ans.}$$

9. **(c)** For second sphere $V_2 = 0 = \frac{1}{4\pi\epsilon_0}\left[\frac{Q/2}{d}+\frac{q_2}{a}\right]$

or $q_2 = -\frac{Qa}{2d}$. **Ans.**

10. **(b)** Here, $\tan\theta = \frac{F_e}{mg}$ or $\tan 45° = \frac{\dfrac{kQ^2}{\left(\dfrac{20\times10^{-2}}{\sqrt{2}}\right)^2}}{1000\times10\times10^{-3}}$

Hence, value of Q = $\sqrt{\frac{20}{9}\times10^{-2}} = 1.5\ \mu C$. **Ans.**



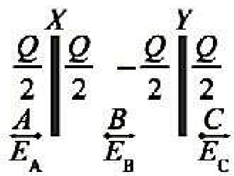
11. **(d)** The charge on longest diagonal of cube, $q = \sqrt{3}a\lambda$.

$\therefore$ $\phi = \frac{q}{\epsilon_0} = \sqrt{3}a\lambda/\epsilon_0$. **Ans.**

12. **(c)** $V = 2\left(\frac{1}{4\pi\epsilon_0}\right)\frac{e}{r}$. **Ans.**

13. **(a, c, d)** $E_A = \frac{Q/2}{2\epsilon_0 A}+\frac{Q/2}{2\epsilon_0 A} = \frac{Q}{2\epsilon_0 A}$

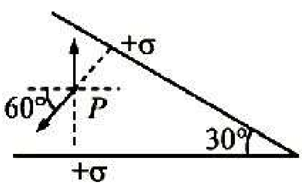
$= E_B = E_C$ **Ans.**



14. **(c)** The Pd between the plates in both halves is same, and also separation between the plates also same, $\therefore$ $E_1 = E_2 = \frac{V}{d}$. **Ans.**

15. **(a)** $\vec{E} = \frac{\sigma}{2\varepsilon_0}\cos 60°(-\hat{x}) + \left[\frac{\sigma}{2\varepsilon_0}-\frac{\sigma}{2\varepsilon_0}\sin 60°\right](\hat{y})$

$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[\left(1-\frac{\sqrt{3}}{2}\right)\hat{y}-\frac{1}{2}\hat{x}\right]$ **Ans.**



16. **(b)** After removing the cell connect a short at the place, so $R_{eff} = 2R$ and $\tau = 2RC$. **Ans.**

17. **(a, b, c)** At steady state, $i = \frac{\varepsilon}{2R}$ in both A and B.

Energy stored in capacitor $= \frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{\varepsilon}{2R}\times R\right)^2 = \frac{C\varepsilon^2}{8}$. **Ans.**

18. **(c)** Each arm in parallel, so Pd across each one is ε.
Now $i = i_1 + i_2 + i_3$

$= \frac{\varepsilon}{R}+\frac{\varepsilon}{R}e^{-t/CR}+\frac{\varepsilon}{R}e^{-t/2CR}$

$= \frac{\varepsilon}{R}(1+e^{-t/CR}+e^{-t/2CR})$. **Ans.**

Current Electricity 16

TOPIC: *Current Density, Drift Speed, Resistance, Ohm's Law, Combination of Resistances, Combination of Cells, Wheatstone Bridge, Meter Bridge, Potentiometer and Conversion of Galvanometer into Ammeter and Voltmeter.*

Review of Formulae

1. **Electric current :** An electric current i in a conductor is defined by

$$i = \frac{dq}{dt}.$$

By convention, the direction of electric current is taken as the direction in which positive charge carriers would move.

2. **Current density :** Current density is related to the current as

$$i = \int \vec{J} \cdot d\vec{A}$$

where $d\vec{A}$ is a vector perpendicular to a surface element of area dA, and the integral is taken over any surface cutting across the conductor. $\vec{J}$ has the same direction as the velocity of the moving positive charges.

3. **Drift speed :** When an electric field $\vec{E}$ is established in a conductor, the charge carriers (assumed positive) acquire a drift speed v_d in the direction of $\vec{E}$; the velocity is related to current density $\vec{J}$ as

$$\vec{J} = (ne)\vec{v}_d$$

4. **Resistance of a conductor :** If V is the p.d. applied across the conductor and i is the corresponding current, then its resistance is defined as :

$$R = \frac{V}{i}.$$

Similarly we can define resistivity ρ and conductivity σ of a material :

$$\rho = \frac{1}{\sigma} = \frac{E}{J}$$

Also $\vec{E} = \vec{J}\rho$

If n is the number of free electrons per unit volume and τ is the relaxation time, then

$$\rho = \frac{m}{ne^2\tau}$$

The resistance R of a conducting wire of length L and uniform cross-section A is

$$R = \frac{\rho L}{A}.$$

5. **Change in ρ or R with temperature :** If R_0 is the resistance of a wire at temperature 0°C, then resistance at any temperature t is

$$R_t \simeq R_0(1 + \alpha t)$$

where α is the temperature coefficient of resistance. It can be defined as

$$\alpha = \frac{1}{R_0}\frac{dR_t}{dt}$$

6. **Ohm's law :** Under given physical conditions the current i produced in the conductor is proportional to the applied potential difference across the conductor.

7. **Combination of resistances :**

(i) In series : $R = R_1 + R_2 + ...$

(ii) In parallel : $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + ...$

8. **Changing the size of the resistor :**

If the conductor is stretched from length ℓ to ℓ', then its resistance changes from R to R' as

$$R' = R\left(\frac{\ell'}{\ell}\right)^2$$

In terms of radius of cross-section of the conductor

$$R' = R\left(\frac{r}{r'}\right)^4.$$

1. **Combination of cells :**

(i) **In series :** If n identical cells each of emf ξ and internal resistance r are connected in series, then current in external resistor R

$$i = \left[\frac{n\xi}{nr + R}\right]$$

In case when $nr << R$,

$$i \simeq n\frac{\xi}{R}$$

(ii) In parallel : $$i = \frac{\xi}{\left[\frac{r}{n} + R\right]}$$

In case $R << nr$,

$$i \simeq n\frac{\xi}{r}.$$

(iii) Series-parallel : If n cells are connected in series and m cells are in parallel, then

$$i = \left[\frac{n\xi}{\frac{nr}{m} + R}\right]$$

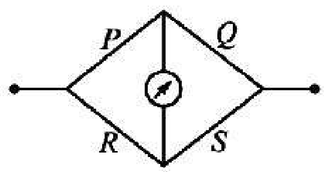
For maximum current

$$\frac{nr}{m} = R.$$

Total number of cells = mn.

2. **Wheatstone bridge :** It is the combination of four resistances in the form of bridge. For the balanced bridge with the resistors P, Q, R and S

$$\frac{P}{Q} = \frac{R}{S} \quad \text{or} \quad \frac{P}{R} = \frac{Q}{S}$$



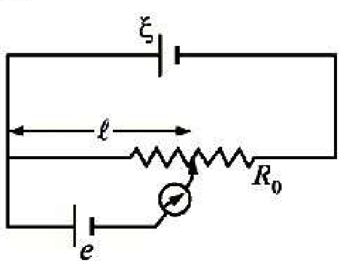
The equivalent resistance across the terminals of the cell

$$\frac{1}{R_{eq}} = \frac{1}{P+Q} + \frac{1}{R+S}$$

3. **Metre bridge :** It is used to find unknown resistance. If ℓ be the balanced length and R is the known resistance, then unknown resistance

$$S = R\left(\frac{1-\ell}{\ell}\right).$$

4. **Potentiometer :** It is an ideal device of finding emf of the cells, internal resistance of the cell etc.



If R_0 is the resistance of the potentiometer wire, then emf of the cell

$$e = \xi \frac{\ell}{\ell_0},$$

where ℓ is the balancing length and ℓ_0 is the length of the potentiometer wire.

Internal resistance

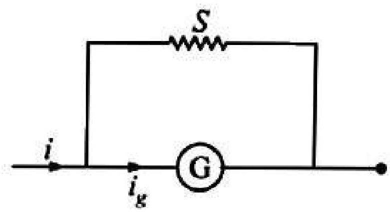
$$r = R\left(\frac{\ell_1 - \ell_2}{\ell_2}\right),$$

where ℓ_1 and ℓ_2 are the balancing lengths without R and with R.

5. **Ammeter :** Galvanometer of resistance G and full scale deflection current i_g can be converted into an ammeter of range i by connecting a shunt of resistance S, such that

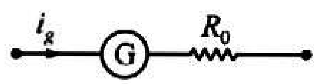
$$i_g = i\frac{S}{S+G}$$

Resistance of ammeter $R_A = \dfrac{SG}{S+G}$



6. **Voltmeter :** A galvanometer of resistance G and full scale deflection current i_g can be converted into a voltmeter of range V by connecting a large resistance R_0 in series, such that

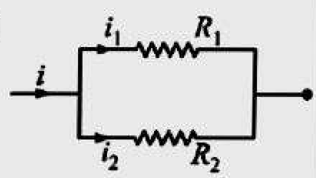
$$V = i_g(G + R_0).$$



Tips and Tricks for Shortcut Solutions

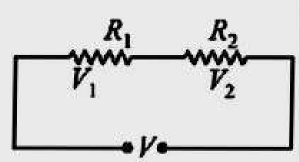
1. Current carrying conductor has no electric field outside the surface, because it has no net charge.
2. When current i is divided into two resistors R_1 and R_2 in parallel,

$$i_1 = \frac{iR_2}{R_1 + R_2} \text{ and } i_2 = \frac{iR_1}{R_1 + R_2}.$$



3. When potential V divided into two resistors R_1 and R_2 in series,

$$V_1 = \frac{VR_1}{R_1 + R_2} \text{ and } V_2 = \frac{VR_2}{R_1 + R_2}$$



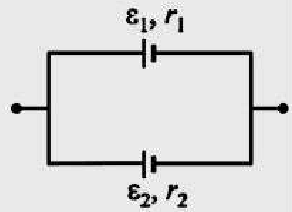
4. The equivalent temperature coefficient of resistance (α) for two resistors; (R_{01}, α_1) and (R_{02}, α_2).

(i) In series: $\alpha = \dfrac{R_{01}\alpha_1 + R_{02}\alpha_2}{R_{01} + R_{02}}$

(ii) In parallel: $\alpha' = \dfrac{R_{02}\alpha_1 + R_{01}\alpha_2}{R_{01} + R_{02}}$

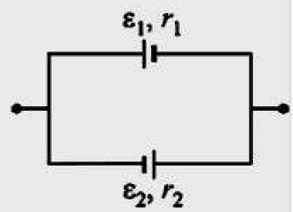
For $R_{01} = R_{02}$, $\alpha = \alpha' = \dfrac{\alpha_1 + \alpha_2}{\ }$

5. If n-cells of different emfs, the equivalent emf



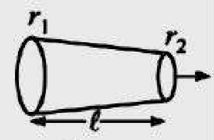
$$\frac{\varepsilon_{eq}}{r_{eq}} = \left[\frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2} + \ldots\ldots + \frac{\varepsilon_n}{r_n}\right].$$

For two cells: $\varepsilon = \left[\dfrac{\varepsilon_1 r_2 + \varepsilon_2 r_2}{r_1 + r_2}\right]$



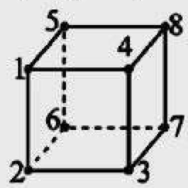
and $\varepsilon' = \left[\dfrac{\varepsilon_1 r_2 - \varepsilon_2 r_1}{r_1 + r_2}\right]$

6. Resistance of a wire (r_1, r_2, ℓ) is



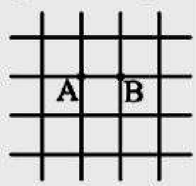
$$R = \frac{\rho\ell}{\pi r_1 r_2}$$

7. If each side of the cube is of resistance r, then



$$R_{17} = \frac{5r}{6}; R_{12} = \frac{7r}{12}, R_{18} = \frac{3r}{4}.$$

8. In an infinite grid, if r is the resistance between two junction, then equivalent between A and B



$$R_{AB} = \left[\frac{2r}{\text{no. of wires connected at the junction}}\right].$$

Illustration 1

Infinite number of point charges; $q, \frac{q}{2}, \frac{q}{4}, \ldots\ldots\infty$ ***are placed on a non-conducting ring of radius R. If it rotates with an angular speed*** ω***, find the equivalent current.***

Short-cut solution :

$$Q = q + \frac{q}{2} + \frac{q}{4} + \ldots\ldots\infty = \frac{q}{1 - \frac{1}{2}} = 2q.$$

Current $i = \dfrac{Q}{T} = \dfrac{2q}{\frac{2\pi}{\omega}} = \dfrac{q\omega}{\pi}.$ ***Ans.***

Illustration 2

A straight conductor of uniform cross-section carries a current I. Let 's' is the specific charge of an electron, find the momentum of all the free electrons per unit length of the conductor due to their drift velocity.

Short-cut solution :

$$P = (mv_d)(nA) = \left(m \times \frac{I}{neA}\right)nA$$

$$= \frac{I}{\left(\frac{e}{m}\right)} = \frac{I}{s}. \qquad \textit{Ans.}$$

Illustration 3

The charge flowing in a conductor varies with time as, $q = at - \frac{bt^2}{2} + \frac{ct^3}{6}$. Find minimum current in the conductor.

Short-cut solution :

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(at - \frac{bt^2}{2} + \frac{ct^3}{6}\right)$$

$$= a - bt + \frac{ct^2}{2}$$

and $$\frac{di}{dt} = -b + ct = 0$$

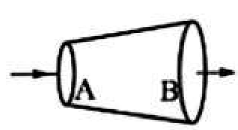
or $$t = \frac{b}{c}.$$

$\therefore$ $$i_{min} = a - b\left(\frac{b}{c}\right) + \frac{1}{2}c\left(\frac{b}{c}\right)^2 = a - \frac{b^2}{2c}. \qquad \textit{Ans.}$$

At $t = 0$, $i = a$, so $a - \frac{b^2}{2c}$ is less than a.

Illustration 4

A steady current flows in a conductor of non-uniform cross-section. Discuss electric field at points A and B. Also compare the resistance of equal width nearby A and nearby B.



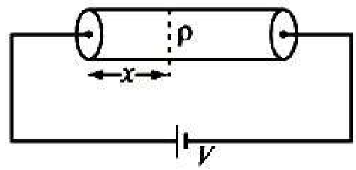
Short-cut solution :

Current density at A is greater than at B, so

$\vec{E} = \vec{j}\rho$ is greater at A. As cross-sectional area at A is smaller, so $R = \frac{\rho\ell}{A}$ is greater at A. *Ans.*

Illustration 5

A cylindrical solid of length L and radius a is having varying resistivity given by $\rho = \rho_0 x$ where ρ_0 is a positive constant and x is measured from left end. The cell connected has emf V and no internal resistance. Find electric field inside solid as a function of x.



Short-cut solution :

Resistance, $$R = \int_0^L \frac{\rho dx}{A} = \int_0^L \frac{\rho_0 x dx}{\pi a^2}$$

$$= \frac{\rho_0 L^2}{2\pi a^2}$$

Current, $$i = \frac{V}{R} = \left(\frac{V}{\frac{\rho_0 L^2}{2\pi a^2}}\right).$$

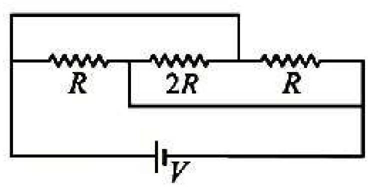
Pd across the element, $$dV = i(dR) = \left(\frac{V}{\frac{\rho_0 L^2}{2\pi a^2}}\right) \times \left(\frac{\rho_0 x dx}{\pi a^2}\right)$$

$$= \frac{2V}{L^2} x dx$$

$$\therefore \quad \frac{dV}{dx} = \frac{2Vx}{L^2}. \qquad \textit{Ans.}$$

Illustration 6

In the circuit shown, find current in the resistor 2R.



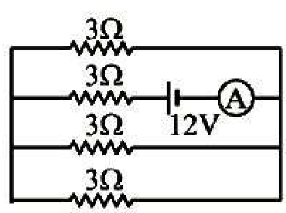
Short-cut solution :

In the circuit all the three resistors are in parallel, so current in $2R$.

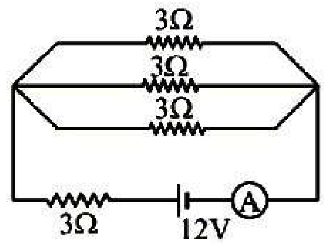
$$i = \frac{V}{2R}. \qquad \textit{Ans.}$$

Illustration 7

In the circuit shown, find reading of ammeter.



Short-cut solution :



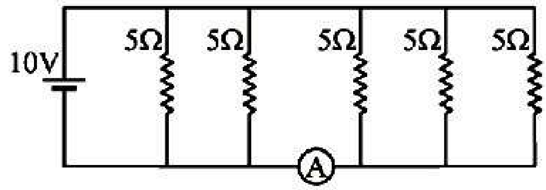
The equivalent resistance across the cell,

$$R = 1 + 3 = 4\Omega.$$

$$\therefore \quad i = \frac{V}{R} = \frac{12}{4} = 3A. \qquad \textit{Ans.}$$

Illustration 8

In the circuit shown find the reading of ammeter.



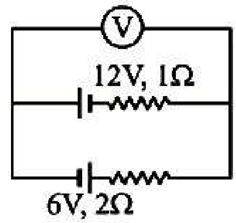
Short-cut solution :

As all the resistors are in parallel, so current in each one, $i = \dfrac{10}{5} = 2A.$

The reading of ammeter, = 3 × 2 = 6A. ***Ans.***

Illustration 9

Two cells 12V, 1Ω and 6V, 2Ω are connected in parallel to a voltmeter V as shown. Find reading of voltmeter.

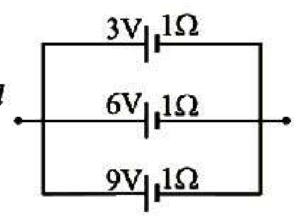


Short-cut solution :

Using,
$$V = \frac{\varepsilon_1 r_2 - \varepsilon_2 r_1}{r_1 + r_2} = \frac{12 \times 2 - 6 \times 1}{1+2} = 6V. \qquad \textit{Ans.}$$

Illustration 10

Three cells of (3V, 1Ω), (6V, 1Ω) and (9V, 1Ω) are connected in parallel. Find P.d. across their ends.



Short-cut solution :

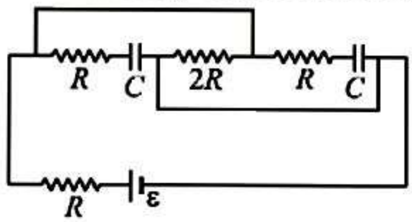
Using,
$$\frac{\varepsilon}{r_{eq}} = \frac{\varepsilon_1}{r_1} + \frac{\varepsilon_2}{r_2} + \frac{\varepsilon_3}{r_3}$$

or
$$\frac{\varepsilon}{\left(\frac{1}{3}\right)} = \frac{3}{1} + \frac{6}{1} + \frac{9}{1}$$

$$\therefore \quad \varepsilon = \frac{1}{3} \times 18 = 6V. \qquad \textit{Ans.}$$

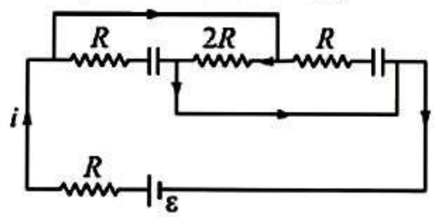
Illustration 11

Find current in resistor 2R at steady state in the circuit shown.



Short-cut solution :

In the branch of capacitors, there is no current so,



$$i = \frac{V}{2R+R} = \frac{V}{3R}.$$ ***Ans.***

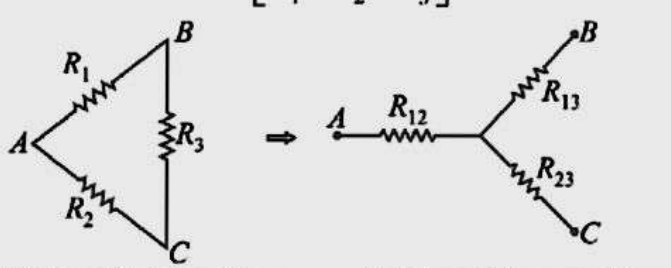
Tips and Tricks on Delta-star Transformation (DST) and Junction Removal Method (JRM)

Delta–star transformation : A combination of three resistors in the form of delta can be effectively converted into star. A delta of three resistors R_1, R_2 and R_3 is equivalent to a star with three resistors R_{12}, R_{13} and R_{23}, where

$$R_{12} = \left[\frac{R_1 R_2}{R_1 + R_2 + R_3}\right]$$

$$R_{13} = \left[\frac{R_1 R_3}{R_1 + R_2 + R_3}\right]$$

$$R_{23} = \left[\frac{R_2 R_3}{R_1 + R_2 + R_3}\right]$$

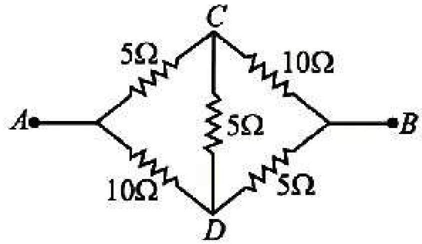


(a) Delta of three resistors. (b) Star of three resistors.

For any two junctions R_{AB} in delta is equal to R_{AB} in star, similarly R_{AC} and R_{BC}.

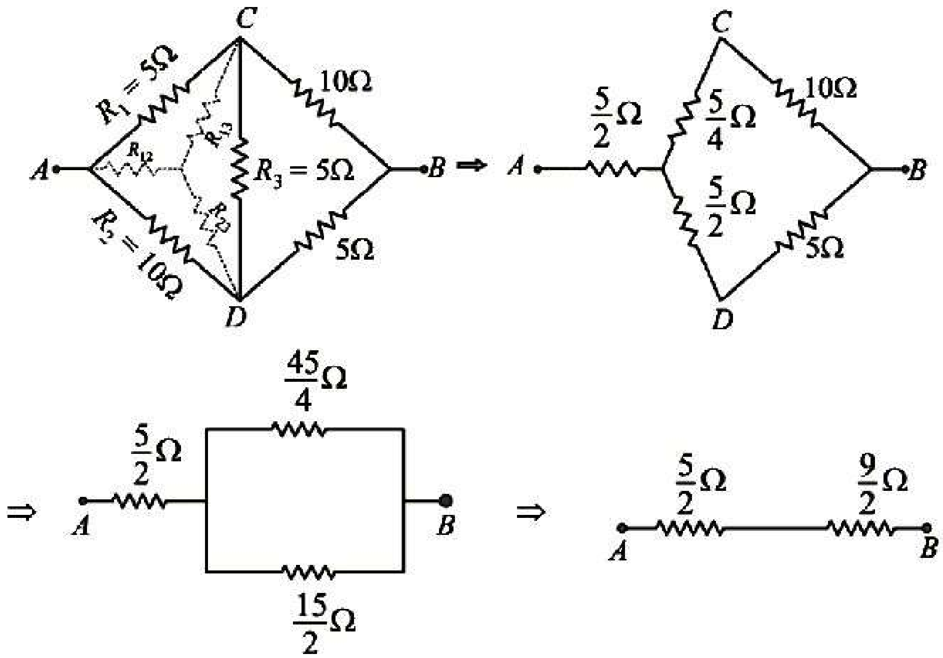
Illustration 12

Find equivalent resistance between the terminals A and B by using delta-star transformation method.



Short-cut solution :

We can simplify the circuit by transforming delta ACD into star as follows.



$$R_{12} = \frac{R_1 R_2}{R_1 + R_2 + R_3} = \frac{5 \times 10}{5 + 10 + 5} = \frac{5}{2}\Omega,$$

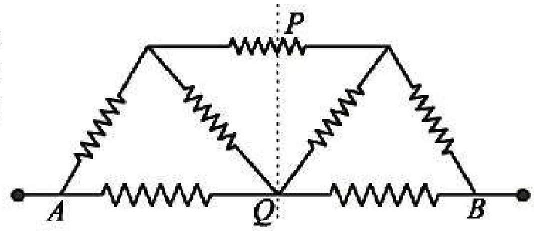
$$R_{13} = \frac{R_1 R_3}{R_1 + R_2 + R_3} = \frac{5 \times 5}{5 + 10 + 5} = \frac{5}{4}\Omega,$$

$$R_{23} = \frac{R_2 R_3}{R_1 + R_2 + R_3} = \frac{10 \times 5}{5 + 10 + 5} = \frac{5}{2}\Omega,$$

$$R_{AB} = \frac{5}{2} + \frac{9}{2} = 7\,\Omega.$$ ***Ans.***

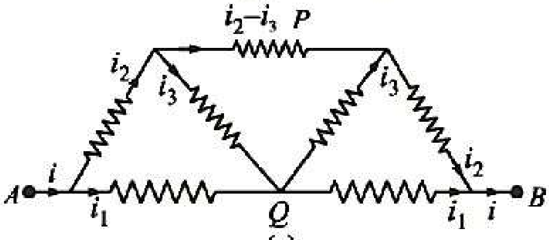
Illustration 13

Find the equivalent resistance of the circuit shown in figure between the points A and B. Each resistor has a resistance 1Ω.

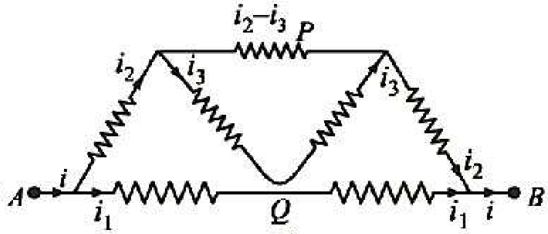


Short-cut solution :

Method I : By junction removal method

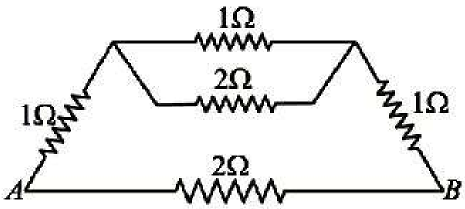


(a)

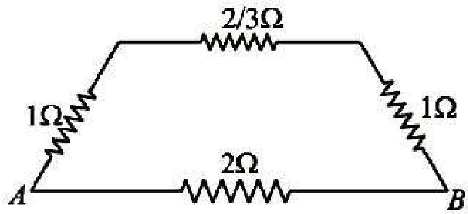


(b)

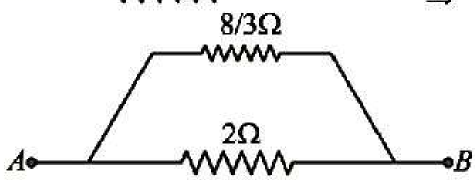
Because of symmetry about dotted line PQ, the currents in left part of circuit are same as the right part of the circuit. The current distribution in the resistors may be as in figure (b). It is very clear from the figure that resistors having current i_3 can be removed from the junction Q without affecting the currents in other resistors as shown in figure (b). The resulting circuit can be simplified as :



$\Rightarrow$

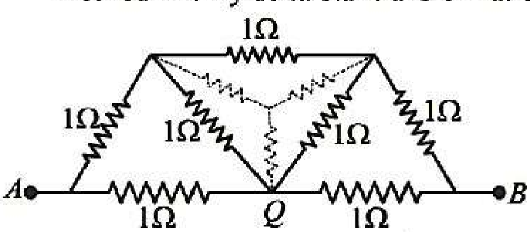


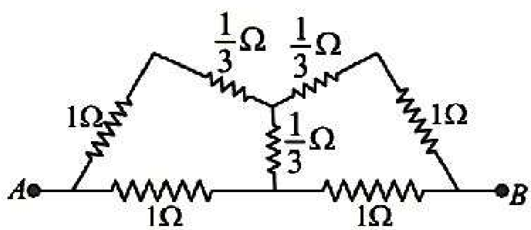
$\Rightarrow$

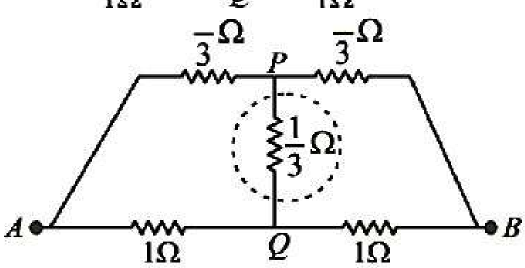


$$\therefore \qquad R_{AB} = \frac{\frac{8}{3}\times 2}{\frac{8}{3}+2} = \frac{8}{7}\Omega$$

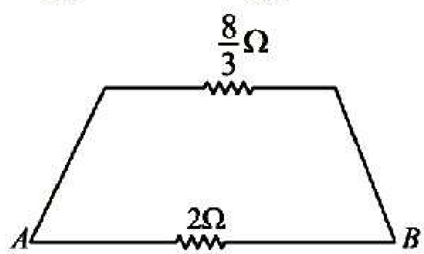
Method II : By delta-star transformation :







$\Rightarrow$

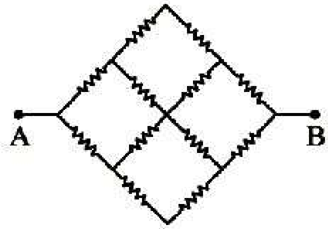


$$\Rightarrow \qquad R_{AB} = \frac{8}{7}\Omega$$

Ans.

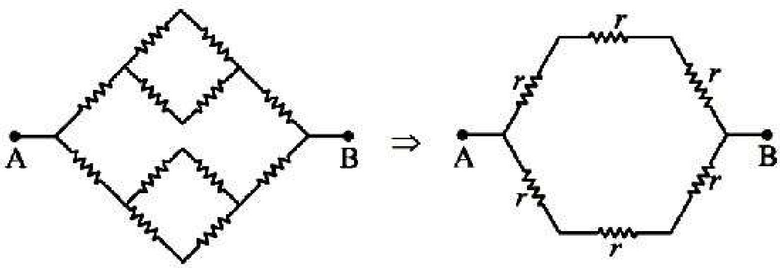
Illustration 14

In the network there are twelve identical resistors, each of resistance r. Find equivalent resistance between A and B.



Short-cut solution :

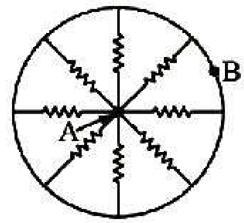
The current distribution and the equivalent circuit after removing the junction is shown in figure.



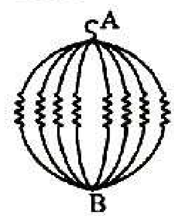
$\Rightarrow R_{AB} = \frac{3r}{2}$. ***Ans.***

Illustration 15

Find equivalent resistance between A and B of the given network. Each resistance is of 1Ω.



Short-cut solution :

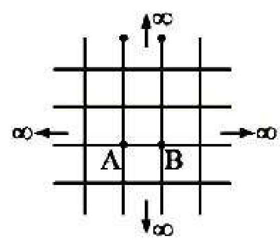


Hang the network from A, so all the resistors become parallel.

$$R_{AB} = \frac{1}{8}\Omega. \qquad \textit{Ans.}$$

Illustration 16

There is an infinite wire grid with square cells as shown. The resistance of each side is r. Find equivalent resistance of whole grid between any two neighbouring points such as A and B.



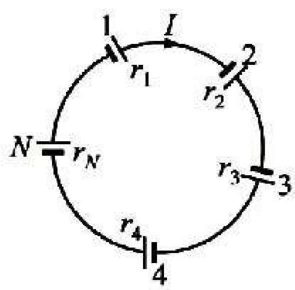
Short-cut solution :

Using, $R_{AB} = \frac{2r}{n}$

Here, $n = 4, \quad \therefore \quad R_{AB} = \frac{2r}{4} = \frac{r}{2}$. ***Ans.***

Illustration 17

A group of N cells whose emf varies directly with the internal resistance as per the equation $E_N = 1.5\, r_N$ are connected as shown in the figure below. The current I in the circuit is



(a) ***0.51 A*** ***(b)*** ***5.1 A***
(c) ***0.15 A*** ***(d)*** ***1.5 A***

Short-cut solution :

$$E_1 = 1.5 \times 1V, E_2 = 1.5 \times 2V, \ldots\ldots$$

Now current
$$I = \frac{E_1 + E_2 + \ldots\ldots + E_n}{r_1 + r_2 + \ldots\ldots + r_n}$$

$$= \frac{1.5(1+2+\ldots\ldots+N)}{(1+2+\ldots\ldots+N)}$$

$$= 1.5\text{ A.}$$

Ans. (d)

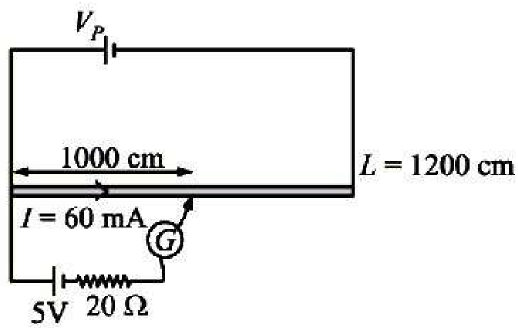
Illustration 18

There is a potentiometer wire of length 1200 cm and 60 mA current is flowing in it. A battery of emf 5 V and internal resistance of 20 Ω is balanced on potentiometer wire with balancing length 1000 cm. The resistance of potentiometer wire is-

Numeric/Integer *[JEE Main 2020]*

(a) *80 Ω* *(b)* *100 Ω* *(c)* *120 Ω* *(d)* *60 Ω*

Short-cut solution :



Potential gradient $= \frac{5}{1000} = \frac{V_P}{1200}$

$$V_P = 6\text{V}$$

and
$$R_P = \frac{V_P}{I} = \frac{6}{60 \times 10^{-3}} = 100\ \Omega$$

Ans. (b)

Video Solution

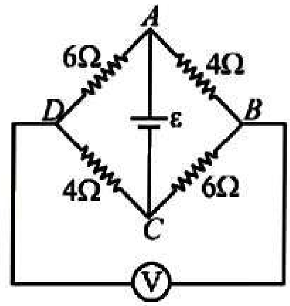
Q. Twelve cells each having the same emf are connected in series and are kept in a closed box. Some of the cells are wrongly connected. This battery is connected in series with an ammeter and two cells identical with each others. The current is 3A when the cells and battery aid each other and 2A when the cells and battery oppose each other. How many cells are wrongly connected ?

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=Gf6n6pVENAw

Illustration 19

In the circuit shown, find reading of voltmeter of large resistance. The emf of the cell is ε.



Short-cut solution :

$$V_A - V_B = \left(\frac{\varepsilon}{4+6}\right)\times 4 = \frac{4\varepsilon}{10}$$

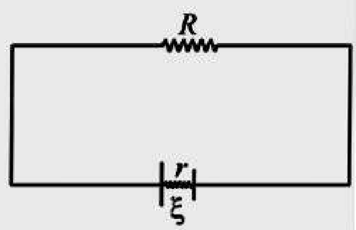
and $$V_A - V_D = \frac{\varepsilon}{6+4}\times 6 = \frac{6\varepsilon}{10}$$

∴ $$V_B - V_D = (V_A - V_D) - (V_A - V_B) = \frac{6\varepsilon}{10} - \frac{4\varepsilon}{10}$$

$$= \frac{\varepsilon}{5}.$$ ***Ans.***

Tips and Tricks for Shortcut Solutions

1. **Maximum power theorem :**



Power generated in the resistor R

$$P = \left(\frac{\xi}{R+r}\right)^2 R.$$

For maximum power

$$r = R \text{ and } P_{max} = \frac{\xi^2}{4R}$$

2. **Electrical appliances :**

The resistance of any electrical appliance of power P_{design} and V_{design} can be obtained by

$$R = \frac{V_{design}^2}{P_{design}}$$

The allowable current

$$i = \frac{P_{design}}{V_{design}}$$

3. In houses the electrical appliances are connected in parallel. If appliances of powers P_1, P_2, ... are connected in parallel across the design voltage V, then total power consumed

$$P = P_1 + P_2 + ...$$

In series :

$$\frac{1}{P} = \frac{1}{P_1} + \frac{1}{P_2} + ...$$

4. **Fuse wire :** In a fuse wire, the change in its temperature $\Delta\theta$ for the constant current i is given by

$$\Delta\theta = \frac{i^2\rho}{2\pi^2 r^3 C}$$

For the given material of fuse wire $r^2 \propto r^3$.

Illustration 20

An electric bulb rated for 500 W at 100 V is used in a circuit having 200 V supply. Find the resistance R that must be put in series with the bulb, so that the bulb draws 500 W.

Short-cut solution :

Current needed for full power consumption

$$i = \frac{P}{V} = \frac{500}{100} = 5\text{A}.$$

and resistance of bulb, $$R_0 = \frac{V^2}{P} = \frac{100^2}{500} = 20\Omega$$

If R is the required resistance, then

$$i = \frac{V_{supply}}{R_0 + R}$$

or $$5 = \frac{200}{20 + R} \Rightarrow R = 20\Omega. \quad \textit{Ans.}$$

Illustration 21

A capacitor of capacitance C has charge Q. It is connected to an identical capacitor through a resistance. Find heat produced in the resistance.

Short-cut solution :

As the capacitors are identical, so final charge on each one is $\frac{Q}{2}$. Therefore heat produced

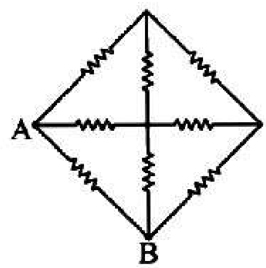
$$= E_i - E_f = \frac{Q^2}{2C} - 2\left[\frac{\left(\frac{Q}{2}\right)^2}{2C}\right]$$

$$= \frac{Q^2}{4C}.$$ ***Ans.***

Concept Booster Exercise

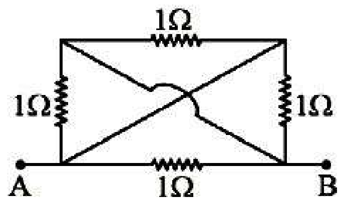
1. The charge flowing in a conductor varies with time as $q = t - t^2$. Then the current,
   (a) remains constant with time (b) varies and will get a maximum value
   (c) decreases with time (d) changes at the rate –2 A/s.

2. In the network shown, each resistance is of 1Ω. The equivalent resistance between A and B is **Numeric/Integer**



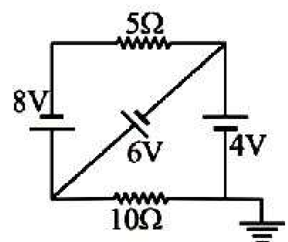
   (a) $1\,\Omega$ (b) $\frac{2}{3}\Omega$
   (c) $\frac{4}{3}\Omega$ (d) $\frac{8}{15}\Omega$

3. In the circuit shown, each of the wire is of 1Ω. The Pd between A and B is $1\,V$. The current flowing between A and B is **Numeric/Integer**



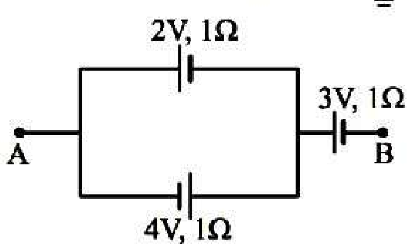
   (a) $1\,A$ (b) $2\,A$
   (c) $3\,A$ (d) $4\,A$

4. The current through the 5Ω resistor in the circuit shown is: **Numeric/Integer**



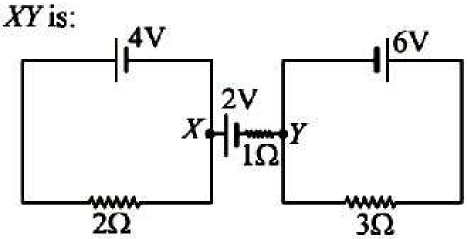
   (a) $0.4\,A$
   (b) $0.8\,A$
   (c) $1.2\,A$
   (d) $2.0\,A$

5. The potential difference between the points A and B is **Numeric/Integer**



   (a) $2\,V$
   (b) $4\,V$
   (c) $6\,V$
   (d) Zero

6. In the circuit shown, the current in conductor XY is: **Numeric/Integer**



   (a) $2\,A$
   (b) $4\,A$
   (c) $6\,A$
   (d) Zero

7. n-identical cells, each of emf ε and internal resistance r are joined in series to form a closed circuit. One cell (A) is joined with reverse polarity. The potential difference across each cell, except A is:

(a) $2\frac{\varepsilon}{n}$ (b) $\frac{n-1}{n}\varepsilon$ (c) $\frac{n-2}{n}\varepsilon$ (d) $\frac{2n}{n-2}\varepsilon$

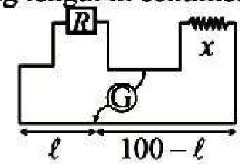
8. Milliammeter of range 10 mA gives full-scale deflection for a current of 100 mA when a shunt of 0.1 Ω is connected in parallel to it. The coil of the milliammeter has a resistance of **Numeric/Integer**

(a) 0.9 Ω (b) 1 Ω (c) 1.1 Ω (d) 0.11 Ω

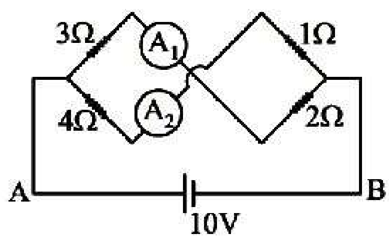
9. A, B and C are voltmeters of resistances R, $1.5R$ and $3R$ respectively. When some potential difference is applied between X and Y, the voltmeter readings are V_A, V_B and V_C respectively, then

(a) $V_A = V_B = V_C$ (b) $V_A \neq V_B = V_C$ (c) $V_A = V_B \neq V_C$ (d) $V_B = \frac{V_A + V_C}{2}$

10. If in a meter bridge experiment, the balancing length l was 25 cm for the situation shown in the figure. If the length and diameter of the of wire of resistance R is made half, then find the new balancing length in centimetre is *[JEE Main 2020]*



11. In the circuit shown, A_1 and A_2 are ideal ammeters. When an ideal cell of emf 10V in applied between A and B, then

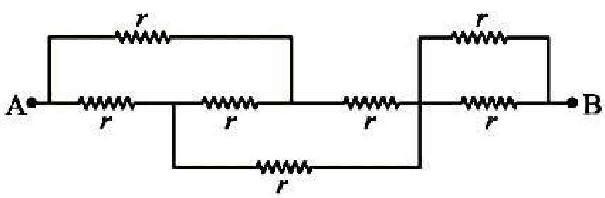


(a) the current drawn from the cell is 4A
(b) the reading of A_1 is 2A
(c) the reading of A_2 is 2A
(d) reading of A_1 is 1A and A_2 is 2A.

12. In full scale deflection current in galvanometer of 100 Ω resistance is 1 mA. Resistance required in series to convert it into voltmeter of range 10 V. *[JEE Main 2020]*

(a) 0.99 KΩ (b) 9.9 KΩ (c) 9.8 KΩ (d) 10 KΩ

13. Find the equivalent resistance of the network shown.



(a) $\frac{3}{2}r$ (b) $2r$ (c) $4r$ (d) $\frac{r}{2}$

14. Two identical cells of the same emf and internal resistance give the same current through an external resistance of 2 Ω, regardless of whether cells are connected in series or parallel. The internal resistance of each cell is: **Numeric/Integer**

(a) 0.5 Ω (b) 1 Ω (c) 3 Ω (d) 2 Ω

15. When a galvanometer is shunted with a 4 Ω resistance, the deflection is reduced to one fifth. If the galvanometer is further shunted with a 2 Ω wire, the further reduction in the deflection will be (the main current remains the same)

(a) $\left(\frac{8}{13}\right)$ of the deflection when shunted with 4 Ω only

(b) $\left(\frac{5}{13}\right)$ of the deflection when shunted with 4 Ω only

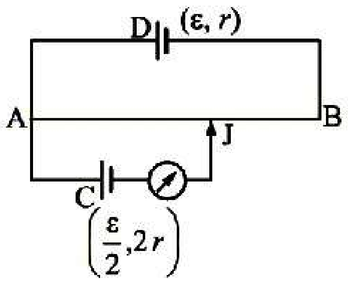
(c) $\left(\frac{3}{4}\right)$ of the deflection when shunted with 4 Ω only

(d) $\left(\frac{3}{13}\right)$ of the deflection when shunted with 4 Ω only

16. A line having a total resistance of 0.5 Ω delivers 15 kW at 240 V to a small factory. The efficiency of transmission is nearly: **Numeric/Integer**

(a) 78% (b) 89% (c) 68% (d) 97%

17. In the potentiometer arrangement shown, the driving cell D has emf ε and internal resistance r. The cell C, whose emf is to be measured, has emf $\frac{\varepsilon}{2}$ and internal resistance $2r$. The potentiometer wire is 100 cm long. If balanced is obtained, the length $AJ = \ell$.



(a) $\ell = 50$ cm

(b) $\ell > 50$ cm

(c) balance will be obtained only if resistance of AB is $> r$.

(d) balance cannot be obtained

18. Two electric bulbs rated at 25 W, 200 V and 100 W, 200 V are connected in series across a 200 -V source. The 25 W and 100 W bulbs now draw P_1 and P_2 powers respectively, then

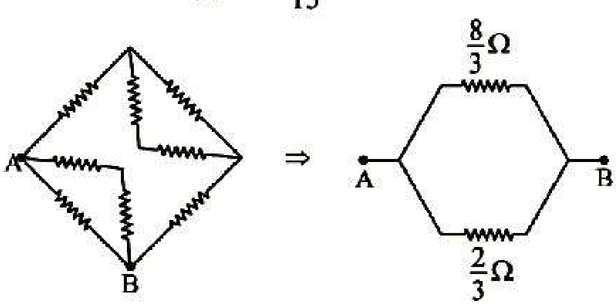
(a) $P_1 = 4$ W (b) $P_2 = 4$ W (c) $P_1 = 16$ W (d) $P_2 = 20$ W

Solutions

1. (c, d) $i = \frac{dq}{dt} = \frac{d(t-t^2)}{dt} = 1 - 2t$. **Ans.**

2. (d) The effective circuit is like:

$\therefore$ $R_{AB} = \frac{8}{15}\Omega$. **Ans.**



3. (d) All the resistors are in parallel, so

$$i = \frac{V}{R} = \frac{1}{\left(\frac{1}{4}\right)} = 4\ A. \qquad \textit{Ans.}$$

4. (a) Take a close loop through 8V and 6V batteries, we have

$$i = \frac{8-6}{5} = 0.4\text{ A}. \qquad \textit{Ans.}$$

5. (a)

$$V_{AB} = \left(\frac{2\times 1 - 4\times 1}{1+1}\right) + 3 = 2V. \qquad \textit{Ans.}$$

6. (d) As the conductor XY does not complete the loop, so $i = 0$. *Ans.*

7. (a)

$$i = \frac{(n-2)\varepsilon}{nr}$$

$$V_B - V_A = \varepsilon - ir = \varepsilon - \frac{(n-2)\varepsilon}{nr}r = \frac{2\varepsilon}{n}. \qquad \textit{Ans.}$$

8. (a) Using,

$$i_g = i\frac{S}{G+S}$$

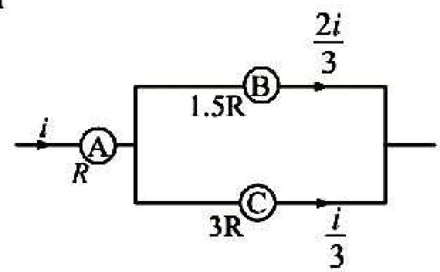
or

$$10 = 100\frac{0.1}{G+0.1} \Rightarrow G = 0.9\ \Omega \quad \textit{Ans.}$$

9. (a) $V_A = iR,\ V_B = \frac{2i}{3}\times(1.5R) = iR$

and

$$V_C = -(3\ \ ) = iR. \ \textit{Ans.}$$



10. (40.00) $\frac{X}{R} = \frac{75}{25} = 3$

$$R = \frac{\rho\ell}{A} = \frac{4\rho\ell}{\pi d^2} \qquad R' = \frac{4\rho\left(\frac{\ell}{2}\right)}{\pi\left(\frac{d}{2}\right)^2} = 2R$$

then $\frac{X}{R'} = \left(\frac{100-\ell}{\ell}\right)$

$$\frac{100-\ell}{\ell} = \frac{X}{2R} = \frac{3}{2} \qquad \ell = 40.00\text{ cm} \qquad \textit{Ans.}$$

11. (a, b, c) 3Ω and 2Ω in series also 4Ω and 1Ω are in series across 10V, so $i_1 = \frac{10}{5} =$ 2A and $i_2 = \frac{10}{5} = 2$A. *Ans.*

12. (b)

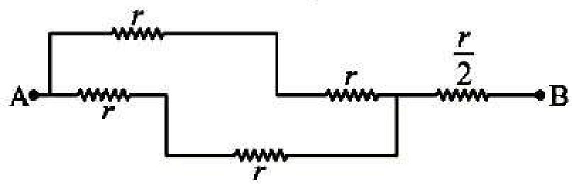
$$V_g = i_g R_g = 0.1\text{ V}$$

$$V = 10\text{ V}$$

$$R = R_g\left(\frac{V}{V_g} - 1\right)$$

$$= 100 \times 99 = 9.9\ K\Omega. \qquad \textit{Ans.}$$

13. (a) The effective circuit is shown in figure.



$$R_{AB} = r+\frac{r}{2} = \frac{3r}{2}.$$ ***Ans.***

14. (d) $\frac{2\varepsilon}{2+2r} = \frac{\varepsilon}{2+\frac{r}{2}} \Rightarrow r = 2\Omega.$ ***Ans.***

15. (b) Using, $i_g = i\frac{S}{S+G} \Rightarrow \frac{1}{5}=\frac{4}{4+G}$

or $G = 16\ \Omega$

$$R_A = \frac{SG}{S+G} = \frac{16\times 4}{20} = 3.2\ \Omega$$

Now, $i_g = i\frac{2}{3.2+2} \Rightarrow \frac{i_g}{i} = \frac{5}{13}.$ ***Ans.***

16. (b) $i = \frac{P}{V} = \frac{15\times 10^3}{240} = 62.5\text{ A}.$

Loss $= i^2R = 62.5^2 \times 0.5 \simeq 2 \times 10^3$ W

Power produced $= 15 + 2 = 17$ kW

Efficiency $= \frac{P_{out}}{P_{in}}\times 100 = \frac{15}{17}\times 100 \simeq 89\%.$ ***Ans.***

17. (b, c) $\left(\frac{\varepsilon}{R+r}\right)\times\frac{R\ell}{L} = \frac{\varepsilon}{2} \Rightarrow \ell = \frac{L(R+r)}{2R} = \frac{1(R+r)}{2R}$

As $\ell < 1$, so $R > r$. Also for r to be non-zero value $\ell > 0.5$. ***Ans.***

18. (b, c) $R_1 = \frac{V^2}{P_1} = \frac{200^2}{25} = 1600\Omega$ and $R_2 = \frac{200^2}{100} = 400\Omega$

In series $i = \frac{V}{R_1+R_2} = \frac{200}{1600+400} = \frac{1}{10}A$

Now $P_1 = i^2R_1 = \left(\frac{1}{10}\right)^2\times 1600 = 16$ W

and $P_2 = \left(\frac{1}{10}\right)^2\times 400 = 4$ W. ***Ans.***

Moving Charges and Magnetism 17

TOPIC 17.1: Motion of a Charged Particle, Cyclotron, Magnetic Force on a Current Carrying Conductor and Torque on a Current Loop.

Review of Formulae

1. When charged particle is projected perpendicular to the magnetic field, its path will be circular.

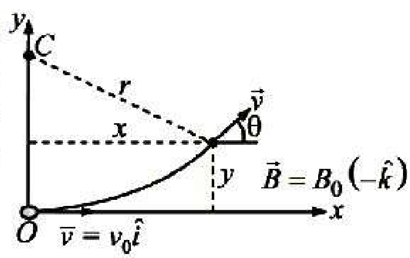
   The radius of path

   $$r = \frac{mv}{qB}$$

   The time to complete the circle

   $$T = \frac{2\pi m}{qB}.$$

2. A charged particle q enters normally in a uniform magnetic field $\vec{B}$. The magnetic field extends to a distance x, which is less than or equal to the radius of the path, then deviation angle θ is given by



   $$\sin\theta = \frac{x}{r}.$$

3. **Cyclotron :** If V is the potential and f is the frequency of the AC source used in cyclotron, then K.E. of the particle q will be

   $$K = 2fqV.$$

4. **Magnetic force on current carrying conductor :**

   $$F = Bi\ell \sin\theta,$$

   and the direction of force can be obtained by Flemings left hand rule. In vector notation $\vec{F} = i\vec{\ell} \times \vec{B}$.

5. **Force on curved conductor :**

   $$\vec{F} = \int_P^Q i\, d\vec{\ell} \times \vec{B}$$

6. Torque on a current loop :

$$\vec{\tau} = \vec{M} \times \vec{B},$$

where $\vec{M} = Ni\vec{A}$, is called magnetic moment of the loop.

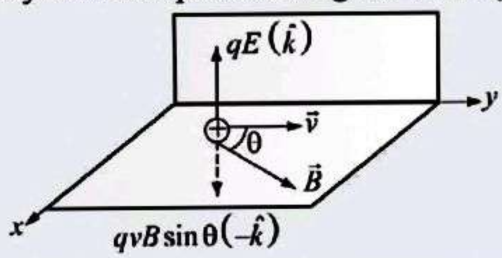
Tips and Tricks for Shortcut Solutions

1. The path of charged particle in $\vec{E}$-field is either straight line or parabolic.
2. The path of charged particle in $\vec{B}$-field is either straight line, circular or helical.
3. When charged particle goes undeviated in a region having $\vec{E}$ and $\vec{B}$-field both, electric field may have component along the velocity of the particle.
4. When charged particle goes without change in velocity in a region having $\vec{E}$ and $\vec{B}$-field both, electric field must be perpendicular to velocity of the particle or

$$\vec{F} = \vec{F}_m + \vec{F}_e = q(\vec{E} + \vec{v} \times \vec{B}) = 0$$

$$\Rightarrow \quad \vec{E} = -\vec{v} \times \vec{B}.$$

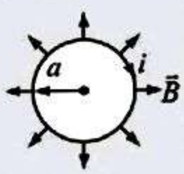
Magnetic field may have component along the velocity of the particle.



5. The ratio of magnetic force to electric force between two charged particles moving parallel

$$\frac{F_m}{F_e} = \frac{v^2}{c^2}.$$

6. The net force on a current carrying close loop of any shape placed in uniform magnetic field with any orientation will be zero.
7. In a radial magnetic field in the plane of the current carrying loop, the magnetic force is



$$F = Bi(2\pi a).$$

8. If a charged body of mass m, and charge q rotates, then its $\vec{M}$ and $\vec{L}$ are related as;

$$\vec{M} = \frac{(\pm q)}{2m}\vec{L}.$$

9. If particle describes angle α inside field, then time spent by it inside field,

$$t = \frac{m\alpha}{qB}, \text{ here } \alpha \to \text{rad.}$$

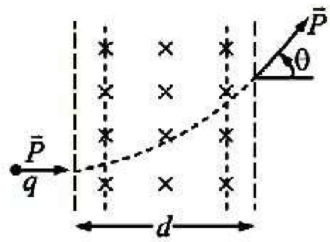
10. For the given perimeter of loop, circular loop has greatest area, so moment of couple will be maximum for circular coil when placed in magnetic field.

$$\tau = NiAB \sin\theta.$$

Illustration 1

A charge particle with a charge q and momentum P, enters into a uniform magnetic field of intensity B normally. If width of the field is 'd' then find deviation angle of the particle. $\left(d < \frac{P}{Bq}\right)$

Short-cut solution :

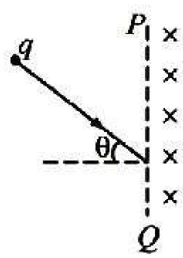


Using, $\sin\theta = \frac{x}{r}$

$$= \frac{d}{\left(\frac{P}{Bq}\right)} = \frac{Bqd}{P}. \quad \textit{Ans.}$$

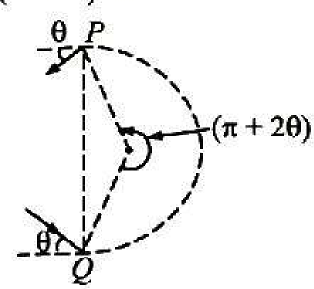
Illustration 2

A particle with charge +q and mass m enters a magnetic field of magnitude B, existing only to the right of the boundary PQ. The direction of the motion of the particle is perpendicular to the direction of $\vec{B}$. Find time spent by the particle in the magnetic field.



Short-cut solution :

The angle moved by the particle in-side magnetic field is (π + 2θ) rad.



Using, $t = \frac{T}{2\pi}(\pi + 2\theta) = \frac{\left(\frac{2\pi m}{qB}\right)}{2\pi}(\pi + 2\theta)$

$$= \frac{m}{qB}(\pi + 2\theta). \quad \textit{Ans.}$$

Illustration 3

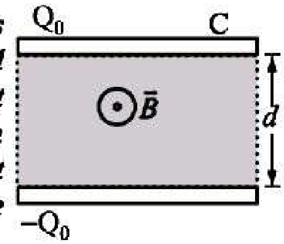
Two protons move parallel to each other with an equal velocity $v = 3 \times 10^5$ m/s. Find ratio of forces of magnetic and electric interaction of the protons.

Short-cut solution :

Using,
$$\frac{F_m}{F_e} = \frac{v^2}{c^2}$$
$$= \frac{(3\times10^5)^2}{(3\times10^8)^2} = 10^{-6}.$$ **Ans.**

Illustration 4

A slab of resistance R is inserted between two parallel plates of a capacitor charged to Q_0. The capacitor is discharged through the solid. A magnetic field is present throughout and is out of the plane of the paper. The total momentum given to the slab after complete discharge is (given that the capacitance of the capacitor is C and the plates are separated by a distance d).



(a) CQ_0B *(b)* dBQ_0 *(c)* zero *(d)* infinity

Short-cut solution :

Momentum,
$$P = \int F dt$$
$$= \int (Bid)dt$$
$$= Bd\int i dt$$
$$= BdQ_0.$$ **Ans. (b)**

Illustration 5

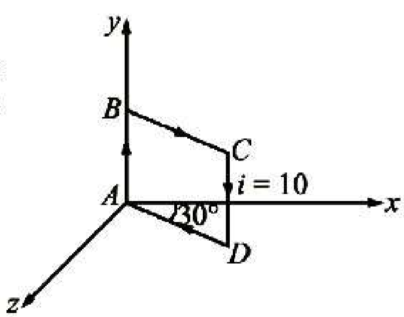
A uniform sphere of mass m = 1kg having charge q = 2μC is rotated about its diameter. Find its ratio of magnetic moment to angular momentum.

Short-cut solution :

Using,
$$\frac{M}{L} = \frac{q}{2m} = \frac{2}{2\times1}$$
$$= 1\ \mu\text{C/kg}.$$ **Ans.**

Illustration 6

Figure shows a square current carrying loop ABCD of side 10 cm and current i = 10A. The magnetic moment $\vec{M}$ of the loop is



(a) $(0.05)\left(\hat{i} - \sqrt{3}\hat{k}\right)$ A-m²

(b) $(0.05)\left(\hat{j} + \hat{k}\right)$ A-m²

(c) $(0.05)\left(\sqrt{3}\hat{i} + \hat{k}\right)$ A-m² *(d)* $\left(\hat{i} + \hat{k}\right)$ A-m²

Short-cut solution :

For constant speed, net force on the particle is zero. So

$$\vec{F} = 0 = q(\vec{E} + \vec{v} \times \vec{B})$$

or $$0 = +e[\vec{E} + 50\hat{i} \times 2 \times 10^{-3}\hat{j}]$$

or $$0 = [\vec{E} + 0.10\hat{k}]$$

$\therefore$ $$\vec{E} = -0.10\hat{k}\ V/m$$ ***Ans.***

Illustration 10

A current carrying conductor lies along the curve $x^2 = 2y$. If the current in the conductor is I then the force acting on the wire lying between $x = -2$ and $x = +2$, due to a magnetic field in the region is $\vec{B} = -B_0\hat{k}$

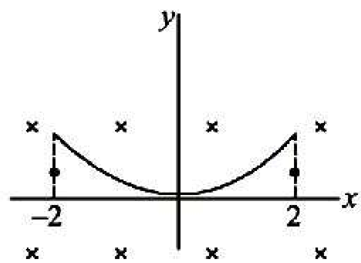
(a) $2IB_0$ *(b)* $4IB_0$ *(c)* IB_0 *(d)* $\frac{IB_0}{2}$

Short-cut solution :

The effective length of the conductor is = 2 + 2 = 4

So $F = Bil$

$= B_0 i \times 4 = 4\,B_0 i.$ ***Ans. (b)***

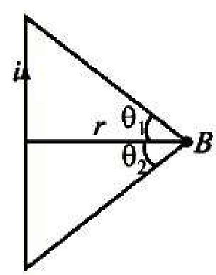


TOPIC 17.2: *Magnetic Field due to a Current Carrying Wire, Biot-Savart's Law, Magnetic Field due to Circular Current Carrying Coil, Solenoid, Force between Two Parallel Current Carrying Wires and Ampere's Law.*

Review of Formulae

1. Magnetic field due to a current carrying wire :

$$B = \frac{\mu_0}{4\pi}\frac{i}{r}(\sin\theta_1 + \sin\theta_2).$$



Magnetic field due to a long straight wire

$$B = \frac{\mu_0 i}{2\pi r},$$

at the middle region of the wire.

and $$B = \frac{\mu_0 i}{4\pi r}$$ at the end of the wire.

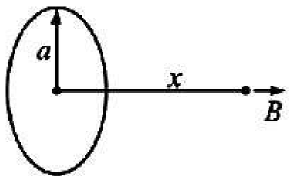
2. **The force between two parallel current carrying wires :**

$$\frac{dF}{d\ell} = \frac{\mu_0}{2\pi}\frac{i_1 i_2}{r}.$$

3. **Magnetic field at the centre of the circular current carrying coil**

$$B = \frac{\mu_0 N i}{2a}$$

The magnetic field on the axis of the circular current carrying coil



$$B = \frac{\mu_0 i a^2}{2\left(a^2+x^2\right)^{3/2}} = \frac{\mu_0}{4\pi}\frac{2\vec{M}}{(R^2+x^2)^{3/2}}$$

4. **Magnetic field due to a solenoid**

$$B = \frac{\mu_0 n i}{2}\left[\cos\theta_1 - \cos\theta_2\right]$$

Magnetic field due to a long solenoid at its centre

$$B = \mu_0 n i.$$

Magnetic field of a toroid of radius R

$$B = \frac{\mu_0 N i}{2\pi R}$$

5. **Ampere's law :** It states that

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0\, i_{in}.$$

The line integral in this equation is evaluated around a closed loop called Amperian loop. The current i_{in} is the net current encircled by the loop.

6. The magnetic field inside a current carrying tube is zero.
7. The magnetic field due to a magnetic dipole on its axis

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2\vec{M}}{x^3}.$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. If a moving charge particle produces electric field $\vec{E}$ at a point, then magnetic field produced by the particle at the same point will be, $\vec{B} = \left[\frac{\vec{v}\times\vec{E}}{c^2}\right]$.

2. The magnetic field in a cavity parallel to the axis of a long straight conductor of current density $\vec{J}$ is given by

$\vec{B} = \frac{\mu_0}{2}(\vec{j} \times \vec{\ell})$. Here $\vec{\ell}$ is the distance between axis of conductor and cavity.

3. Magnetic field of a large current carrying sheet nearby it, carrying current i_0 per unit width is given by

$$B = \frac{\mu_0 i_0}{2}.$$

4. When a circular coil produces magnetic field at the centre is folded into a coil of n-turns, the magnetic field at the centre due to same current

$$B' = n^2 B.$$

5. The magnetic field at the centre is zero in the following cases :

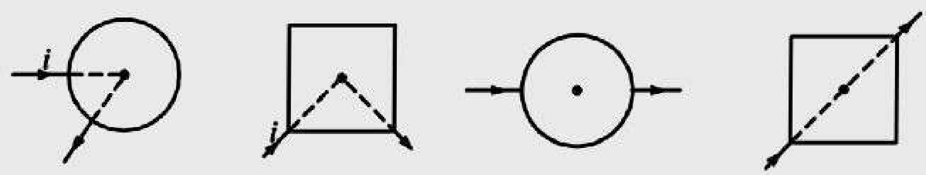


Illustration 11

A charge particle is moving with a velocity $\vec{v} = (\hat{i} + \hat{j})$ and produces an electric field at a point equal to $\vec{E} = 2\hat{k}$. Find magnetic field produced by it at the same point.

Short-cut solution :

Using,
$$\vec{B} = \frac{\vec{v} \times \vec{E}}{c^2}$$

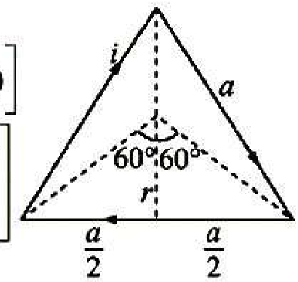
$$= \frac{(\hat{i} + \hat{j}) \times (2\hat{k})}{c^2}$$

$$= \frac{2(\hat{i} + \hat{j})}{c^2}.$$ ***Ans.***

Illustration 12

An equilateral triangular loop of side a carries current i. Find magnetic field at its centre.

Short-cut solution :



$$B = 3\left[\frac{\mu_0}{4\pi}\frac{i}{r}(\sin\theta_1 + \sin\theta_2)\right]$$

$$= \left[\frac{\mu_0}{4\pi}\frac{i}{\left(\frac{a}{2\sqrt{3}}\right)} \times 2\sin 60°\right]$$

$$= \frac{9}{2\pi}\frac{\mu_0 i}{a}.$$ ***Ans.***

Illustration 13

Two long current carrying conductors carry equal currents are placed parallel. Draw magnetic field along the line joining them in the following cases:

(i) Currents in the same direction.

(ii) Currents in opposite direction. ***[JEE Main 2010]***

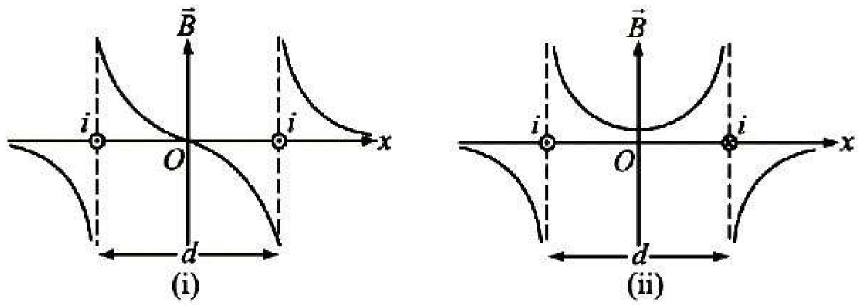
Short-cut solution :

Assuming the direction of $\vec{B}$ positive along +*ve*-*x*-axis and *y*-axis.

(i) If d is the seperation between them, then at a distance x from left conductor, magnetic field

$$B = \frac{\mu_0 i}{2\pi}\left[\frac{1}{x} - \frac{1}{d-x}\right]$$

At $x \to 0, B \to \infty$. Also $x \to d, B \to \infty$. At $x = \frac{d}{2}, \; B = 0.$

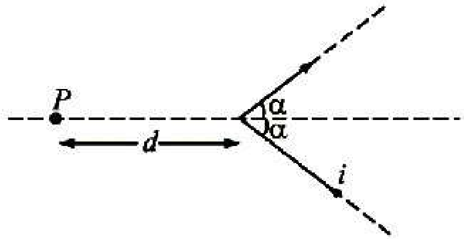


(ii) $B = \frac{\mu_0 i}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right].$

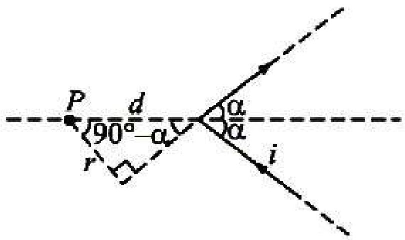
$x = 0$ and d, $B \to \infty$ and at $x = \frac{d}{2}, \; B = \frac{2\mu_0 i}{\pi d}.$ ***Ans.***

Illustration 14

An infinite long wire carries current i bent as shown. Find magnetic field at point P.



Short-cut solution :



$$r = d\sin\alpha$$

$$B = 2\left[\frac{\mu_0 i}{4\pi}\frac{(\sin 90° - \sin(90° - \alpha))}{d\sin\alpha}\right]$$

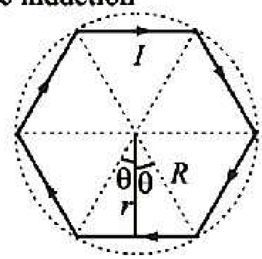
$$= \frac{\mu_0 i}{2\pi d}\frac{(1-\cos\alpha)}{\sin\alpha} = \frac{\mu_0 i}{2\pi d}\tan\frac{\alpha}{2}. \qquad \textit{Ans.}$$

Illustration 15

A current I flows along a thin wire shaped as a regular polygon with n sides which can be inscribed into a circle of radius R. Find the magnetic induction at the centre of the polygon. Analise the obtained expression at n → ∞.

Short-cut solution :

In a regular polygon of n-sides, each of its side subtends an angle $\frac{2\pi}{n}$ at the centre of the polygon. The angle $\theta = \frac{\pi}{n}$. If B_1 is the field produced by each side of the polygon, then total magnetic induction



$$B = nB_1$$

$$= n\frac{\mu_0 I}{4\pi}\frac{\left[\sin\frac{\pi}{n} + \sin\frac{\pi}{n}\right]}{r}$$

where $\qquad r = R\cos\frac{\pi}{n}.$

Thus $\qquad B = n\frac{\mu_0}{4\pi}I \times \frac{2\sin\pi/n}{R\cos\pi/n}$

$$= \frac{\mu_0 nI}{2\pi R} \tan\frac{\pi}{n} \qquad \textit{Ans.}$$

For $n \to \infty$, we can write

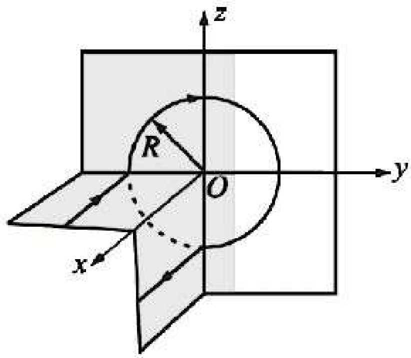
$$B = \frac{\mu_0 I}{2R} \lim_{n\to\infty}\left[\frac{\tan \pi / n}{\pi / n}\right]$$

$$= \frac{\mu_0 I}{2R} \qquad \textit{Ans.}$$

Polygon of infinite sides is a circle, and so its magnetic induction at the centre is equal to that due to a circular loop.

Illustration 16

Find the magnetic induction of the field at the point O if the wire carrying a current I has the shape shown in figure. The radius of the curved part of wire is R, the linear parts of the wire are very long.



Solution :

In this case current in the curved conductor is divided into two parts. The length of upper curved part is thrice that of hidden lower part and so currents in them are $\frac{I}{4}$ and $\frac{3I}{4}$ respectively. Thus

$$\vec{B} = \vec{B}_{\text{straight}} + \vec{B}_{\text{curved}}$$

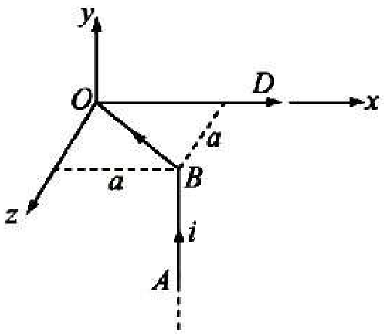
$$= \left[\frac{\mu_0 I}{4\pi R}(-\hat{k}) + \frac{\mu_0 I}{4\pi R}(-\hat{j})\right]$$

$$+ \left[\frac{\mu_0\left(\frac{3}{4}\right)\left(\frac{I}{4}\right)}{2R}(-\hat{i}) + \frac{\mu_0\left(\frac{1}{4}\right)\left(\frac{3I}{4}\right)}{2R}\hat{i}\right]$$

$$= -\frac{\mu_0 I}{4\pi R}(\hat{j} + \hat{k}). \qquad \textit{Ans.}$$

Illustration 17

An infinite long conductor carries current i is bent as shown in figure. Find magnetic field at the origin.



Short-cut solution :

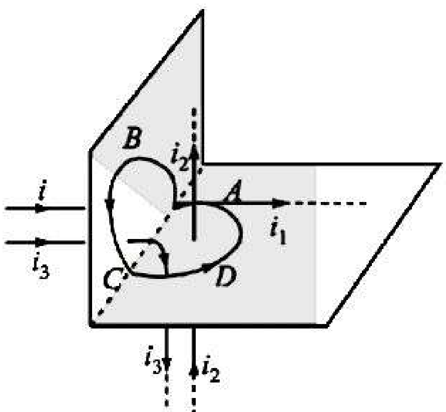
$$\vec{B} = \vec{B}_{AB} + \vec{B}_{BO} + \vec{B}_{OD}$$

$$= \frac{\mu_0 i}{4\pi a\sqrt{2}}\left[\sin 45°(-\hat{i}) + \sin 45°\hat{k}\right] + 0 + 0$$

$$= \frac{\mu_0 i}{8\pi a}(-\hat{i} + \hat{k}).$$ ***Ans.***

Illustration 18

Figure shows an Amperian path ABCDA. The loop lies in two perpendicular planes as shown. Find the value of $\oint \vec{B}\cdot d\vec{\ell}$ for the path ABCDA.



Short-cut solution :

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 i_{in}$$

$$= \mu_0(i_1 + i_3 + i_2 - i_3)$$

$$= \mu_0(i_1 + i_2).$$ ***Ans.***

***TOPIC 17.3:** Magnetic Moment, Magnetic Field due to a Bar Magnet, Work done, Potential Energy in Magnetic Field, Torque, Moving Coil Galvanometer, Vibration Magnetometer and Gauss's Law in Magnetism.*

Review of Formulae

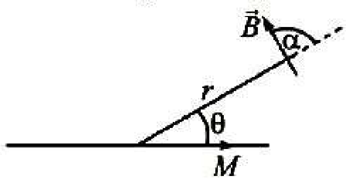
1. **Bar magnet :** If m is the magnetic charge and ℓ is the length of the magnet, then magnetic moment of the magnet

$$\vec{M} = m\vec{\ell}$$

2. **Magnetic field due to a magnetic charge *m* at a distance *r***

$$B = \frac{\mu_0}{4\pi}\frac{m}{r^2}$$

3. Magnetic field due to a short magnetic dipole *M* at a distance *r*



$$B = \frac{\mu_0}{4\pi}\frac{M}{r^3}\sqrt{3\cos^2\theta + 1}$$

and $$\tan\alpha = \frac{\tan\theta}{2}$$

(i) On the axis of the dipole, $\theta = 0$

$$B = \frac{\mu_0}{4\pi}\frac{2M}{r^3}$$

(ii) On the equator, $\theta = 90°$

$$B = \frac{\mu_0}{4\pi}\frac{M}{r^3}$$

4. **Torque on a magnet in magnetic field**

$$\vec{\tau} = \vec{M} \times \vec{B}$$

5. **Work done in increasing angle from θ_1 to θ_2**

$$W = MB(\cos\theta_1 - \cos\theta_2).$$

6. **The potential energy of magnet in magnetic field**

$$U = -MB\cos\theta.$$

7. **True dip and apparent dip :** If θ is the true dip and θ′ is the apparent dip in a plane making angle α with the meridian plane, then

$$\tan\theta' = \frac{\tan\theta}{\cos\alpha}$$

8. Moving coil galvanometer : If i is the current in galvanometer, then

$$i = \frac{C}{NBA}\alpha$$

9. Vibration magnetometer : The time period of magnet of magnetometer

$$T = 2\pi\sqrt{\frac{I}{MB_H}}$$

For magnets of magnetic moments M_1 and M_2

$$\frac{M_1}{M_2} = \frac{T_2^2 + T_1^2}{T_2^2 - T_1^2}$$

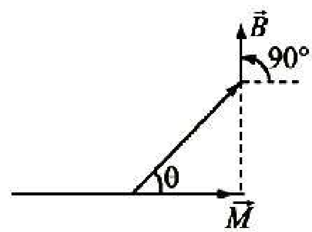
10. Gauss's law in magnetism :

$$\oint \vec{B} \cdot d\vec{A} = 0$$

Illustration 19

Find the angular position at which magnetic field is perpendicular to the magnetic dipole, provided it to be short enough.

Short-cut solution :



$$\theta + \alpha = 90°$$

Also $$\tan\alpha = \frac{\tan\theta}{2}$$

or $$\tan(90 - \theta) = \frac{\tan\theta}{2}$$

or $$\frac{1}{\tan\theta} = \frac{\tan\theta}{2} \Rightarrow \theta = \tan^{-1}(\sqrt{2}).$$ ***Ans.***

Video Solution

Q. If θ_1 and θ_2 are the angles of dip observed in two vertical planes right angles to each other and θ be the true angle of dip, show that $\cot^2\theta = \cot^2\theta_1 + \cot^2\theta_2$.

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=Z7qULsmGLck

Illustration 20

The angle of dip at the place is 45°. Find apparent angle of dip in a plane 30° from meridian plane.

Short-cut solution :

Using, $\tan\theta' = \dfrac{\tan\theta}{\cos\alpha}$

$$= \frac{\tan 45^\circ}{\cos 30^\circ} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

or $\theta' = \tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$. ***Ans.***

Illustration 21

A bar magnet vibrates in a magnetometer with a period T. Find the time period if an identical piece of wood is placed of the magnet.

Short-cut solution :

$$T = 2\pi\sqrt{\frac{I}{MB_H}}$$

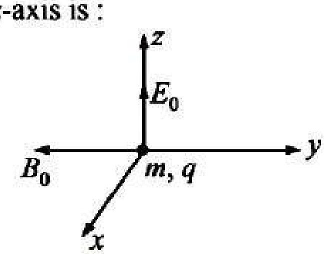
When identical piece of wood (non-magnetic) is placed on the magnet

$$I' = 2I \text{ and } M' = M + 0 = M$$

$\therefore$ $T' = 2\pi\sqrt{\dfrac{2I}{MB_H}} = \sqrt{2}T$. ***Ans.***

Concept Booster Exercise

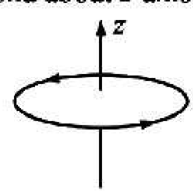
1. A particle of mass *m* and charge *q* is released from rest at origin as shown. The speed of the particle when it has travelled a distance '*d*' along *z*-axis is :



(a) $\sqrt{\dfrac{2m}{qE_0d}}$ (b) $\sqrt{\dfrac{2qE_0d}{m}}$

(c) $\sqrt{\dfrac{E_0d}{B_0}}$ (d) zero

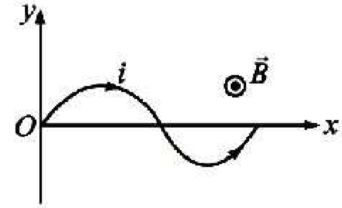
2. An electron in a circular orbit of radius *r* makes *n* rotations per second about *z*-axis in counter clockwise sense, its magnetic moment



(a) $-\pi r^2 ne\hat{k}$ (b) $\pi r^2 ne\hat{k}$

(c) $\dfrac{r^2ne}{2\pi}\hat{i}$ (d) $-\dfrac{\pi}{2}r^2ne\hat{j}$

3. A wire carrying current I is placed in a uniform magnetic field $\vec{B}$ is in the form of the curve $y = a\ \sin\left(\frac{\pi x}{L}\right)$; $0 \le x \le 2L$. The force acting of the wire is :



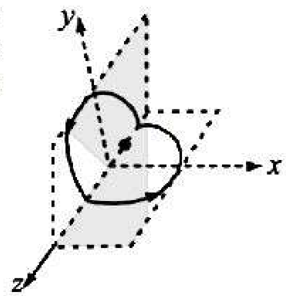
(a) $\frac{IBL}{\pi}$ (b) $IBL\pi$

(c) $2BIL$ (d) zero

4. A non-conducting rod of length ℓ has a total charge q. The rod is rotated about an axis passing through its centre of mass with constant angular velocity ω. The magnetic moment of the rod is:

(a) $\frac{q\omega\ell^2}{6}$ (b) $\frac{q\omega\ell^2}{2}$ (c) $\frac{q\omega\ell^2}{12}$ (d) $\frac{q\omega\ell^2}{24}$

5. A circular loop of wire of radius R is bent about its diameter along two mutually perpendicular planes as shown in the figure. If the loop carries current I, then its magnetic moment is:



(a) $\frac{\pi R^2 I}{2}(\hat{i}+\hat{j})$ (b) $\pi R^2 I(\hat{i})$

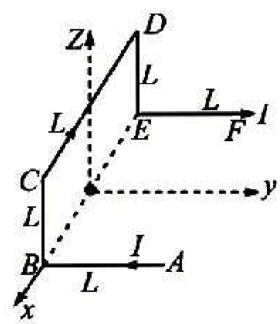
(c) $\frac{\pi R^2 I}{2}(\hat{j}+\hat{k})$ (d) zero

6. The magnetic field at the centre of current carrying loop is B. When it folded into two turns, the magnetic field due to the same current at the centre of the loop is $x\ B$: Find the value of x.

Numeric/Integer

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) 4

7. A conductor $ABCDEF$, with each side of length L, is bent as shown. It is carrying a current I in a uniform magnetic field $\vec{B}$ parallel to the positive y direction. The force experienced by the wire is $\frac{\mu_0 I}{x\pi a}$. Find the value of x.



(a) BIL in the positive y-direction

(b) BIL in the position x-direction

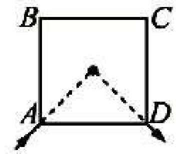
(c) 3 BIL along positive y-direction

(d) zero

8. A particle of mass m and positive charge q is projected with a speed of v_0 in y-direction in the presence of electric and magnetic field are in x-direction. Find the instant of time at which the speed of particle becomes double the initial speed. *[JEE Main 2020]*

(a) $t = \frac{mv_0\sqrt{3}}{qE}$ (b) $t = \frac{mv_0\sqrt{2}}{qE}$ (c) $t = \frac{mv_0}{qE}$ (d) $t = \frac{mv_0}{2qE}$

9. *ABCD* is a square loop made of uniform conducting wire. A current enters the loop at *A* and leaves at *D*. The magnetic field is

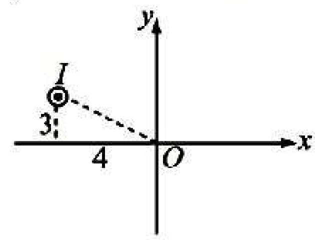


(a) zero at all points inside the loop
(b) zero at the centre of the loop
(c) maximum at the centre of loop
(d) zero at each point

10. Electric field in space is given by $\vec{E}(t) = E_0 \frac{(\hat{i}+\hat{j})}{\sqrt{2}} \cos(\omega t + Kz)$. A positively charged particle at $(0, 0, \pi/K)$ is given velocity $v_0\hat{k}$ at $t = 0$. Direction of force acting on particle is *[JEE Main 2020]*

(a) $f = 0$
(b) Antiparallel to $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$
(c) Parallel to $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$
(d) $\hat{k}$

11. An infinitely long straight with current *I* flowing along the positive *z*-axis is located at (–4, 3) as shown. The magnetic field at point *O* is:

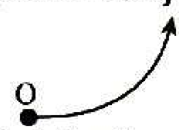


(a) $\frac{\mu_0}{2\pi} I(\hat{i}+\hat{j})$
(b) $\frac{\mu_0}{4\pi} I(3\hat{i}+4\hat{j})$
(c) $\frac{\mu_0}{2\pi} I\left(\frac{3\hat{i}+4\hat{j}}{25}\right)$
(d) none of these

12. A length ℓ of a wire is bent to form a circular coil of some turns. A current *I* is then established in the coil and is placed in uniform $\vec{B}$. The maximum torque acts on the coil is $\frac{IB\ell^2}{x\pi}$. Find the value of *x*. **Numeric/Integer**

(a) 0
(b) 2
(c) 4
(d) 1.5

13. A charged particle initially at rest at *O*, when released follows a trajectory as shown. Such a trajectory is possible in the presence of : *[KVPY - 2014]*



(a) electric field of constant magnitude and varying direction.
(b) magnetic field of constant magnitude and varying direction.
(c) electric field of constant magnitude and constant direction.
(d) electric and magnetic fields of constant magnitudes and constant directions which are parallel to each other

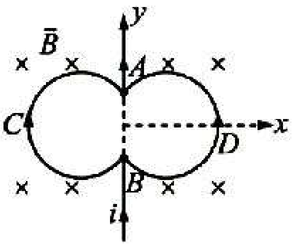
14. A particle of charge *q* and mass *m* starts moving from origin under the action of an electric field $\vec{E} = E_0\hat{i}$ and magnetic field $\vec{B} = B_0\hat{k}$. Its velocity at $(x, 3, 0)$ is $(4\hat{i} + 3\hat{j})$. The value of *x* is:

(a) $\frac{10m}{qE_0}$
(b) $\frac{18E_0}{qB_0 m}$
(c) $\frac{25E_0B_0}{m}$
(d) $\frac{25m}{2qE_0}$

15. The electric field and the magnetic field in a region are given by $\vec{E} = \hat{i}E_0$ and $\vec{B} = \hat{j}B_0$. Consider a frame of reference moving with a velocity $v_0\hat{k}$. The electric field in this frame will be zero if v_0 is equal to :

(a) $\frac{E_0}{B_0}$ (b) $\frac{2E_0}{B_0}$ (c) $\frac{E_0}{2B_0}$ (d) none of these

16. The figure shows a conducting loop *ACBDA* placed in the plane perpendicular to a uniform magnetic field $\vec{B}$. The two parts *ACB* and *ADB* are circular arcs of radius '*a*'. The points *A* and *B* are at $(0, +\ell)$ and $(0, -\ell)$ respectively. The current in the loop is *i*. The magnetic force on the loop due to the field is $x\ Bil$. Find the value of *x*.



Numeric/Integer

(a) 1 (b) 2 (c) 3 (d) zero

17. A positive charge *Q* is distributed over a circular ring of radius '*r*'. It is placed in a horizontal plane and is rotated about its axis at a frequency '*f*'. A horizontal field $\vec{B}$ exists in the space. The torque acting on the ring due to the magnetic field is:

(a) $\pi r^2 QfB$ (b) $\frac{\pi}{2}r^2 QfB$ (c) $2\pi r^2 QfB$ (d) zero

18. A long cylindrical wire of radius 10 cm caries an electric current. The magnetic field at a point outside the wire at a distance 5 cm from the surface is 4.0 μT. The magnetic field at a point inside the wire at a distance 5 cm from the surface will be:

(a) $3\mu T$ (b) $4\mu T$ (c) $5\mu T$ (d) $12\mu T$

Solutions

1. **(b)** Magnetic field does not change the speed. It is only by $\vec{E}$-field, so $a_Z = \frac{E_0 q}{m}$ and $v^2 = 0^2 + 2a_Z d$.

2. **(a)** $i = \frac{e}{T} = ne \quad \vec{M} = i\vec{A} = (ne) \times \pi r^2(-\hat{k})$

3. **(c)** $F = Bi\ell \sin\theta = Bi(2L)\sin 90° = 2BIL.$

4. **(d)** $M = \int i(dA) = 2\int_0^{\frac{\ell}{2}} \frac{\left(\frac{q}{\ell}dx\right)}{\left(\frac{2\pi}{\omega}\right)} \times \pi x^2 = \frac{q\omega\ell^2}{24}.$

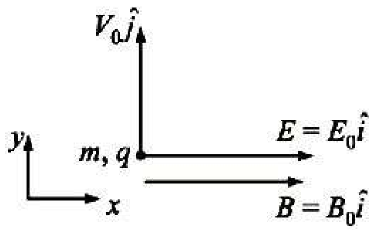
5. (a) $\vec{M} = iA(\hat{i}+\hat{j}) = \dfrac{\pi R^2}{2} I(\hat{i}+\hat{j})$.

6. (d) Using, $B' = n^2 B = 2^2 B = 4B$.

7. (a) Side AB and EF, BC and DE have equal and opposite forces. So net force is only on side CD,

$$\vec{F}_{CD} = BIL(\hat{j}).$$

8. (a)



As $\vec{v} = v_0\hat{j}$ (magnitude of velocity does not change in y-z plane)

$$(2v_0)^2 = v_0^2 + v_x^2;\ v_x = \sqrt{3}v_0$$

$\therefore$ $$\sqrt{3}v_0 = 0 + \frac{qE}{m}t; \qquad t = \frac{mv_0\sqrt{3}}{qE}$$ ***Ans.***

9. (b) $B = k3\left(\dfrac{i}{4}\right) - k\left(\dfrac{3i}{4}\right) = 0$

10. (c) Force due to electric field is in direction $-\dfrac{(\hat{i}+\hat{j})}{\sqrt{2}}$

because at $t = 0$, $E = -\dfrac{(\hat{i}+\hat{j})}{\sqrt{2}}E_0$

Force due to magnetic field is in direction $q(\vec{v}\times\vec{B})$ and $\vec{v} \parallel \hat{k}$

$\therefore$ it is parallel to $\vec{E}$

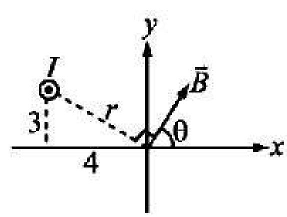
$\therefore$ net force is antiparallel to $\dfrac{(\hat{i}+\hat{j})}{\sqrt{2}}$. ***Ans.***

11. (c)

$$\vec{B} = \frac{\mu_0}{2\pi}\cdot\frac{I}{r}(\cos\theta\hat{i} + \sin\theta\hat{j})$$

$$= \frac{\mu_0}{2\pi}\cdot\frac{I}{r}\left(\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}\right)$$

$$= \frac{\mu_0}{2\pi}\cdot\frac{I}{r^2}(3\hat{i}+4\hat{j}) = \frac{\mu_0}{2\pi}I\left[\frac{(3\hat{i}+4\hat{j})}{25}\right].$$



12. (b) For given length, the area is maximum for circular coil, so

$$\tau_{max} = IAB\sin 90° = I\times\left(\frac{\ell}{2\pi}\right)^2 \pi B = \frac{IB\ell^2}{4\pi}.$$

13. **(a)** Magnetic force on static charge will be zero.

14. **(d)** $\vec{v} = 4\hat{i} + 3\hat{j}$, so $v = 5$ m/s,

using, $v^2 = 0 + 2a_x x \Rightarrow x = \dfrac{v^2}{2a_x} = \dfrac{5^2}{2\left(\dfrac{E_0 q}{m}\right)} = \dfrac{25m}{2qE_0}$.

15. **(a)** For zero electric field,

$$\vec{F}_e + \vec{F}_m = 0$$

or $q\vec{E} + q(\vec{v}_0 \times \vec{B}) = 0$

or $v_0 = \dfrac{E_0}{B_0}$

16. **(b)** $F = B\left(\dfrac{i}{2}\right) \times 2\ell + B\left(\dfrac{i}{2}\right) \times 2\ell = 2Bi\ell$.

17. **(a)** $\tau = iAB \sin 90° = \left(\dfrac{Q}{T}\right)\pi r^2 B = \pi r^2 QfB$.

18. **(a)** $B = \dfrac{\mu_0}{2\pi} \cdot \dfrac{i}{r}$

or $4 = \dfrac{\mu_0}{2\pi} \cdot \dfrac{i}{0.15}$

And $B' = \dfrac{\mu_0}{2\pi} \cdot \dfrac{ir}{a^2} = \dfrac{\mu_0}{2\pi} \cdot \dfrac{i \times 0.05}{(0.1^2)}$

On solving above equations, we get $B' = 3\mu T$.

Electromagnetic Induction 18

TOPIC 18.1: *Magnetic Flux, Faraday's Law of Electromagnetic Induction and Motional emf.*

Review of Formulae

1. **Magnetic flux :** Magnetic flux of the magnetic field $\vec{B}$ through the normal area A is

$$\phi_B = B_{\perp}A = BA\cos\theta.$$

2. **Faraday's law :** Whenever there is change in magnetic flux linked with circuit, there induces an emf in the circuit. The rate of change of magnetic flux is equal to the induced emf. Thus

$$e = -\frac{d\phi_B}{dt},$$

here negative sign indicates that induced emf opposes the change in flux.

Induced charge in time Δt,

$$\Delta Q = \frac{\Delta\phi}{R}.$$

3. **Motional emf :** When a metallic conductor moves in a magnetic field $\vec{B}$ with velocity an emf induces across its ends. The induced emf

$$e = Bv\ell\sin\theta.$$

In general, it can be written as :

$$e = \int(\vec{v}\times\vec{B})\cdot d\vec{\ell}.$$

4. When a metallic conductor is rotated in normal magnetic field about its one ends, the induced emf across the ends

$$e = \frac{B\omega\ell^2}{2}.$$

5. Induced electric field : If E_n is the induced electric field, then by Faraday's law

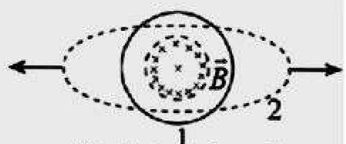
$$e = \oint\vec{E}\cdot d\vec{\ell} = -\frac{d\phi_B}{dt}.$$

6. Induced emf in a coil rotating in uniform magnetic field

$$e = NBA\omega\sin\omega t.$$

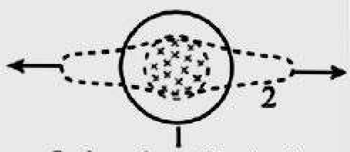
TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. For induced electric field or emf, there are two basic mechanisms;
   (i) The motion of conductor relative to magnetic field, called motional emf. The angle between $\vec{v}\times\vec{B}$ and $\vec{\ell}$ should not be 90° or between $\vec{v}$ and $\vec{B}$ should not be zero.
   (ii) Change in magnetic flux with time.
2. The induced emf in a loop is distributed around the loop in proportional to the resistance of its parts.
3. Induced electric field is of non-conservative nature so, its integral over close loop is not zero, $\oint \vec{E}\cdot d\ell = -\frac{d\phi_B}{dt}$. In close loop potential difference has no meaning so we use induced emf.
4. When the field $\vec{B}$ varies with time as well as the distance, the induced emf should be calculated by taking into consideration of two factors. So induced emf can be written as :
$$e = \oint \vec{E}\cdot d\vec{\ell} = -\frac{\partial\phi}{\partial t} + \oint\left[\vec{v}\times\vec{B}\right]\cdot d\vec{\ell}$$
   Here the first term is due to the time variation of the magnetic field, while the second is due to the motion of the loop.
5. When magnetic field changes with both time and space:
$$e = -\frac{d\phi_B}{dt} = -\left[\frac{\partial\phi_B}{\partial t} + \frac{\partial\phi_B}{\partial x}\frac{\partial x}{\partial t}\right] = -A\left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial x}v\right].$$
6. In apparent change in magnetic flux, there is no induced emf. See following cases:
   (i) Let us take a steady magnetic field inside a conducting loop 1. If loop changes in area as like shape 2, there is no induced emf in the loop because there is apparent change in magnetic flux.



No induced emf

   (ii) In above case, if any part of loop crosses the magnetic field, then there will be induced emf in it (see figure).

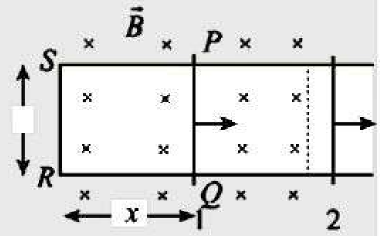


Induced emf in the loop

   (iii) Now let us take a conductor PQ moves on a U-shape conductor placed in steady magnetic field. The emf will be induced in the loop till the conductor PQ moves in the field (position 1).

Induced emf, $|e| = \frac{dQ}{dt} = B\frac{d(\ell x)}{dt}$

$= B\ell\frac{dx}{dt} = Bv\ell.$



When conductor PQ moves outside the magnetic field, there is apparent change in magnetic flux, there is no induced emf in the loop when conductor in position 2.

(iv) Take a conducting loop of radius r is placed in time varying magnetic field $\left(\frac{dB}{dt}\right)$ in a cylindrical region R ($r > R$). Here the change in flux in region of area πr^2. Therefore induced emf in the loop

$$|e| = \frac{d\phi_B}{dt} = \pi R^2\left(\frac{dB}{dt}\right)$$

Conducting loop

If E_n is the induced electric field, then

$$\oint \vec{E}_n . d\vec{\ell} = -\frac{d\phi_B}{dt}$$

or $$E_n \times 2\pi r = -\pi R^2\frac{dB}{dt}$$

or $$E_n = -\frac{R^2}{2r}\left(\frac{dB}{dt}\right).$$

7. When current in one of the loops is increased, when they placed parallel, they will repel and if current is decreased they will attract.

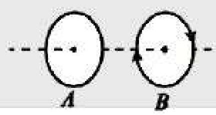
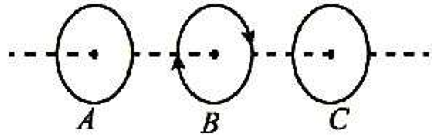


Illustration 1

Three identical conducting circular loops are placed parallel as shown in figure if current in loop B increases, then



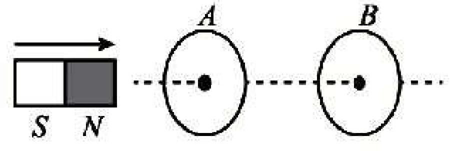
(a) there is repulsion between B and C, attraction between A and B.
(b) there is attraction between B and C, repulsion between A and B.
(c) there is repulsion between B and C, also between A and B.
(d) none of the above.

Short-cut solution :

When current in loop B increases, its field causes increase in magnetic flux in both the loops A and C. So both the loops move away from B so as to compensate the increased flux. ***Ans. (c)***

Illustration 2

A bar magnet moves towards two conducting identical parallel circular loops with constant velocity as shown. Then



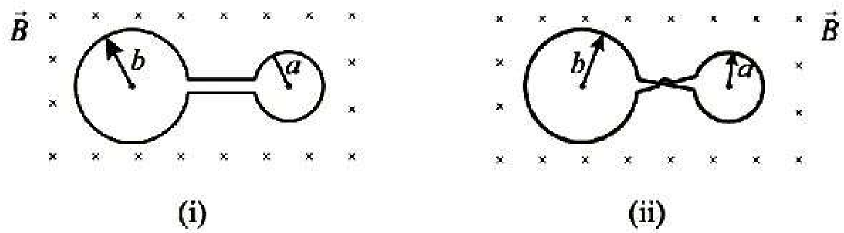
(a) ***loop A attracts towards magnet, while loop B repels***
(b) ***loop A repels from magnet while loop B attracts***
(c) ***both the loops repel from the magnet***
(d) ***both the loops will attract each other***

Short-cut solution :

As magnet moves towards the loops, so magnetic flux in both of them increase. To compensate increased flux, both the loops move away from the magnet. The direction of induced current in both of them is in the same sense, so they attract each other. ***Ans. (c, d)***

Illustration 3

Figure shows two conducting loops of radii a and b ($a < b$) joined together with wires of negligible resistance. They are placed in time varying magnetic field $\frac{dB}{dt} = k$, perpendicular to the plane of the loops. Find induced emf in the loops in the following cases. Also indicate the direction of induced current in both the loops.



Short-cut solution :

$$\text{Induced emf } |e| = \frac{d\phi}{dt} = A\left(\frac{dB}{dt}\right) = kA$$

So $|e_a| = k(\pi a^2)$

and $|e_b| = k(\pi b^2)$

Net emf: (i) $e = e_b + e_a = \pi k(b^2 + a^2)$.

(ii) $e = e_b - e_a = \pi k(b^2 - a^2)$. ***Ans.***

The direction of induced current/ emf is to compensate the increased flux, so the direction of current is counter clockwise in bigger loop in both the cases (see figures)

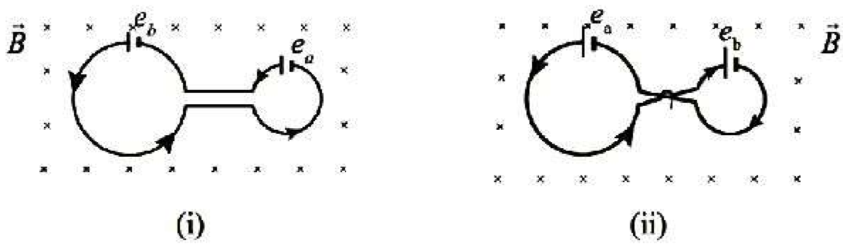
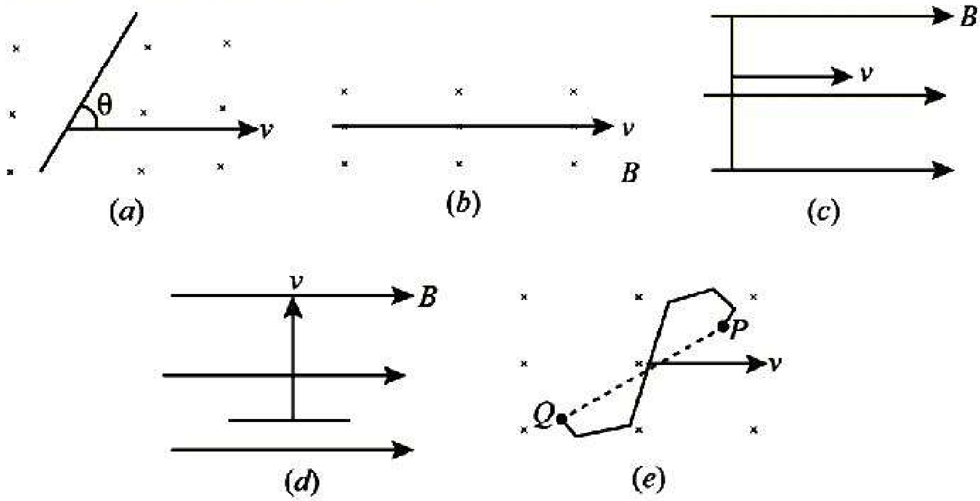


Illustration 4

Find the induced emf across the ends of the conducting rod in the following situations. The length of the conductor is ℓ.



Short-cut solution :

(a) In this case we can make $\vec{v}$ perpendicular to length of the rod or $\vec{\ell}$ perpendicular to v. Thus

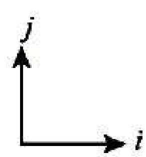
$$e = B(v \sin\theta)\ell \text{ or } Bv(\ell \sin\theta)$$

$$= Bv\ell \sin\theta \qquad \textbf{\textit{Ans.}}$$

(b) If we take the plane of motion of the rod as xy, then

$$e = \int\left[v\hat{i} \times \left(-B\hat{j}\right)\right]\cdot\left(\ell\hat{i}\right)$$

$$= -vB\ell\left(\hat{k}\cdot\hat{i}\right) = 0$$



(c) $e = \int\left[v\hat{i}\times B\hat{i}\right]\cdot\left(\ell\hat{j}\right)=0$

(d) $e = \int\left(v\hat{j}\times B\hat{i}\right)\cdot\ell\hat{i}$

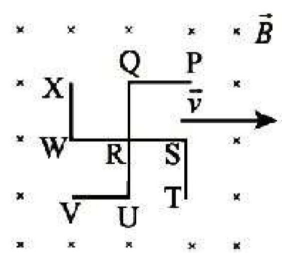
$= vB\ell\left(-\hat{k}\cdot\hat{i}\right)=0$

(e) If PQ line makes θ with the velocity vector, then

$e = B(v\sin\theta)PQ.$ ***Ans.***

Illustration 5

A copper conductor is bent in the shape of 'Swastika' moves with constant velocity v in a perpendicular magnetic field $\vec{B}$ directed downwards. The each side of the swastika is of length ℓ. Find Pd between P and V, between X and V, between P and X.



Short-cut solution :

The length of conduction perpendicular to velocity vector

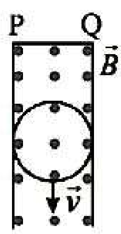
$\ell_{PV} = 2\ell,\ \therefore\ e_{PV} = Bv(2\ell).$

And $\ell_{XV} = 2\ell, \therefore\ e_{XV} = Bv(2\ell).$

$\ell_{PX} = 0, \qquad \therefore\ e_{PX} = 0.$ ***Ans.***

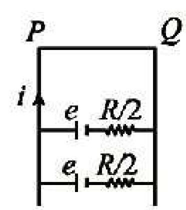
Illustration 6

A vertical ring of radius r and resistance R falls vertically. It is in contact with two vertical rails which are joined at the top. The rails are without friction and resistance. There is a horizontal uniform magnetic field of magnitude $\vec{B}$ perpendicular to the plane of the ring and the rails. Find the current in the section PQ when the speed of the ring is v.



Short-cut solution :

Spliting the ring into two parts, each of them becomes a cell of emf, $e = Bv\ell = Bv(2r)$.



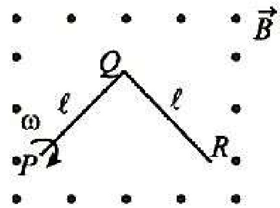
$$i = \frac{e}{R_{eq}} = \frac{2Bvr}{(R/4)} = \frac{8Bvr}{R}.$$

Ans.

Illustration 7

Figure shows an L-shaped metal rod rotating about its end P in a plane perpendicular to the magnetic field $\vec{B}$. Find the P.d between P and R, between Q and R.

Short-cut solution :



Distance PR, $\ell_{PR} = \sqrt{2}\ell$

$$\therefore \quad V_P - V_R = \frac{B\omega\ell_{PR}^2}{2}$$

$$= \frac{B\omega(\sqrt{2}\ell)^2}{2} = B\omega\ell^2.$$

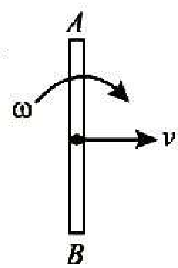
[Here point P is at higher potential and point R is at lowest potential].

$$\text{And} \quad V_Q - V_R = \frac{B\omega}{2}\left[\ell^2_{PR} - \ell^2_{PQ}\right]$$

$$= \frac{B\omega}{2}\left[(\sqrt{2}\ell)^2 - \ell^2\right] = \frac{B\omega\ell^2}{2}. \textit{Ans.}$$

Illustration 8

A metal rod of length ℓ, moving with an angular velocity ω and velocity of its centre is v. Find potential difference between points A and B at the instant shown in figure. A uniform magnetic field of strength $\vec{B}$ exist perpendicular to the plane of paper:



(a) $Bv\ell$

(b) $Bv\ell + \frac{1}{2}B\omega\ell^2$

(c) $B\omega\ell - \frac{1}{2}B\omega\ell^2$

(d) $Bv\ell + B\omega(\ell/2)^2$

Short-cut solution :

$$V_A - V_B = (V_A - V_B)_{translation} + (V_A - V_B)_{rotation}$$

$$= Bv\ell + \frac{B\omega}{2}\left[\left(\frac{\ell}{2}\right)^2 - \left(\frac{\ell}{2}\right)^2\right]$$

$$= Bv\ell. \qquad \textit{Ans. (a)}$$

Illustration 9

A square loop of side 12 cm with its side parallel to x and y axis is moved with a velocity of 8 cm/s in the positive x-direction in a magnetic field pointing towards positive z-direction. The field has a gradient of 10^{-3} T/cm. Find the magnitude of the induced emf if field changes at the rate of 0.1 T/s.

Short-cut solution :

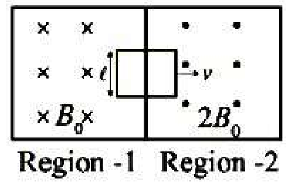
We know that, the induced emf in this case is given by

$$|e| = A\left[\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial x}\right]$$

$$= 144\times10^{-4}\left[0.1+8\times10^{-3}\right]$$

$$= 155 \text{ mV.}$$ ***Ans.***

Illustration 10

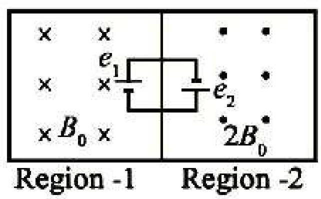


A conducting loop is being pulled with speed v from region I of magnetic field to region II. If resistance of the loop is R, current induced in the loop at the instant shown is

(a) $\frac{B_0\ell v}{R}$, *clockwise* *(b)* $\frac{B_0\ell v}{R}$, *anticlockwise*

(c) $\frac{3B_0\ell v}{R}$, *clockwise* *(d)* $\frac{3B_0\ell v}{R}$, *anticlockwise*

Short-cut solution :



$$e = e_1 + e_2$$

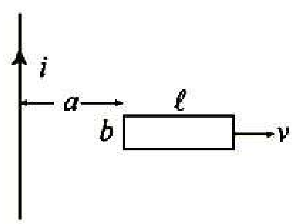
$$= B_0 v\ell + 2B_0 v\ell$$

$$= 3B_0 v\ell$$

$\therefore$ $$i = \frac{e}{R} = \frac{3B_0 v\ell}{R}.$$ ***Ans.(c)***

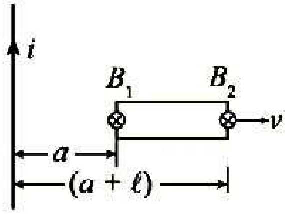
Illustration 11

A rectangular metallic loop of length ℓ and width b is placed coplanerly with a long wire carrying a current i. The loop is moved perpendicular to the wire with a speed v in the plane containing the wire and the loop. Calculate the emf induced in the loop when the rear end of the loop is at a distance a from the wire.



Short-cut solution :

When $x = a$, the position of the loop is shown in figure. The magnetic field at left arm of the loop



$$B_1 = \frac{\mu_0}{2\pi}\frac{i}{a}$$

Similarly, magnetic field at right arm of the loop

$$B_2 = \frac{\mu_0}{2\pi}\frac{i}{(a+\ell)}$$

Now induced emf across left and right arms are

$$e_1 = B_1 v\, b \text{ and } e_2 = B_2 v\, b$$

$$\therefore \quad e_{net} = e_1 - e_2$$

$$= vb\,[B_1 - B_2]$$

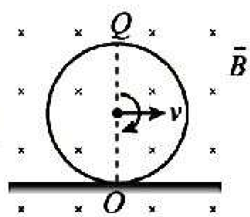
$$= vb\left[\frac{\mu_0}{2\pi}\frac{i}{a} - \frac{\mu_0}{2\pi}\frac{i}{a+\ell}\right]$$

$$= \frac{\mu_0 i(b\ell v)}{2\pi a(a+\ell)}.$$

Ans.

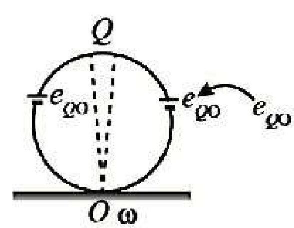
Illustration 12

A conducting ring of radius r and resistance R rolls on a horizontal surface with constant velocity v. The magnetic field $\vec{B}$ is perpendicular to the plane of the ring. Find P.d between points O and Q. Also find current in the ring.



Short-cut solution :

Split ring into two equal half and rotates each one about O with $\omega = \frac{v}{r}$. So induced emf across each one becomes, (Here Q point will be at higher potential)



$$e_{QO} = \frac{B\omega\ell^2}{2} = \frac{B\omega(2r)^2}{2}$$

$$= 2B\omega r^2 = 2B(\omega r)r$$

$$= 2Bvr.$$

It is clear that net emf of the ring becomes

$$e_{net} = e_{QO} - e_{QO} = 0.$$

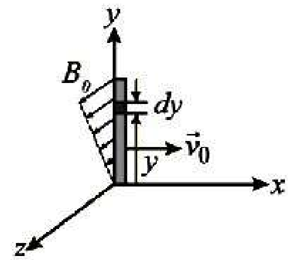
Therefore, there is no current in the ring. ***Ans.***

Illustration 13

The magnetic field in a region is given by $\vec{B} = \vec{k}\,\frac{B_0}{L}y$ where L is a fixed length. A conducting rod of length L lies along y-axis between the origin and the point (0, L, 0). If the rod moves with a velocity $\vec{v} = v_0\vec{i}$, find the emf induced between the ends of the rod.

Short-cut solution :

Consider a small element of width dy at a distance y from the origin. The induced emf across it



$$de = B\,v\,(dy)$$

$$= \left(\frac{B_0 y}{L}\right) v_0\, dy$$

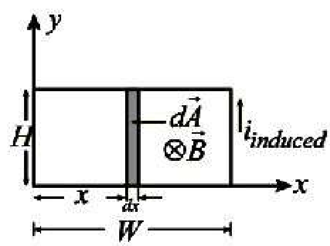
The induced emf across whole length of the rod

$$e = \int_0^L \frac{B_0}{L} v_0\, y\, dy$$

$$= \frac{B_0 v_0 L}{2}.$$ ***Ans.***

Illustration 14

Figure shows a rectangular loop of wire immersed in a non-uniform and varying magnetic field $\vec{B}$ that is perpendicular to and directed into the page. The field's magnitude is given by B = 4t2x2, with B in tesla, t in second, and x in metre. The loop has width W = 3.0 m and height H = 2.0 m. What are the magnitude and direction of the induced emf ξ around the loop at t = 0.10 s ?



Solution :

Take an element of thickness dx, its area $dA = Hdx$. The magnetic flux through this area

$$d\phi = \vec{B}\cdot d\vec{A} = BdA\cos 0^\circ = BdA$$

$$= B(Hdx)$$

$$= 4t^2x^2\, Hdx$$

The total flux through the entire loop

$$\phi = \int d\phi = 4t^2 H \int_0^{3.0} x^2 dx$$

$$= 4t^2H\left|\frac{x^3}{3}\right|_0^{3.0} = 72t^2 \quad (H = 2.0 \text{ m})$$

Now by Faraday's law, the magnitude of induced emf

$$e = \frac{d\phi}{dt} = \frac{d\left[72t^2\right]}{dt}$$

$$= 144\,t$$

At $\quad t = 0.10$ s,

$$e = 144 \times 0.10$$

$$= 14.4 \text{ V} \qquad \textit{Ans.}$$

Direction of induced emf : The flux of $\vec{B}$ through the loop is into the page and is increasing in magnitude with time. According to Lenz's law, the field of the induced current must oppose this increase and so is directed out of the page. Therefore the direction of induced current or emf is counterclockwise in the loop (see figure).

Induced Electric Field:

$$E_n = -\frac{r}{2}\left(\frac{dB}{dt}\right) \text{ for } r \le R$$

and

$$E_n = -\frac{R^2}{2r}\left(\frac{dB}{dt}\right) \text{ for } r \ge R$$

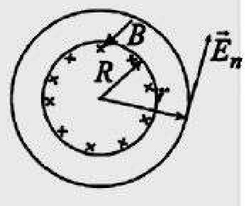


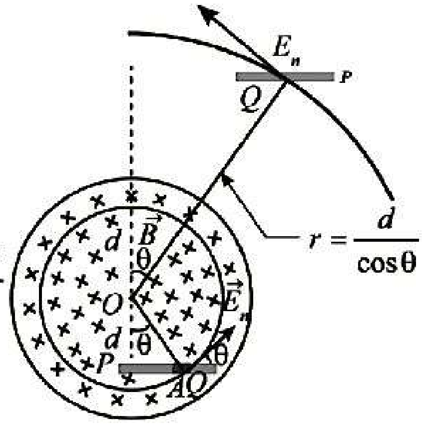
Illustration 15

A conducting rod is placed in a changing magnetic field as shown in figure. Find induced emf across its length.

Short-cut solution :

Let us consider a rod of length ℓ is placed in magnetic field, which is perpendicular to the plane of paper and pointing into it. The field is changing at the constant rate of dB/dt.

Now consider a point A in the rod, it is at a distance $r = \dfrac{d}{\cos\theta}$ from the centre of field.



The induced electric field at A,

$$E_n = -\frac{r}{2}\cdot\frac{dB}{dt} = -\frac{d}{2\cos\theta}\cdot\frac{dB}{dt}$$

The component of field along the rod

$$E = E_n\cos\theta = \frac{d}{2}\frac{dB}{dt}$$

∴ Potential difference between ends of the rod

$$e = E\ell = \frac{d}{2}\ell\frac{dB}{dt}$$

Case (1) : If rod is placed along the diameter

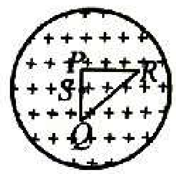
$$d = 0, \therefore e = 0.$$

Case (2) : If the rod is outside the changing field

$$E = \frac{R^2}{2d}\frac{dB}{dt}\cos\theta.$$ ***Ans.***

Illustration 16

Consider a region of cylindrical magnetic field, changing with time at the rate of x. A triangular conducting loop PQR is placed in the field such that mid point of side PQ coincides with axis of the magnetic field region. PQ = PR = 2ℓ. Find emf induced across PQ, QR and PR of the loop.



Short-cut solution :

Using, $$e = \frac{d}{2}\ell\left(\frac{dB}{dt}\right)$$

For PQ, $d = 0$, $\therefore e_{PQ} = 0$

For PR, $d = \ell$, $\therefore e_{PR} = \frac{\ell}{2}\times 2\ell\times x = x\ell^2$

For whole loop,
$$|e| = \frac{d\phi}{dt} = A\left(\frac{dB}{dt}\right)$$
$$= \frac{1}{2}(2\ell\times 2\ell)\times x$$
$$= 2\ell^2 x$$

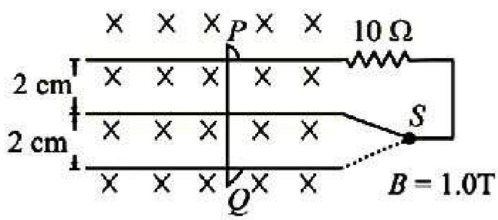
As $e_{PQ} + e_{PR} + e_{RQ} = e$

or $0 + x\ell^2 + e_{RQ} = 2\ell^2 x$

$\therefore e_{RQ} = \ell^2 x.$ ***Ans.***

Illustration 17

Consider the situation shown in figure. The wire PQ has a negligible resistance and is made to slide on the three rails with a constant speed of 5 cm/s. Find the current in the 10Ω resistor when the switch S is thrown to (a) the middle rail (b) the bottom rail.



Short-cut solution :

(a) The induced emf,

$$e = Bv\ell = 1 \times 0.05 \times 0.02 = 1 \times 10^{-3}\text{ V}$$

Thus current,

$$i = \frac{e}{R} = \frac{1\times10^{-3}}{10} = 0.1 \text{ mA.} \qquad \textit{Ans.}$$

(b) The induced emf,

$$e = Bv\ell = 1 \times 0.05 \times 0.04 = 2 \times 10^{-3}\text{ V}$$

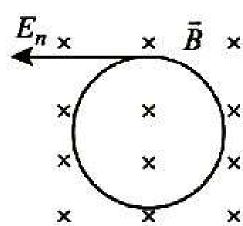
Thus current,

$$i = \frac{e}{R} = \frac{2\times10^{-3}}{10} = 0.2 \text{ mA.} \qquad \textit{Ans.}$$

Illustration 18

A non-conducting ring of radius r has charge Q. A magnetic field perpendicular to the plane of the ring changes at the rate of $\frac{dB}{dt} = x$. ***Find torque experienced by the ring.***

Short-cut solution :



Using, $E_n = \frac{r}{2}\left(\frac{dB}{dt}\right) = \frac{rx}{2}$

As the field at each point is tangential so, its force

$$F = E_n Q = \frac{rxQ}{2}.$$

Now, torque about centre

$$\tau = Fr = \frac{rxQ}{2} \times r$$

$$= \frac{1}{2}Qr^2x. \qquad \textit{Ans.}$$

***TOPIC 18.2**: Self and Mutual Inductance, Energy Stored in an Inductor, RL-DC Circuit, LC-Oscillations, Combination of Inductors, AC Generator and Transformer.*

Review of Formulae

1. **Self induction :** The induced emf in the coil itself due to change in current in it is called self induction. For the coil of N turns

$$N\phi_B = Li,$$

where L is called self inductance.

2. **Self induction of circular coil of N turns and radius r**

$$L = \frac{\mu_0 \pi N^2 r}{2}.$$

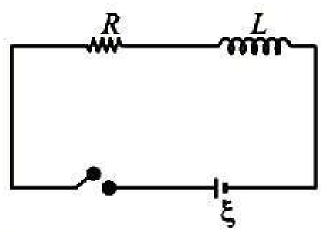
3. **Energy stored in an inductor :** Energy in the inductor stores due to magnetic field in it. For any current i in the inductor, the energy stored

$$U = \frac{1}{2} Li^2.$$

If B is the magnetic field in the coil, then

$$U = \frac{B^2}{2\mu_0} \times Vol.$$

4. **RL-DC circuit : For a circuit with time constant τ, the growing current i in circuit at any time t**



$$i = i_0\left(1 - e^{-t/\tau}\right)$$

where $i_0 = \frac{\xi}{R}$ and $\tau = \frac{L}{R}$

The decay current in the circuit, with initial current i_0

$$i = i_0 e^{-t/\tau}.$$

5. **LC-oscillations :** When a capacitor C with a charge q is connected to an inductor L, the energy of the circuit oscillates between C and L. If ω is the angular frequency of oscillations, then

$$\frac{d^2 q}{dt^2} + \omega^2 q = 0,$$

where $$\omega = \sqrt{\frac{1}{LC}}.$$

Also $$T = 2\pi\sqrt{LC}.$$

Electrical energy of capacitor will store in inductor after time $T/4$ and vice-versa.

6. **Mutual inductance :** The induced emf in the second coil due to change in current in first coil, is called mutual induction. If M is the mutual inductance between the coils, then

$$N_2\phi_{21} = Mi_1,$$

or

$$N_1\phi_{12} = Mi_2.$$

7. **Mutual induction between two circular coils of radii R_1 and R_2 with turns N_1 and N_2 is**

$$M = \left[\frac{\mu_0 \pi N_1 N_2 R_2^2}{2R_1}\right].$$

8. **If L_1 and L_2 are the self inductances of two coils, then mutual induction between them**

$$M = \sqrt{L_1 L_2}.$$

9. **Combination of inductors :**

(i) **In series :** $L = L_1 + L_2$

(ii) **In parallel :** $\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}.$

10. **AC generator :** If i_0 be the maximum current (current amplitude), then

$$i = i_0 \sin\omega t,$$

where

$$i_0 = \frac{NBA\omega}{R}.$$

11. **Transformer :** For the transformer with the turns N_p and N_s in the primary and secondary coils, the ratio of output and input potentials

$$\frac{V_s}{V_P} = \frac{N_s}{N_P} = \frac{i_P}{i_s}.$$

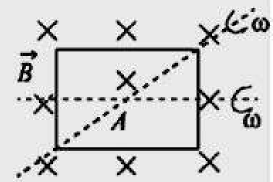
For the ideal transformer

$$V_P i_P = V_s i_s.$$

TIPS! & TRICKS **Tips and Tricks for Shortcut Solutions**

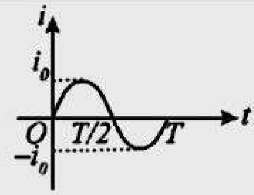
1. Maximum induced emf in a coil rotating in uniform $\vec{B}$

$$e_{max} = NBA\omega.$$



2. Maximum induced emf in an inductor due to AC of frequency f and peak value i_0.

$$e_{max} = L\left(\frac{\Delta i}{\Delta t}\right) = L\frac{2i_0}{(T/2)} = \frac{4Li_0}{T} = 4Lfi_0.$$



3. Reciprocity Theorem : Experiments confirm that

$$\phi_{12} = \phi_{21}.$$

Also $$M_{12} = M_{21}.$$

4. In LC-oscillations *EE* changes into *ME* and vice-versa, each after *T*/4. Both the energies are equal after *T*/8 starting from $t = 0$, when charge on the capacitor $q = \frac{Q}{\sqrt{2}}$.

5. When mutual induction between the coils of different radii are asked, we should take smaller coil for total flux due to the current in larger coil.

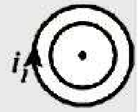
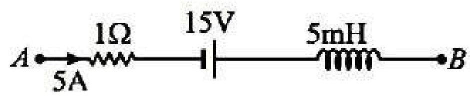


Illustration 19

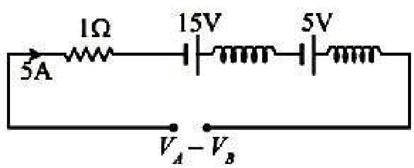
The network shown in the figure is part of a complete circuit. At some instant, the current in inductor is 5A and is decreasing at a rate 10^3A/s, then find $V_A - V_B$



Short-cut solution :

The induced emf in the inductor, $e = L\frac{di}{dt} = 5\times10^{-3}\times10^{3} = 5V$.

As the current is decreasing, so this induced emf will compensate the current. The effective circuit is as follows;



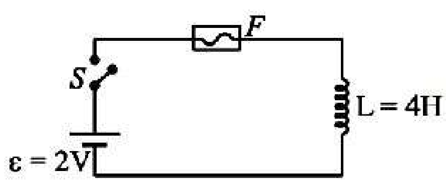
In close loop,

$$-5 \times 1 + 15 + 5 + (V_A - V_B) = 0$$

or $$V_A - V_B = -15\text{ V}.$$ ***Ans.***

Illustration 20

In the circuit shown, the cell is ideal. The coil has an inductance of 4H and zero resistance F is a fuse of zero resistance and will blow when the current through it is 5A. The switch is closed at t = 0, find the time when fuse will blow.



Short-cut solution :

Using, $\varepsilon = L\dfrac{di}{dt}$

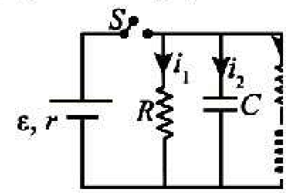
or $di = \dfrac{\varepsilon}{L}dt \Rightarrow \int_0^i di = \dfrac{\varepsilon}{L}\int_0^t dt$

$\therefore$ $i = \dfrac{\varepsilon t}{L}$.

or $5 = \dfrac{2}{4}t \Rightarrow t = 10s.$ ***Ans.***

Illustration 21

In the circuit shown, find the value of currents, i_1, i_2 and i_3 at
(i) $t = 0$ (ii) $t = \infty$, after closing of the switch



Short-cut solution :

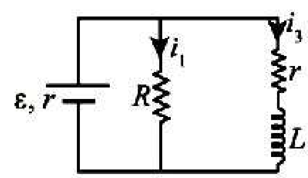
(i) At $t = 0$, inductor behaves like infinite resistor and capacitor like zero resistor for *DC*.

$\therefore$ $i_1 = 0,\ i_2 = \dfrac{\varepsilon}{r},\ i_3 = 0$

(ii) After t = ∞, inductor offers no resistance while capacitor like infinite resistor for DC.

$i_2 = 0$

$\therefore$ $i_1 + i_3 = \dfrac{\varepsilon}{\left[\dfrac{Rr}{R+r} + r\right]}$



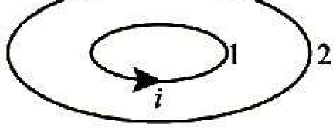
Also $i_1 R = i_3 r$

On solving above equations, we get

$i_1 = \dfrac{\varepsilon}{2R+r}$ and $i_3 = \dfrac{\varepsilon R}{2Rr + r^2}$. ***Ans.***

Illustration 22

(i) Two circular loops 1 and 2 whose centres coincide lie in a plane (see figure). The radii of the loops are a_1 and a_2. Current i flows in loop 1. Find the magnetic flux ϕ_2 associated by loop 2, if $a_1 << a_2$.



(ii) Also find mutual induction between the loops.

Short-cut solution :

(i) The direct calculation of the flux ϕ_2 is clearly a rather complicated problem since the configuration of the field itself is complicated. However, the application of the reciprocity theorem greatly simplifies the solution of the problem. Let us pass the same current i through loop 2. Then the magnetic flux ϕ_1 created by this current through loop 1 can be easily found, provided that $a_1 << a_2$: it is sufficient to multiply the magnetic field B at the centre of the loop $\left(B = \frac{\mu_0 i}{2a_2}\right)$ by the area $\pi a_1{}^2$ of the circle.

Thus
$$\phi_2 = \phi_1$$
$$= B_2 A_1$$
$$= \frac{\mu_0 i}{2a_2}\pi a_1^2. \qquad \textit{Ans.}$$

(ii) On comparing $\phi_2 = Mi$, we have
$$M = \frac{\mu_0 \pi a_1{}^2}{2a_2}. \qquad \textit{Ans.}$$

Illustration 23

At $t = 0$, an inductor of zero resistance is joined to a cell of emf ε_0 through a resistance. The current increases with a time constant τ. Find emf across the coil after time t.

Short-cut solution :

Using,
$$i = i_0(1 - e^{-t/\tau})$$

$\therefore$
$$\frac{di}{dt} = -i_0\left(-\frac{1}{\tau}\right)e^{-t/\tau} = \frac{\varepsilon_0}{R}.\frac{R}{L}e^{-t/\tau} = \frac{\varepsilon_0}{L}e^{-t/\tau}$$

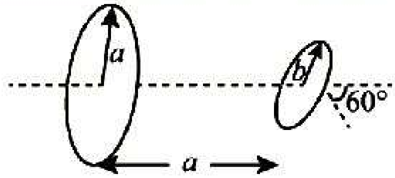
or
$$L\frac{di}{dt} = \varepsilon_0 e^{-t/\tau}$$

Here
$$L\frac{di}{dt} = \varepsilon\text{, emf}$$

$\therefore$
$$\varepsilon = \varepsilon_0 e^{-t/\tau}. \qquad \textit{Ans.}$$

Illustration 24

Two circular conducting loops of radii a and b (b << a) are placed on the same axis as shown. Find mutual induction between the loops. [JEE Adv. 2012]



Short-cut solution :

Flux in smaller loop due to current in larger loop

$$\phi = BA\cos 60°$$

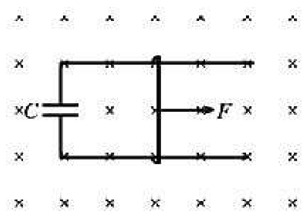
$$= \frac{\mu_0 i a^2}{2(a^2+a^2)^{3/2}} \times \pi b^2 \times \frac{1}{2}$$

$$= \left(\frac{\mu_0 \pi b^2}{8\sqrt{2}a}\right) i$$

$$\therefore \quad M = \frac{\phi}{i} = \frac{\mu_0 \pi b^2}{8\sqrt{2}a}. \qquad \textit{Ans.}$$

Video Solution

Q. A wire of mass m and length ℓ can freely slide on a pair of parallel, smooth, horizontal rails placed in a vertical magnetic field B. The rails are connected by a capacitor of capacitance C. The electric resistance of the rails and the wire is zero. If a constant force F acts on the wire as shown in the figure, find the acceleration of the wire.

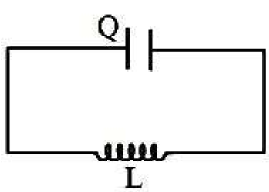


To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=jAowGt65I3Q

Illustration 25

A charged capacitor C is connected to an ideal inductor L at t = 0. Find the time when electrical energy stored in capacitor becomes equal the magnetic energy of the inductor.



Short-cut solution :

Supposing initial charge on the capacitor is Q and energy stored is $Q^2/2C$. If q is the charge remains on the capacitor, when both the energies becomes equal, then

$$\frac{q^2}{2C} = \frac{1}{2}\left[\frac{Q^2}{2C}\right]$$

or $$q = \frac{Q}{\sqrt{2}}$$

Now using, $$q = Q\cos\omega t$$

or $$\frac{Q}{\sqrt{2}} = Q\cos\frac{2\pi}{T}t$$

or $$\frac{2\pi t}{T} = \frac{\pi}{4}$$

or $$t = \frac{T}{8} = \frac{2\pi\sqrt{LC}}{8} = \frac{\pi}{4}\sqrt{LC}.$$ ***Ans.***

Illustration 26

The secondary coil of an ideal step down transformer is delivering 500 W power at 12.5 A current. If the ratio of turns in the primary to the secondary is 5 : 1, then find the current flowing in the primary coil.

Short-cut solution :

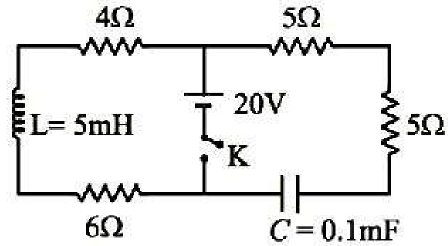
For secondary coil, $$V_s = \frac{P}{i_s} = \frac{500}{12.5} = 40V$$

Now $$\frac{V_s}{V_p} = \frac{N_s}{N_p} \Rightarrow V_p = \frac{N_p}{N_s}V_s = \frac{5}{1}\times 40 = 200V$$

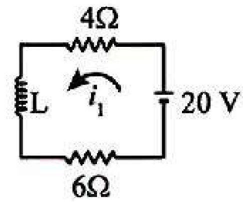
$\therefore$ $$i_p = \frac{P}{V_p} = \frac{500}{200} = 2.5\ A.$$ ***Ans.***

Illustration 27

In the circuit show, the key (K) is closed at $t = 0$, find the current through the key at the instant $t = 10^{-3}\ell n_2$ s.



Short-cut solution :

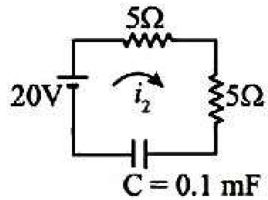


Using , $i_1 = \frac{\varepsilon}{R}(1 - e^{-t/\tau_1})$; Here $\tau_1 = \frac{L}{R}$

Here $\frac{\varepsilon}{R} = \frac{20}{10} = 2A$, $\tau_1 = \frac{5\times10^{-3}}{10} = 5\times10^{-4}s$

On substituting $t = 10^{-3}\ell n\, 2$, we get

$$i_1 = 1.5\text{ A}$$



And $i_2 = i_0 e^{-t/\tau} = \frac{\varepsilon}{R} e^{-t/\tau_2}$

Here $\frac{\varepsilon}{R} = \frac{20}{10} = 2A$,

$$\tau_2 = CR = 0.1 \times 10^{-3} \times 10 = 10^{-3}\text{s}.$$

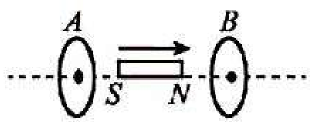
On substituting $t = 10^{-3}\ell n\, 2$, we get

$$i_2 = 1.0\text{A}$$

Therefore, $i = i_1 + i_2 = 1.5 + 1.0 = 2.5$ A. ***Ans.***

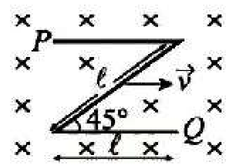
Concept Booster Exercise

1. A bar magnet is moved between two parallel circular conducting loops with constant velocity v as shown in figure.



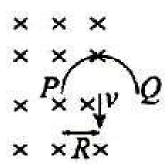
   (a) There is induced current in loop B only.
   (b) There is induced current in loop A only.
   (c) The current in each loop flows in same direction.
   (d) The loops will repel each other.

2. A conducting rod in the shape of letter Z of equal arm each ℓ move in a perpendicular magnetic field $\vec{B}$ with constant velocity $\vec{v}$ as shown. Find potential difference between P and Q



   (a) $\frac{Bv\ell}{\sqrt{2}}$ (b) $\sqrt{2}Bv\ell$ (c) $2Bv\ell$ (d) $3Bv\ell$

3. A semicircular conducting ring of radius R is falling with its plane vertical in a horizontal magnetic induction $\vec{B}$. At some instant its velocity is v, find potential difference between its ends P and Q.



   (a) $2BvR$ (b) BvR (c) πBvR (d) Zero

4. A uniform magnetic field exists in a region is given by $\vec{B} = 3\hat{i} + 4\hat{j} + 5\hat{k}$. A rod of length 5 m is placed along y-axis is moved along x-axis with constant speed 1 m/s. The induced emf in the rod will be **Numeric/Integer**

   (a) 15 V (b) 20 V (c) 25 V (d) Zero

5. A coil of self-inductance L is placed in an external magnetic field (no current flows in the coil). The total magnetic flux linked with the coil is ϕ. The magnetic field energy stored in the coil is

(a) $\frac{\phi^2}{L}$ (b) $\frac{2\phi}{L}$ (c) $\frac{\phi^2}{2L}$ (d) Zero

6. The magnetic field of the earth at a place is B_0 and the angle of dip is θ. A horizontal conductor of length ℓ, lying north-south, moves eastwards with a velocity v. The emf induced across the rod is:

(a) Zero (b) $B_0 v\ell$ (c) $B_0 v\ell\sin\theta$ (d) $B_0 v\ell\cos\theta$

7. A long solenoid of N-turns has a self-inductance L and area of cross-section A. When a current i flows through the solenoid, the magnetic field inside it has magnitude B. The current i is equal to:

(a) $\frac{BAN}{L}$ (b) $BANL$ (c) $\frac{BN}{AL}$ (d) $\frac{BA}{NL}$

8. Two coils, A and B, are linked such that emf ε is induced in B when the current in A is changing at the rate I. If i current is now made to flow in B, the flux linked with A will be :

(a) $(\varepsilon / I)i$ (b) $\varepsilon i I$ (c) $(\varepsilon I) i$ (d) $i I / \varepsilon$

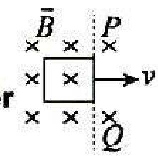
9. At $t = 0$, an inductor of zero resistance is joined to a cell of emf ε through a resistance. The current increases with a time constant τ. The time when P.d. across coil becomes equal to that of resistor:

(a) τ (b) $\tau \ell n\, 2$ (c) $\tau(1 - \ell n\, 2)$ (d) $\frac{\tau}{\ell n 2}$

10. When a coil carrying a steady current is short-circuited, the current in it decreases η times in time t_0. The time constant of the circuit is :

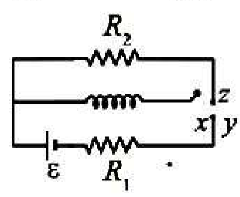
(a) $t_0 \ell n\, \eta$ (b) $\frac{t_0}{\ell n \eta}$ (c) $\frac{t_0}{\eta}$ (d) $\frac{t_0}{\eta - 1}$

11. Figure shows a square loop of side 1 m and resistance 1Ω. The magnetic field on left side of line PQ has a magnitude $B = 1.0$ T. The work done in pulling the loop out of the field uniformly in 1s is: **Numeric/Integer**



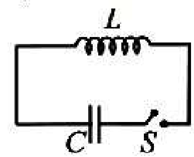
(a) 0.1 J (b) 2 J (c) 1 J (d) 5 J

12. In the circuit shown, x is joined to y for a long time, and then x is joined to z. The total heat produced in R_2 is :



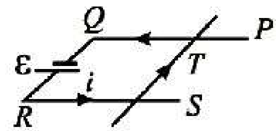
(a) $\frac{\varepsilon^2 L}{2R_1}$ (b) $\frac{\varepsilon^2 L}{2R_2^2}$ (c) $\frac{\varepsilon^2 L}{2R_1 R_2}$ (d) $\frac{\varepsilon^2 L R_2}{2R_1^3}$

13. The capacitor of capacity C is given charge Q and then connected to the coil of inductance L by closing the switch S. The maximum current flowing in the circuit at any time will be :



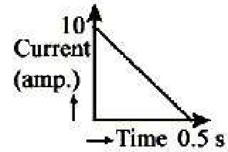
(a) $\frac{Q}{2\sqrt{LC}}$ (b) $\frac{Q}{\sqrt{LC}}$ (c) $\frac{2Q}{\sqrt{LC}}$ (d) $\frac{\;}{\sqrt{LC}}$

14. *PQ* and *RS* are smooth, parallel, horizontal rails on which a conductor *T* can slide. A cell ε drives current *i* through the rails and *T*.



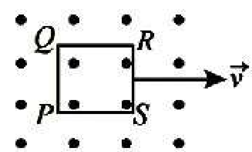
(a) T will experience no force
(b) T will experience a force to the left
(c) The current in the rails will set up a magnetic field over T
(d) T will experience a force to the right

15. In a coil of resistance 100Ω, a current is induced by changing the magnetic flux through it as shown in the figure. The magnitude of change in flux through the coil is **Numeric/Integer**



(a) 250 Wb (b) 275 Wb (c) 200 Wb (d) 225 Wb

16. A conducting square loop *PQRS* of side ℓ moves in a perpendicular magnetic field $\vec{B}$ with constant velocity $\vec{v}$. Choose correct alternative(s)



(a) The induced emf across each side of the loop is zero
(b) The induced emf of the loop is $2Bv\ell$
(c) The induced emf across *PQ* and *RS* equal to $Bv\ell$ across each one and no induced emf across *QR* and *PS*.
(d) The net emf of the loop is zero.

Solutions

1. **(c)** The induced current in both the loops will be in anticlockwise direction. ***Ans.***

2. **(a)** $e_{PQ} = Bv(\ell \sin 45°) = \dfrac{Bv\ell}{\sqrt{2}}$. ***Ans.***

3. **(b)** The length of the ring perpendicular to velocity vector and inside magnetic field is *R*. $\therefore\ e = BvR$. ***Ans.***

4. **(c)** $e = (\vec{v} \times \vec{B}).\vec{\ell} = [\hat{i} \times (3\hat{i} + 4\hat{j} + 5\hat{k})].5\hat{j}$
$= 25$ V. ***Ans.***

5. **(c)** $\phi = \text{L}i \Rightarrow i = \dfrac{\phi}{L}$. $U = \dfrac{1}{2}Li^2 = \dfrac{1}{2}L\left(\dfrac{\phi}{L}\right)^2 = \dfrac{\phi^2}{2L}$. ***Ans.***

6. **(c)** The horizontal conductor will cross vertical component of earth's field, $Bv = B_0 \sin\theta$.
$\therefore\ e = B_v v\ell = B_0 v\ell \sin\theta$. ***Ans.***

7. **(a)** $\phi = \text{L}i = \text{NBA} \Rightarrow i = \dfrac{NBA}{L}$. ***Ans.***

8. **(a)** Let M = mutual inductance between *A* and *B*.

$$e_B = \text{M}\frac{di_A}{dt} = MI \quad \text{or} \quad M = \frac{\varepsilon}{I}$$

Now, $\phi_A = Mi_B = \left(\dfrac{\varepsilon}{I}\right)i$ ***Ans.***

9. **(b)** $i = i_0(1 - e^{-t/\tau})$. $\frac{di}{dt} = -i_0\left(-\frac{1}{\tau}\right)e^{-t/\tau} = \frac{\varepsilon}{R} \times \frac{R}{L} e^{-t/\tau}$

or $L\frac{di}{dt} = \varepsilon e^{-t/\tau}$

or $e = \varepsilon e^{-t/\tau}$.

Now $V_{Coil} = V_{Resistor}$

or $\varepsilon e^{-t/\tau} = iR$

$= \frac{\varepsilon}{R}(1 - e^{-t/\tau})R$

$\therefore$ $t = \tau \ell n 2$. ***Ans.***

10. **(b)** Using, $i = i_0 e^{-t/\tau}$ or $\frac{i_0}{\eta} = i_0 e^{-t/\tau}$

$\Rightarrow$ $t_0 = \tau \ell n \eta \Rightarrow \tau = \frac{t_0}{\ell n \eta}$. ***Ans.***

11. **(c)** $W = Fs = Bi\ell \times \ell = B\left(\frac{Bv\ell}{R}\right)\ell^2$

$= \frac{B^2 v \ell^3}{R} = \frac{1^2 \times 1 \times 1^3}{1} = 1J.$ ***Ans.***

12. **(a)** In steady state energy stored in inductor

$U = \frac{1}{2}Li_0^2 = \frac{1}{2}L\left(\frac{\varepsilon}{R_1}\right)^2 = \frac{\varepsilon^2 L}{2R_1}$

= heat produced in R_2 during discharge. ***Ans.***

13. **(b)** $\frac{1}{2}Li_0^2 = \frac{Q^2}{2C}$. ***Ans.***

14. **(c, d)** Because of the current in the rails, there is an upward magnetic field in which *T* experiences a force to right. ***Ans.***

15. **(a)** According to Faraday's law of electromagnetic induction, $\varepsilon = \frac{d\phi}{dt}$

Also, $\varepsilon = iR$

$\therefore$ $iR = \frac{d\phi}{dt} \Rightarrow \int d\phi = R\int i dt$

Magnitude of change in flux ($d\phi$) = R × area under current vs time graph

or, $d\phi = 100 \times \frac{1}{2} \times \frac{1}{2} \times 10 = 250$ Wb. ***Ans.***

16. **(c, d)** The sides *PQ* and *RS* are only perpendicular to velocity vector, so $e_{PQ} = e_{SR} = Bv\ell$. And $e_{net} = Bv\ell - Bv\ell = 0$. ***Ans.***

Alternating Current and EM-Waves 19

TOPIC 19.1*: *RMS and Peak Value of AC, Impedence and Admittance, RLC Circuit, Resonance and Power in AC Circuits.

Review of Formulae

1. **RMS value of AC :** For sinusoidal AC, $i = i_0 \sin \omega t$, the RMS value

$$i_{\text{rms}} = \frac{i_0}{\sqrt{2}} = 0.707\, i_0.$$

Similarly

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = 0.707\, V_0.$$

2. **Three simple circuits :**
   (i) Circuit having resistor only : The alternating potential difference across a resistor has amplitude $V_R = iR$; the current is in phase with the potential.
   (ii) Circuit having capacitor only : $V_c = iX_c$ in which $X_c = \frac{1}{\omega C}$ is the capacitive reactance : the current leads the potential by 90°.
   (iii) Circuit having inductor only : $V_L = iX_L$, in which $X_L = \omega L$ is the inductive reactance; the current lags the potential by 90°.

3. **Impedance and admittance :** The total resistance of AC circuit is called impedance Z, and reciprocal of impedance is called admittance A. Thus $A = \frac{1}{Z}$.

4. **Series RL C circuit :** For a series RLC circuit,

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

and

$$\tan \phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

current $$i = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

5. **Resonance :** The current amplitude i_0 in a series *LRC* circuit driven by a sinusoidal external emf is a maximum $\left(i = \frac{V_0}{R}\right)$ when the driving angular frequency ω equals the natural angular frequency ω_0 (resonance frequency) of the circuit. Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the potential.

$$\omega_0 = \sqrt{\frac{1}{LC}}.$$

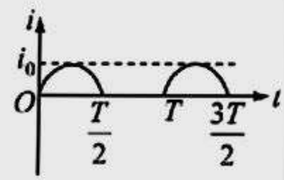
6. **Power in AC circuits :** In a series *RLC* circuit, the average power P_{av} equal to the production rate of thermal energy in the resistor :

$$P_{av} = V_{rms}\, i_{rms} \cos\phi.$$

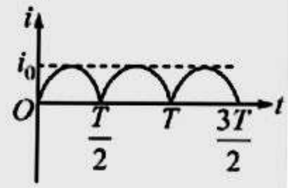
Here $\cos\phi$ is called power factor and is equal to $\frac{R}{Z}$. Its value $\cos\phi \le 1$. For purely inductive or capacitive circuit $\cos\theta = 0$ and so $P_{av} = 0$.

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

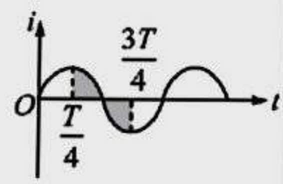
1. The average value of half rectified AC
   (i) from 0 to $\frac{T}{2}, i_{av} = \frac{2i_0}{\pi}$
   (ii) from 0 to T, $i_{av} = \frac{i_0}{\pi}$



2. The average value of full wave rectified AC from 0 to $\frac{T}{2}$ or from 0 to T will be $\frac{2i_0}{\pi}$.



3. The average value of AC over half cycle will be zero from $\frac{T}{4}$ to $\frac{3T}{4}$.



4. $\frac{1}{T}\int_0^T \sin^2\omega t\, dt = \frac{1}{2}$; $\frac{1}{T}\int_0^T \cos^2\omega t\, dt = \frac{1}{2}$.
5. The RMS value of square wave is equal to its peak value.
6. $V = iR$ is the general relation between i, V & R, so it can be used for both DC and AC.
7. The power consumed in the circuit is never negative.

Illustration 1

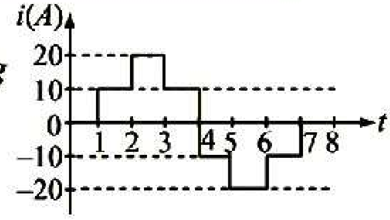


Figure shows the wave form of an alternating current. Find rms value of the current.

Short-cut solution :

Since the waveform is symmetrical, therefore rms value for half wave is same as that of full wave. So

$$\overline{i^2} = \frac{10^2 + 20^2 + 10^2}{3} = 200$$

Now $$i_{rms} = \sqrt{\overline{i^2}} = \sqrt{200} = 10\sqrt{2}\ A. \qquad \textit{Ans.}$$

Illustration 2

Find the effective or rms value of i = 5 sin 100 πt + 5 $\cos\left(100\pi t + \frac{\pi}{6}\right)$.

Short-cut solution :

The given value can be written as

$$i = 5\sin 100\pi t + 5\sin\left(100\pi t + \frac{2\pi}{3}\right)$$

$\therefore$ $$i_0 = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5\cos\left(\frac{2\pi}{3}\right)}$$

$$= \sqrt{25 + 25 + 50\left(-\frac{1}{2}\right)}$$

$$= 5.$$

Now $$i_{rms} = \frac{i_0}{\sqrt{2}} = \frac{5}{\sqrt{2}}\ A. \qquad \textit{Ans.}$$

Illustration 3

The time varying current is given by i = 10 + 5 $\sin\left(\omega t + \frac{\pi}{6}\right)$ ***A. Find rms value of the current.***

Short-cut solution :

Here 10 *A* is the steady current (DC) and 5 $\sin\left(\omega t + \frac{\pi}{6}\right)$ is AC, so

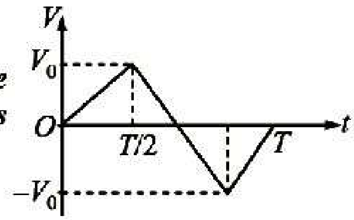
$$i_1 = 10\ A \text{ and } i_2 = \frac{5}{\sqrt{2}}\ A$$

Now $$i_{rms} = \sqrt{i_1^2 + i_2^2} = \sqrt{10^2 + \left(\frac{5}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{225}{2}} = 10.6\ A. \qquad \textbf{Ans.}$$

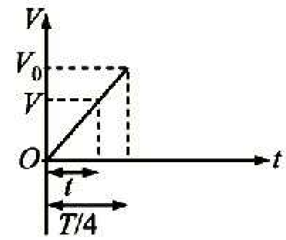
Illustration 4

The voltage-time (V-t) graph for triangular wave having peak value V_0 is as shown in figure. Find rms value of V from t = 0 to $\frac{T}{4}$.



Solution :

The value of V at any time



$$\frac{V}{t} = \frac{V_0}{\left(\frac{T}{4}\right)}$$

or $$V = \frac{4V_0 t}{T}$$

Now $$V^2 = \left(\frac{4V_0 t}{T}\right)^2 = \frac{16V_0^2}{T^2}t^2$$

$$\bar{V}^2 = \frac{\frac{16V_0^2}{T^2}\int_0^{T/4} t^2 dt}{\left(\frac{T}{4}\right)} = \frac{16V_0^2}{T^2}\times\frac{4}{T}\left|\frac{t^3}{3}\right|_0^{\frac{T}{4}}$$

$$= \frac{16\times 4V_0^2}{T^3}\times\left(\frac{T}{4}\right)^3\times\frac{1}{3}$$

$$= \frac{V_0^2}{3}$$

$\therefore$ $$V_{rms} = \sqrt{\bar{V}^2} = \frac{V_0}{\sqrt{3}}. \qquad \textbf{Ans.}$$

Illustration 5

The voltage of an AC supply varies with time as $V = 120 \sin 100\pi t \cos 100\pi t$. Find maximum value of voltage and frequency.

Short-cut solution :

$$V = 60(2 \sin 100\,\pi t \cos 100\,\pi t)$$

$$= 60 \sin 200\,\pi t.$$

On comparing with $V = V_0 \sin \omega t$, we have

$$V_0 = 60\ V \text{ and } \omega = 200\ \pi$$

or $$f = \frac{\omega}{2\pi} = 100 \text{ Hz.}$$ ***Ans.***

Illustration 6

An electrical heater and a capacitor are joined in series across a 220 V, 50 Hz AC supply. The potential difference across the heater is 90V. The potential difference across capacitor will be about.

(a) 100 V (b) 110 V (c) 140 V (d) 200 V

Short-cut solution :

As $$V^2 = V_R^2 + V_C^2 \Rightarrow V_C = \sqrt{V^2 - V_R^2}$$

$$= \sqrt{220^2 - 90^2}$$

$$\simeq 200 \text{ V.}$$ ***Ans. (d)***

Illustration 7

A coil of inductance 0.50 H and resistance 100 Ω is connected to a 240 V, 50 Hz AC supply.

(i) What is the maximum current in the coil?

(ii) What is the time lag between voltage maximum and current maximum?

Short-cut solution :

(i) $$\omega = 2\pi f = 2\pi \times 50 = 100\pi \text{ rad/s,}$$

$$X_L = \omega L = 100\pi \times 0.5 = 50\ \pi\Omega.$$

Impedance, $$Z = \sqrt{R^2 + X_L^2} = \sqrt{100^2 + (50\pi)^2} = 186\ \Omega$$

Maximum current, $$i_0 = \frac{V_0}{Z} = \frac{\sqrt{2} \times 240}{186} = 1.82\ A.$$ ***Ans.***

(ii) $$\tan \phi = \frac{X_L}{R} = \frac{50\pi}{100} = \frac{\pi}{2} = 1.57$$

or $$\phi = 57°$$

Therefore, time lag, $$\Delta t = \left(\frac{\phi}{2\pi}\right)T = \frac{\phi}{2\pi f}$$

$$= \left(57 \times \frac{\pi}{180}\right) \times \frac{1}{2\pi f} = \frac{57}{180 \times 2 \times 50} = 3.2 \times 10^{-3}\ s.$$ ***Ans.***

Illustration 8

A series LCR circuit with L = 0.12 H, C = 480 nF, R = 23 Ω is connected to a 230 V variable frequency supply.

(i) What is the source frequency for which current amplitude is maximum? Obtain this maximum value.

(ii) What is the source frequency for which average power absorbed by the circuit is maximum. Obtain the value of this maximum power.

(iii) For which frequencies of the source is the power transferred to the circuit half the power at resonant frequency? What is the current amplitude at these frequencies?

(iv) What is the Q-factor of the given circuit?

Solution :

(i) The current will be maximum at resonance, so

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.12 \times 480 \times 10^{-9}}} = 4167 \text{ rad/s}$$

and $f_0 = \frac{\omega_0}{2\pi} = 663$ Hz. $I_0^{max} = \frac{V_0}{R} = \frac{\sqrt{2} \times 230}{23} = 14.1$ A. ***Ans.***

(ii) Average power will also be maximum at the same frequency.

$$[P_{av}]_{max} = \frac{i_0^2 R}{2} = \frac{14.1^2 \times 23}{2} = 2300 \text{ W.}$$ ***Ans.***

(iii) Bandwidth, $\Delta\omega = \frac{R}{L} = \frac{23}{0.12} = 191.7$ rad/s

Now $\omega_1 = \omega_0 - \frac{\Delta\omega}{2} = 4167 - \frac{191.7}{2} = 4071.2$ rad/s

or $f_1 = \frac{\omega_1}{2\pi} = \frac{4071.2}{2\pi} = 648$ Hz.

And $\omega_2 = \omega_0 + \frac{\Delta\omega}{2} = 4167 + \frac{191.7}{2} = 4262.8$ rad/s

or $f_2 = \frac{\omega_2}{2\pi} = \frac{4262.8}{2\pi} = 678$ Hz.

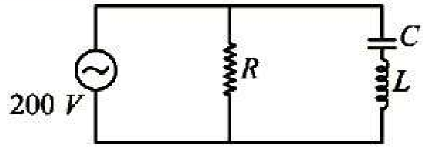
At these frequencies, $i = \frac{1}{\sqrt{2}} i_0^{max} = \frac{14.1}{\sqrt{2}} = 10\ A.$ ***Ans.***

(iv) $Q\text{-factor} = \frac{\omega_0}{\Delta\omega}$

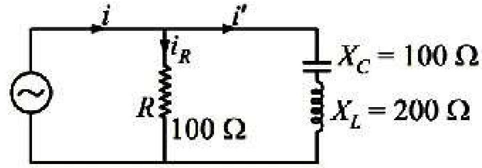
$$= \frac{4167}{191.7} = 21.7.$$ ***Ans.***

Illustration 9

In the circuit shown; $R = 100\ \Omega$, $X_C = 100\ \Omega$ and $X_L = 200\ \Omega$. Find effective current through the source.

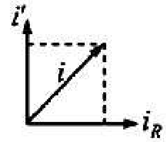


Short-cut solution :



$$i_R = \frac{200}{100} = 2\ A.$$

And $$i' = \frac{V}{X_L - X_C} = \frac{200}{200-100} = 2A$$



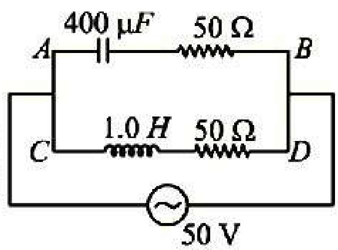
Resultant current, $$i = \sqrt{i_R^2 + i'^2} = \sqrt{2^2 + 2^2}$$

$$= 2\sqrt{2}\ A.$$

Ans.

Illustration 10

In the given circuit, the AC source has $\omega = 50$ rad/s. Considering the inductor and capacitor to be ideal, find impedance of the circuit and current drawn from the source. *[JEE Adv.2012]*



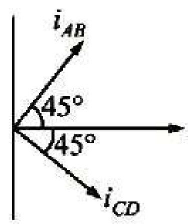
Short-cut solution :

$$X_L = \omega L = 50 \times 1 = 50\ \Omega$$

and $$X_C = \frac{1}{\omega C} = \frac{1}{50 \times 400 \times 10^{-6}} = 50\ \Omega$$

Now, $$Z_{AB} = \sqrt{R^2 + X_C^2} = \sqrt{50^2 + 50^2}$$

$$= 50\sqrt{2}\ \Omega$$



and $$Z_{CD} = \sqrt{R^2 + X_L^2} = \sqrt{50^2 + 50^2}$$

$$= 50\sqrt{2}\ \Omega$$

Current, $$i_{AB} = \frac{V}{Z_{AB}} = \frac{50}{50\sqrt{2}} = \frac{1}{\sqrt{2}}\ A$$

$$\tan \phi_1 = \frac{X_C}{R} = \frac{50}{50} = 1 \quad \text{or} \quad \phi_1 = 45^\circ$$

and $$i_{CD} = \frac{V}{Z_{CD}} = \frac{50}{50\sqrt{2}} = \frac{1}{\sqrt{2}}\ A$$

$$\tan\phi_2 = \frac{X_L}{R} = \frac{50}{50} = 1 \quad \text{or} \quad \phi_2 = 45°$$

Resultant impedance

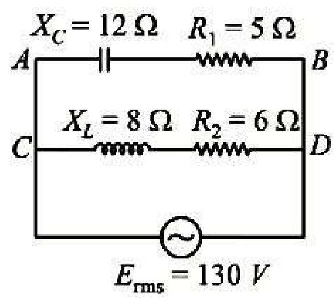
$$Z = \sqrt{Z_{AB}^2 + Z_{CD}^2} = \sqrt{(50\sqrt{2})^2 + (50\sqrt{2})^2}$$
$$= 100\ \Omega. \qquad \textbf{\textit{Ans.}}$$

and

$$i = \sqrt{i_{AB}^2 + i_{CD}^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$
$$= 1\ A. \qquad \textbf{\textit{Ans.}}$$

Illustration 11

What amount of power delivered by the AC source in the circuit shown?



Short-cut solution :

$$Z_{AB} = \sqrt{R_1^2 + X_C^2} = \sqrt{5^2 + 12^2} = 13\ \Omega$$

and

$$i_{1\,rms} = \frac{\varepsilon_{rms}}{Z_{AB}} = \frac{130}{13} = 10\ A$$

$$Z_{CD} = \sqrt{R_2^2 + X_L^2} = \sqrt{6^2 + 8^2} = 10\ \Omega$$

and

$$i_{2\,rms} = \frac{rms}{CD} = \frac{130}{10} = 13\ A$$

$$\text{Power dissipated} = i_{1rms}^2 R_1 + i_{2rms}^2 R_2$$
$$= 10^2 \times 5 + 13^2 \times 6$$
$$= 1514\ \text{W}. \qquad \textbf{\textit{Ans.}}$$

TOPIC 19.2: Electromagnetic Waves, Maxwell's Equations, Energy of EM-Waves and Radiation Pressure.

Review of Formulae

1. **EM-waves :** An EM-wave consists of oscillating electric and magnetic fields. An EM-wave travelling along an x-axis has an electric field $\vec{E}$ and a magnetic field $\vec{B}$ which depend on x and t as;

$$E = E_0 \sin(kx - \omega t)$$

and

$$B = B_0 \sin(kx - \omega t).$$

Electric field induces the magnetic field and vice-versa. The speed of EM-waves is c, which can be written as

$$c = \frac{E}{B} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

2. **Maxwell's equations :** Maxwell discovered that all the basic principles of electromagnetism can be formulated in terms of four fundamental equations, called Maxwell's equations. These are :
   (i) Gauss's law for electricity

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}.$$

   (ii) Gauss's law for magnetism

$$\oint \vec{B} \cdot d\vec{A} = 0$$

   (iii) Faraday's law of induction

$$\oint \vec{E} \cdot d\vec{\ell} = \frac{-d\phi_B}{dt}$$

   (iv) Ampere's law

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \left(i_C + \epsilon_0 \frac{d\phi_E}{dt} \right)$$

3. **Energy flow :** The rate per unit area at which energy is transported via an electromagnetic wave is given by the Pointing vector

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

4. **Intensity of EM-wave :** The time averaged rate per unit area at which energy is transported, is called the intensity of wave :

$$I = \frac{1}{2} \epsilon_0 E_0^2 c.$$

The intensity of the waves at distance r from a point source of power P is

$$I = \frac{P}{4\pi r^2}$$

5. **Radiation pressure :** When a surface intercepts electromagnetic radiation, a force is exerted on the surface. If the radiation is totally absorbed by the surface, the force is

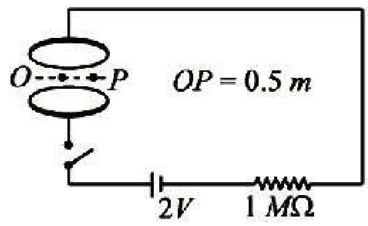
$$F = \frac{IA}{c}$$

where A is the area of the surface perpendicular to the path of the radiation. If the radiation is totally reflected back along its original path, the force is

$$F = \frac{2IA}{c}$$

Illustration 12

A parallel plate capacitor with circular plates of radius 1 m has a capacitance of 1 nF. At t = 0, it is connected for charging in series with a resistor R = 1 MΩ across a 2V battery (see figure). Calculate the magnetic field at a point P, half way between the centre and the periphery of the plates, after $t = 10^{-3}$ s. [The charge on the capacitor at time t is given by $q = CV(1 - e^{-t/\tau})$, where $\tau = CR$.]



Short-cut solution :

Electric field between the plates

$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{A\epsilon_0} = \frac{CV(1 - e^{-t/CR})}{A\epsilon_0}$$

Displacement current,

$$i_d = \epsilon_0 \frac{d\phi_E}{dt} = \epsilon_0 \frac{d}{dt}(E \times \pi r^2)$$

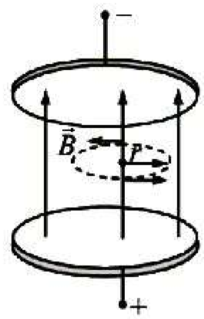
$$= \epsilon_0 \pi r^2 \left(\frac{dE}{dt}\right) = \frac{\epsilon_0 \pi r^2}{A\epsilon_0}\frac{d}{dt}\left[CV\left(1 - e^{-t/CR}\right)\right]$$

$$= \frac{\pi r^2}{\pi R^2} \times CV \times \frac{1}{CR} \times e^{-t/CR}$$

$$= \left(\frac{r}{R}\right)^2 \times \frac{V}{R} e^{-t/CR}$$

$$= \left(\frac{0.5}{1}\right)^2 \times \frac{2}{10^6} \times \frac{-10^{-3}}{e(10^{-9} \times 10^6)}$$

$$= \frac{1}{(2 \times 10^6)} e^{-1}.$$



Now using Ampere's law

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 (i_c + i_d)$$

or

$$B \times 2\pi r = \mu_0 \left(0 + \frac{1}{2 \times 10^6} e^{-1}\right)$$

or

$$B \times 2\pi \times \left(\frac{1}{2}\right) = \mu_0 \times \frac{1}{2 \times 10^6} e^{-1}$$

$\Rightarrow$

$$B = 0.74 \times 10^{-13}\ T.$$

Ans.

Video Solution

Q. A box contains L, C and R. When 250 dc is applied to the terminals of the box, a current of 1.0 A flows in the circuit. When an ac source of 250 V rms at 2250 rad/s is connected, a current of 1.25 A rms flows. It is observed that the current rises with frequency and becomes maximum at 4500 rad/s. Find the values of L, C and R. Draw the circuit diagram.

To see the video solution, scan the QR code:
OR Visit https://www.youtube.com/watch?v=qsU4VFEGuVI

Illustration 13

The magnetic field in a plane electromagnetic wave is given by

$$B_y = 2 \times 10^{-7} \sin(0.5 \times 10^{-3}x + 1.5 \times 10^{11} t)\ T.$$

(i) What is the wavelength and frequency of the wave?

(ii) Write the expression of electric field.

Short-cut solution :

(i) On comparing, with

$$B = B_0 \sin(kx + \omega t), \text{ we have}$$

$$B_0 = 2 \times 10^{-7}\ T, k = 0.5 \times 10^{-3} \text{ or } \frac{2\pi}{\lambda} = 0.5 \times 10^{-3}$$

$$\therefore \quad \lambda = \frac{2\pi}{0.5 \times 10^{-3}} = 0.126 \text{ m}$$

Also $\omega = 1.5 \times 10^{11}$

or $f = \frac{\omega}{2\pi} = \frac{1.5 \times 10^{11}}{2\pi} = 23.9 \text{ GHz}$

(ii) Using, $E_0 = cB_0 = 3 \times 10^8 \times (2 \times 10^{-7}) = 60$ V/m.

For direction of magnetic field, using

$$\vec{S} \to \frac{\vec{E} \times \vec{B}}{\mu_0}, \quad \text{or} \quad (-\hat{i}) \to \vec{E} \times (\hat{j}) \text{ gives } \vec{E} \to \hat{k}$$

$$\vec{E}_Z = (\hat{k})60 \sin(0.5 \times 10^3 x + 1.5 \times 10^{11} t). \qquad \textit{Ans.}$$

Illustration 14

Calculate the electric and magnetic fields produced by the radiation coming from a 100 W bulb at a distance of 3m. Assume that the efficiency of the bulb is 2.5% and it is a point source.

Short-cut solution :

$$I = \frac{\text{Power}}{\text{area}} = \frac{100 \times 2.5\%}{4\pi(3)^2} = 0.022 \text{ W/m}^2.$$

Half of this intensity is provided by the electric field and half by the magnetic field, so

$$\frac{1}{2}\left(\epsilon_0 E_{rms}^2 c\right) = \frac{I}{2}$$

$$= \frac{0.022}{2}$$

$$\therefore \quad E_{rms} = \sqrt{\frac{0.022}{(8.85\times10^{-12})\times(3\times10^8)}}$$

$$= 2.9 \text{ V/m}$$

and $$E_0 = \sqrt{2}\, E_{rms} = \sqrt{2}\times 2.9 = 4.07 \text{ V/m.} \quad \textbf{Ans.}$$

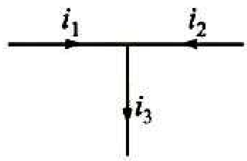
Now $$B_{rms} = \frac{E_{rms}}{c} = \frac{2.9}{3\times10^8} = 9.6\times10^{-9}\ T.$$

and $$B_0 = \sqrt{2}B_{rms} = \sqrt{2}\times 9.6\times10^{-9}$$

$$= 1.4\times10^{-8}\text{ T.} \quad \textbf{Ans.}$$

Concept Booster Exercise

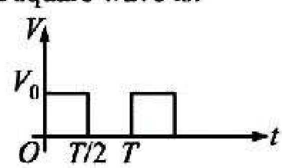
1. If $i_1 = 3\sin\omega t$ and $i_2 = 4\cos\omega t$, then i_3 is:



(a) $5\sin(\omega t + 53°)$ (b) $5\sin(\omega t + 37°)$

(c) $5\sin(\omega t + 45°)$ (d) $5\cos(\omega t + 53°)$

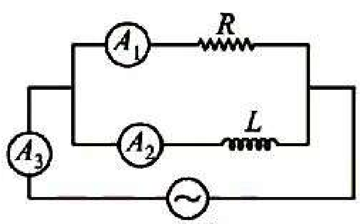
2. The rms value of potential difference V of half wave rectified square wave is:



(a) V_0 (b) $V_0/\sqrt{2}$

(c) V_0 (d) $\frac{V_0}{4}$

3. In the circuit shown, assuming all ammeters to be ideal, if readings of the hot wire ammeters A_1 and A_2 are i_1 and i_2 respectively, then reading of the hot wire ammeter A_3 is :



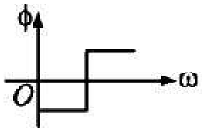
(a) $(i_1 - i_2)$ (b) $(i_1 + i_2)$

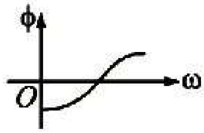
(c) $\sqrt{i_1^2 + i_2^2}$ (d) zero

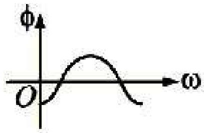
4. An electric lamp designed for operation on 110 V, AC is connected to a 220 V, AC supply, through a choke coil of inductance 2H, for proper operation. The angular frequency of the AC is $100\sqrt{2}$ rad/s. If a capacitor is to be used in place of the choke coil, its capacitance must be **Numeric/Integer**

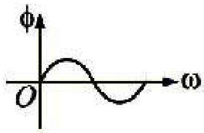
(a) 1 μF (b) 2 μF (c) 5 μF (d) 10 μF

5. An inductor and a capacitor are joined in series to an AC source. The frequency of the AC is gradually increased. The phase difference ϕ between the emf and the current is plotted against the angular frequency ω. Which of the following best represents the resulting curve?

(a) 

(b) 

(c) 

(d) 

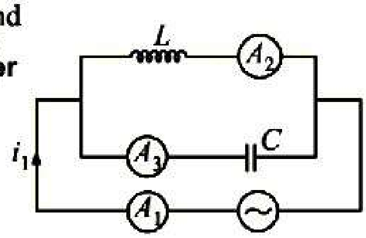
6. In an AC circuit a resistance R is connected in series with an inductance L. If the phase difference between the voltage and current is $\frac{\pi}{4}$, the value of inductive reactance is xR. Find the value of x. **Numeric/Integer**

(a) 1 (b) 0.5 (c) 0 (d) 1.5

7. The current in resistance R in resonance is

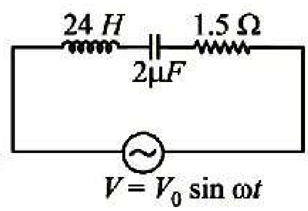
(a) minimum but finite (b) maximum but finite
(c) zero (d) none of these

8. For the circuit shown, the ammeter A_2 reads 1.6A and ammeter A_3 reads 0.4 A. Then i_1 is : **Numeric/Integer**



(a) 1.6 A
(b) 0.4 A
(c) 1.2 A
(d) 2 A

9. An LCR circuit as shown in figure is connected to a voltage source whose frequency can be varied. The frequency, at which the voltage across the resistor is maximum? **Numeric/Integer**

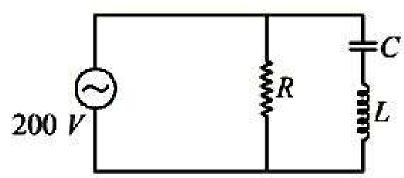


(a) 23 Hz (b) 143 Hz
(c) 345 Hz (d) 902 Hz

10. A transformer having efficiency of 90% is working on 200 V and 3 kW power supply. If the current in the secondary coil is 6 A, the voltage across the secondary coil and the current in the primary coil respectively are :

(a) 300 V, 15 A (b) 450 V, 15 A
(c) 450 V, 13.5 A (d) 600 V, 15 A

11. In the circuit shown, $R = 100\ \Omega$, $X_C = 100\ \Omega$ and $X_L = 200\ \Omega$. The effective current through the source is: **Numeric/Integer**

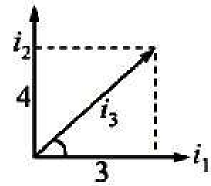


(a) 2 A (b) $2\sqrt{2}A$
(c) 0.5 A (d) zero

Solutions

1. **(a)** $i_3 = \sqrt{i_1^2 + i_2^2} = \sqrt{3^2 + 4^2} = 5\ A$

and $\tan\phi = \frac{4}{3}$ or $\phi = 53°$. ***Ans.***



2. **(b)** $V_{rms} = \sqrt{\frac{V_1^2 + V_2^2}{2}} = \sqrt{\frac{V_0^2 + 0^2}{2}} = \frac{V_0}{\sqrt{2}}$. ***Ans.***

3. **(c)** $i_3 = \sqrt{i_1^2 + i_2^2}$. ***Ans.***

4. **(c)** $\omega L = \frac{1}{\omega C}$ or $C = \frac{1}{\omega^2 L} = \frac{1}{(100\sqrt{2})^2 \times 2} = 5 \times 10^{-6}$ F. ***Ans.***

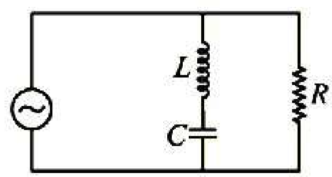
5. **(a)** $\tan\phi = \frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$. For $R = 0$, $\phi = \pm\frac{\pi}{2}$.

At low frequencies, $\frac{1}{\omega C} > \omega L$, $\phi = -\frac{\pi}{2}$

At high frequencies, $\omega L > \frac{1}{\omega C}$, $\phi = \frac{\pi}{2}$. ***Ans.***

6. **(a)** $\tan\frac{\pi}{4} = \frac{X_L}{R}$ $\Rightarrow$ $X_L = R$. ***Ans.***

7. **(c)** At resonance, inductor and capacitor together offers zero resistance, so entire current will pass through them. ***Ans.***



8. **(c)** $i = 1.6 - 0.4 = 1.2\ A$. ***Ans.***

9. **(c)** $\omega_0 = \sqrt{\frac{1}{LC}}$. ***Ans.***

10. **(b)** $\eta = 0.9 = \frac{V_s i_s}{V_p i_p} = \frac{V_s(6)}{3 \times 10^3}$ $\Rightarrow$ $V_s = 450$ V

and $V_p i_p = 3000$ or $i_p = \frac{3000}{V_p} = \frac{3000}{200} = 15\ A$. ***Ans.***

11. **(b)** $i_1 = \frac{200}{100} = 2\ A$

and $i_2 = \frac{200}{(200 - 100)} = 2\ A$

$\therefore$ $i = \sqrt{i_1^2 + i_2^2} = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}A$. ***Ans.***

Ray Optics 20

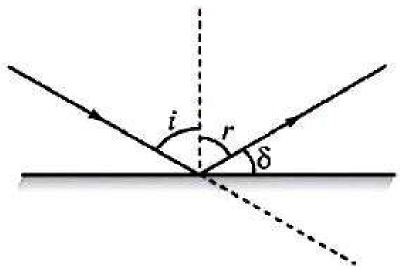
TOPIC 20.1: *Angle of Incident and Deviation Produced by a Plane Mirror, Number of Images Formed by a Plane Mirror and Image Velocity.*

Review of Formulae

1. **Deviation produced by a mirror**

   (i) By single mirror; at an angle of incidence i, it is

   $$\delta = 180° - 2i.$$



   (ii) By two mirrors at an angle θ, the deviation

   $$\delta = 360° - 2\theta.$$

   It is independent of the angle of incidence of ray.

2. **Number of images**

   Suppose θ is the angle between the mirrors, then

   (i) if $\frac{360°}{\theta}$ is even integer, then number of images $n = \frac{360°}{\theta} - 1$ for all positions of the object.

   (ii) if $\frac{360°}{\theta}$ is odd integer, then

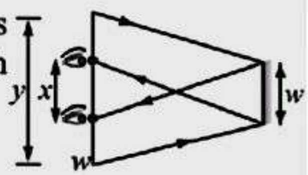
   $n = \frac{360°}{\theta}$ if the object is placed off the bisector of the mirrors, and

   $\left(\frac{360°}{\theta} - 1\right)$, when object is placed on the axis.

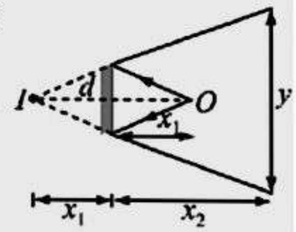
   $n = 3$ for $\theta = 90°$ and 5 for 60° and 72°.

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. The minimum size (y) of mirror (plane or spherical) needed to make full image is $y \to 0$.
2. The minimum height of mirror needed to see the full image of the observer himself is half the height of the observer.
3. If y is the width of the face of the observer and x is the distance between his eyes, then minimum width of mirror needed to see full face is $\left(\frac{y-x}{2}\right)$.



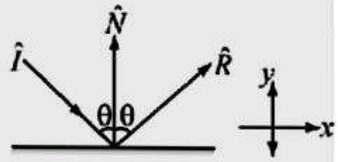
4. If an observer moves parallel to plane mirror of width 'd' at a distance x_2 and if object at a distance x_1 from mirror, then the maximum distance upto which he can see the image is y.



From similar triangles, we have

$$\frac{y}{x_1 + x_2} = \frac{d}{x_1} \Rightarrow y = \left(1 + \frac{x_2}{x_1}\right)d.$$

5. If ray incident at an angle θ with the normal, then unit vector along incident ray,



$$\hat{I} = \sin\theta\hat{i} - \cos\theta\hat{j},$$

and unit vector along reflected ray $\hat{R} = \sin\theta\hat{i} + \cos\theta\hat{j}$.

6. If vector $\vec{I}$ and $\vec{N}$ given, then angle of incidence θ

$$\cos(180° - \theta) = \frac{\vec{I} \cdot \vec{N}}{IN}. \text{ Also } \cos\theta = \frac{\vec{R} \cdot \vec{N}}{RN}.$$

7. If vector $\vec{I}$ and $\vec{R}$ are given, then angle of incidence

$$\cos(180° - 2\theta) = \frac{\vec{I} \cdot \vec{R}}{IR}.$$

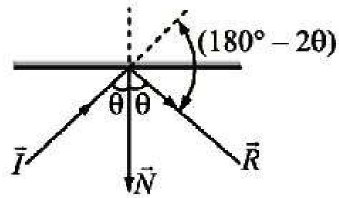
8. As $\vec{I}, \vec{R}$ and $\vec{N}$ are in the same plane, so their scalar product, $[\vec{N}\vec{R}\vec{I}] = 0$.
9. The number of images by two mirrors placed at 60° and 72° may be 5, and for angle 72 and 75 may be 4.
10. A person standing at the middle of the room looking in a plane mirror hung on a front wall can see the image of the back wall if the height of mirror is equal to one third of the height of the wall.

Illustration 1

A ray of light travelling in the direction $\frac{1}{2}(\hat{i}+\sqrt{3}\hat{j})$ is incident on a plane mirror. After reflection, it travels along the direction $\frac{1}{2}(\hat{i}-\sqrt{3}\hat{j})$. Find angle of incident.

[JEE Adv. 2013]

Short-cut solution :



If θ is the angle of incident, then

$$\cos(180° - 2\theta) = \frac{\vec{I}\cdot\vec{R}}{IR}$$

$$= \frac{\frac{1}{2}(\hat{i}+\sqrt{3}\hat{j})\cdot\frac{1}{2}(\hat{i}-\sqrt{3}\hat{j})}{\left(\frac{1}{2}\sqrt{1^2+(\sqrt{3})^2}\right)\left(\frac{1}{2}\sqrt{1^2+(\sqrt{3})^2}\right)}$$

$$= \frac{1-3}{2\times 2} = -\frac{1}{2}$$

∴ $180 - 2\theta = 120°$

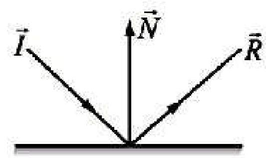
or $\theta = 30°$. *Ans.*

Illustration 2

A ray of light is incident on a plane mirror along a vector in $(\hat{i}+\hat{j})$. The normal at incident point is along $(\hat{i}+\hat{k})$. The unit vector along the reflected ray is:

(a) $\frac{\hat{j}+\hat{k}}{\sqrt{2}}$ *(b)* $\frac{\hat{j}-\hat{k}}{\sqrt{2}}$ *(c)* $\frac{\hat{i}+\hat{k}}{\sqrt{2}}$ *(d)* $\frac{\hat{i}-\hat{k}}{\sqrt{2}}$

Short-cut solution :



As $\vec{I}, \vec{N}$ and $\vec{R}$ in the same plane, so

$$[\vec{N}\vec{R}\vec{I}] = 0$$

If we take $\vec{R} = (\hat{j}-\hat{k})$, then

$$\vec{N}\vec{R}\vec{I} = (\hat{i}+\hat{k})\cdot(\hat{j}-\hat{k})\times(\hat{i}+\hat{j})$$

$$= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}$$

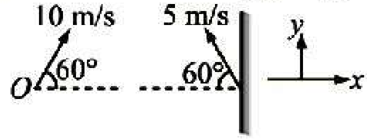
$$= 1(0+1) + 0 + 1(0-1)$$

$$= 0.$$

Therefore correct option is (b). *Ans. (b)*

Illustration 3

Velocity of object and mirror are shown in figure, find the velocity of image.



Short-cut solution :

Along x-direction

$$\vec{v}_{om} = \vec{v}_o - \vec{v}_m = 10\cos 60°\hat{i} + 5\cos 60°\hat{i}$$

$$= \left(5 + 5 \times \frac{1}{2}\right)\hat{i} = 7.5\hat{i}$$

$\therefore$
$$\vec{v}_{im} = -\vec{v}_{om} = -7.5\hat{i} \text{ m/s.}$$

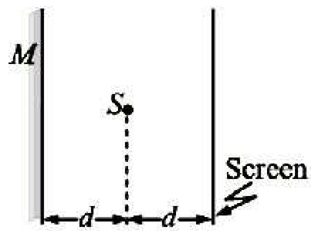
Along y-direction

$$\vec{v}_i = \vec{v}_o = 10\sin 60°\hat{j} = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}\hat{j} \text{ m/s.}$$

$\therefore$ Velocity of image $= \left(-7.5\hat{i} + 5\sqrt{3}\hat{j}\right)$ m/s. ***Ans.***

Illustration 4

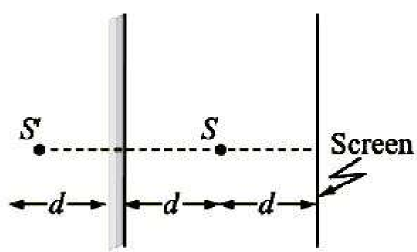
A point source of light S is placed at a distance d from a screen; the intensity at the centre of the screen is I. When a perfectly reflecting mirror M is placed a distance d behind the source, then intensity at the centre of the screen becomes :



(a) I ***(b)*** $2I$ ***(c)*** $\frac{10}{9}I$ ***(d)*** $4I$

Short-cut solution :

Intensity,
$$I = \frac{k}{r^2}$$



Intensity at the screen without mirror

$$I_1 = \frac{k}{d^2}$$

Intensity at the screen with mirror

$$I_2 = \frac{k}{d^2} + \frac{k}{(3d)^2}$$

$$= \frac{10k}{9d^2}.$$

Ans. (c)

***TOPIC 20.2**: Spherical Mirror Formula, Magnification Laws of Reflection in Vector Form.*

Review of Formulae

1. **Spherical mirror formula (for small size mirror)**

$$\frac{1}{v}+\frac{1}{u} = \frac{1}{f}=\frac{2}{R}.$$

2. **Lateral magnification**

$$m = \frac{-v}{u} = \frac{f}{f-u} = \frac{f-v}{f}.$$

Longitudinal magnification

$$m_L = -\frac{\text{length of image}}{\text{length of object}}$$

For short object (δu), its length of image (δv)

$$m_L = \frac{\delta v}{\delta u} = -\frac{v^2}{u^2}.$$

Superficial magnification

$$m_{\text{area}} = \frac{A_i}{A_o} = -\frac{v^2}{u^2}.$$

3. **Velocity of image:**

$$\vec{v}_{im} = -\frac{v^2}{u^2}\vec{v}_{om}.$$

4. If the object and image distances are measured from focus, then $x_i x_o = f^2$.

Illustration 5

A point object is placed 60 cm from pole of a concave mirror of focal length 10 cm on the principal axis. Find

(i) Position of the image.

(ii) If object is shifted 1mm towards the mirror along principal axis, find the shift in position of image.

Short-cut solution :

(i) Using, $$\frac{1}{v}+\frac{1}{u} = \frac{1}{f}$$

or $$\frac{1}{v}+\frac{1}{-60} = \frac{1}{-10}$$

or $$v = -12 \text{ cm.}$$ ***Ans.***

(ii) Using,

$$\delta v = -\frac{v^2}{u^2}(\delta u)$$

$$= -\left(\frac{-12}{-60}\right)^2 \times 1 = -\frac{1}{25} \text{ mm.} \quad \textbf{Ans.}$$

Illustration 6

When an object is placed at a distance of 25 cm from a mirror, the magnification is m_1. The object is moved 15 cm further away with respect to the earlier position, and the magnification is m_2. If $\frac{m_1}{m_2} = 4$, the calculate the focal length of the mirror.

Short-cut solution :

Using,
$$m = \frac{f}{f-u}$$

or
$$m_1 = \frac{f}{f-(-25)} = \frac{f}{f+25}$$

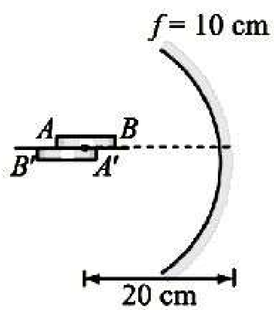
and
$$m_2 = \frac{f}{f+(25+15)} = \frac{f}{f+40}$$

$\therefore$
$$\frac{m_1}{m_2} = 4 = \frac{f+40}{f+25} \quad \Rightarrow \quad f = -20 \text{ cm.} \quad \textbf{Ans.}$$

Illustration 7

An object of length 4 cm is placed symmetrically on the centre of curvative of a concave mirror of focal length 10 cm. Find length of its image.

Short-cut solution :



For end A:
$$\frac{1}{v_A} + \frac{1}{-22} = -\frac{1}{10}$$

or
$$v_A = -18.3 \text{ cm}$$

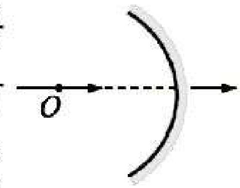
For end B:
$$\frac{1}{v_B} + \frac{1}{-18} = \frac{1}{-10}$$

or
$$v_B = -22.5 \text{ cm}$$

Length of image $(A'B') = 22.5 - 18.3 = 4.2$ cm. **Ans.**

Illustration 8

An object kept on the principal axis is moving in the same direction as that of mirror as shown in figure. Speed of the object and mirror is 10 m/s and $\frac{40}{13}$ m/s. The radius of curvature of the mirror is 20 cm. What should be the distance of object from the mirror at this instant so that the image is stationary?



Short-cut solution :

Using,
$$\vec{v}_{im} = -\frac{v^2}{u^2} \times \vec{v}_{om}$$
$$= -\left(\frac{f}{f-u}\right)^2 \times \vec{v}_{om}$$

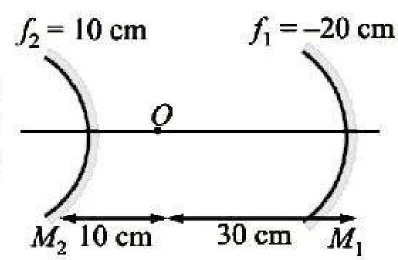
or
$$0 - \frac{40}{13} = -\left(\frac{-10}{-10-u}\right)^2 \times \left(10 - \frac{40}{13}\right)$$

or
$$\frac{-10}{-10-u} = \pm\frac{2}{3}$$

On solving, we get u = –25 cm. ***Ans.***

Illustration 9

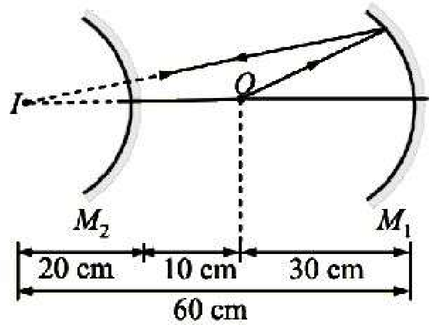
In figure there are two mirrors; one convex mirror, $f_1 = 20$ cm and other of $f_2 = 10$ cm. Find the total magnification after two reflections first on M_1 and then on M_2.



Solution :

For M_1 :
$$\frac{1}{v_1} + \frac{1}{-30} = \frac{1}{-20} \Rightarrow v_1 = -60 \text{ cm}$$

$\therefore$
$$m_1 = \frac{-v_1}{u_1} = -\frac{(-60)}{(-30)} = -2.$$



For M_2 : $\frac{1}{v_2}+\frac{1}{+20} = \frac{1}{+10} \Rightarrow v_2 = +20$ cm

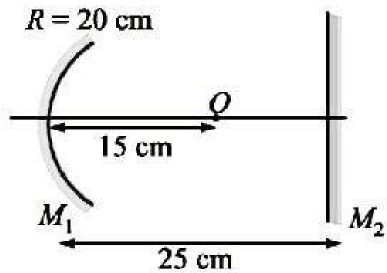
$\therefore$ $m_2 = \frac{-v_2}{u_2} = \frac{-20}{20} = -2.$

Total magnification, $m = m_1 \times m_2 = (-2)(-2)$

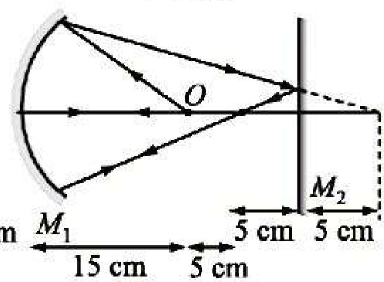
$= 4.$ ***Ans.***

Illustration 10

Find the position of final image after three successive reflections taking first on M_1.



Short-cut solution :



For M_1 : $\frac{1}{v}+\frac{1}{u} = \frac{1}{f}$

or $\frac{1}{v}+\frac{1}{-15} = \frac{1}{-10} \Rightarrow v = -30$ cm

For M_2 : Object distance, $u = (30 - 25) = 5$ cm, and so $v = -5$ cm.

Now again for M_1 :

$u = -20$ cm

$\therefore$ $v = \frac{uf}{u-f} = \frac{(-20)\times(-10)}{-20+10} = -20$ cm.

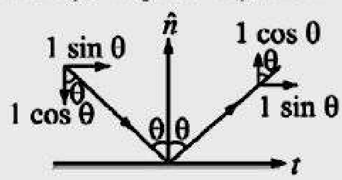
So final image will be 20 cm from M_1. ***Ans.***

Law of reflection in vector form

If $\hat{I}$ and $\hat{R}$ are the unit vectors along incident and reflected rays respectively, then

$\hat{I} = 1 \sin\theta\,\hat{t} - 1\cos\theta\,\hat{n}$

and $\hat{R} = 1 \sin\theta\,\hat{t} + 1\cos\theta\,\hat{n}$

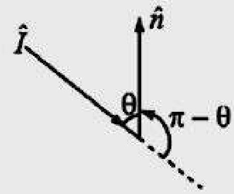


$\therefore$ $\hat{R}-\hat{I} = (2\cos\theta)\,\hat{n}$

or $\hat{R} = \hat{I} + (2\cos\theta)\,\hat{n}$ ...(i)

Also $\hat{I}\cdot\hat{n} = 1 \times 1 \times \cos(\pi - \theta)$

$= -\cos\theta$ ...(ii)



From above equations, we have

$$\hat{R} = \hat{I} - 2(\hat{I}\cdot\hat{n})\hat{n}$$

Illustration 11

A ray of light is incident on the yz-plane mirror along a unit vector $\hat{I} = \frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$. Find unit vector along the reflected ray.

Short-cut solution :

The unit vector normal to yz-plane is $\hat{n} = \hat{i}$.

Using, $\hat{R} = \hat{I} - 2(\hat{I}\cdot\hat{n})\hat{n}$

Here $\hat{I}\cdot\hat{n} = \left(\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right)\cdot\hat{i} = \frac{1}{\sqrt{3}}.$

$\therefore$ $\hat{R} = \left(\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right) - 2\left(\frac{1}{\sqrt{3}}\right)\hat{i}$

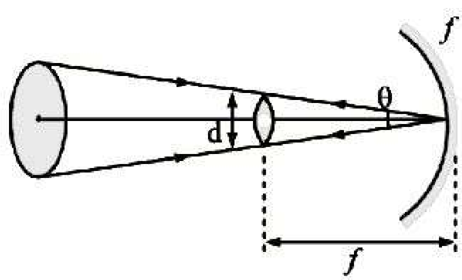
$= -\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}.$ ***Ans.***

Illustration 12

Calculate the diameter of the image of sun formed by the concave mirror of focal length f, if θ is the angle subtends by sun's rays at the mirror.

Short-cut solution :

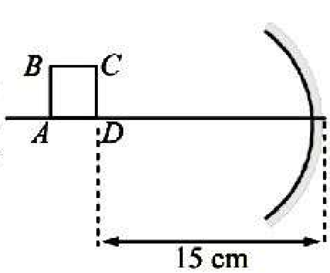
The image of sun will be formed at the focus of the mirror. As θ is very small (θ ~ 1°), so



$$\frac{d}{f} = \theta \quad \text{or} \quad d = f\theta.$$ ***Ans.***

Illustration 13

A square ABCD of side 1 mm is kept at distance 15 cm infront of the concave mirror as shown in figure. The focal length of the mirror is 10 cm. Find the length of the perimeter of its image.



Short-cut solution :

As side of the square is small in comparison to the distance $u = 15$ cm, so we assume AB and CD at same distance. So

$$\frac{1}{v}+\frac{1}{-15} = \frac{1}{-10} \quad \Rightarrow \quad v = -30 \text{ cm.}$$

For sides AB and CD,

$$A'B' = m(AB) = \left(\frac{30}{15}\right)\times 1 = 2\text{mm.}$$

For sides AD and BC,

$$A'D' = m^2(AD) = \left(\frac{30}{15}\right)^2 \times 1 = 4 \text{ mm.}$$

Now perimeter $= A'B' + B'C' + C'D' + D'A' = 2 + 4 + 2 + 4 = 12$ mm. ***Ans.***

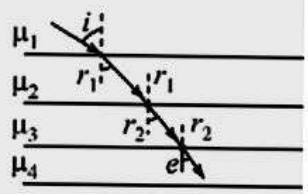
TOPIC 20.3: *Refraction of Light, Laws of Reflection, Real and Apparent Depth and Vector Form of Snell's Laws.*

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. When light passes from one to other medium, its frequency will not change but wavelength gets changed.
2. If light ray travels through many mediums, then angle of emergence in any medium depends on angle of incidence, RI of first and last medium.

$$\frac{\mu_2}{\mu_1} = \frac{\sin i}{\sin r_1},$$

$$\frac{\mu_3}{\mu_2} = \frac{\sin r_1}{\sin r_2},$$



and
$$\frac{\mu_4}{\mu_3} = \frac{\sin r_2}{\sin e}$$

$$\therefore \quad \frac{\mu_2}{\mu_1}\times\frac{\mu_3}{\mu_2}\times\frac{\mu_0}{\mu_3} = \frac{\sin i}{\sin r_1}\times\frac{\sin r_1}{\sin r_2}\times\frac{\sin r_2}{\sin e}$$

or
$$\frac{\mu_4}{\mu_1} = \frac{\sin i}{\sin e}.$$

3. If first and last mediums are same, then

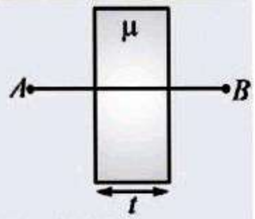
$$\mu_4 = \mu_1 \quad \Rightarrow \quad e = i.$$

4. If light travels a distance x_m in medium of RI, μ_m in any time, then distance travelled by light in air (vacuum) in the same duration will be $\mu_m x_m$. So optical path length

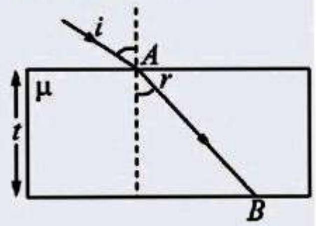
$$x_{air} = \mu_m x_m.$$

5. Optical path of light from A to B travelling through a glass slab of thickness t will be;

Optical path of $AB = AB + (\mu - 1)t$.



6. Time taken by light to travel through a slab of thickness t and RI, μ:



Distance, $$AB = \frac{t}{\cos r} = \frac{t}{\sqrt{1-\sin^2 r}}$$

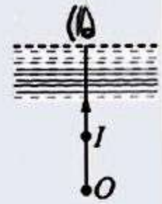
and $$\sin r = \frac{\sin i}{\mu}$$

$\therefore$ $$AB = \frac{t}{\sqrt{1-\frac{\sin^2 i}{\mu^2}}}$$

$$= \frac{\mu t}{\sqrt{\mu^2 - \sin^2 i}}.$$

Time take in $= \frac{AB}{v_m} \frac{AB}{c/\mu}$

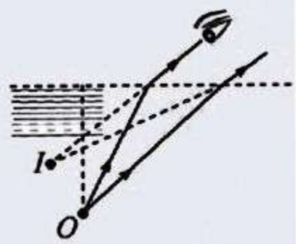
7. Apparent depth



(i) For overhead observer, $AD = \frac{RD}{{}_{\text{observer medium}}\mu_{\text{object medium}}}$

$$= \frac{RD}{{}_o\mu_m}$$

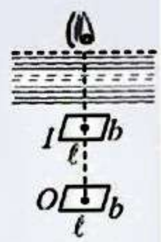
Apparent shift $$S = RD\left(1 - \frac{1}{{}_o\mu_m}\right)$$



(ii) For oblique observer

$$AD = \frac{(RD)_o\mu_m^2 \cos^3\theta}{(\mu_m^2 - \sin^2\theta)^{3/2}}.$$

(iii) For oblique line of sight the AD will be smaller than that of overhead observer.



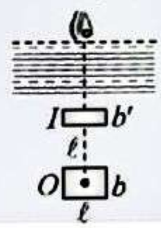
8. Area of plate (overhead observer)

(i) When plate is situated parallel to refracting surface

Area of image, $A_i = A_o$.

(ii) When plate is situated normal to refracting surface

In this case height of image



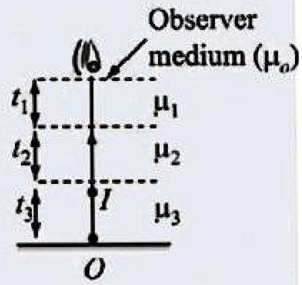
$$b' = \frac{b}{{}_o\mu_m}$$

$$\therefore \qquad A_i = \ell' b' = \frac{\ell b}{{}_o\mu_m} = \frac{A_o}{{}_o\mu_m}.$$

9. Apparent depth in general

$$AD = \frac{t_1}{{}_o\mu_1} + \frac{t_2}{{}_o\mu_2} + \ldots\ldots + \frac{t_n}{{}_o\mu_n}.$$

and $S = t_1\left(1 - \frac{1}{{}_0\mu_1}\right) + t_2\left(1 - \frac{1}{{}_0\mu_2}\right) + \ldots\ldots + t_n\left(1 - \frac{1}{{}_0\mu_n}\right).$

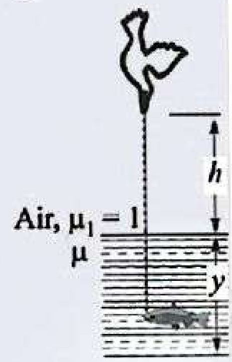


10. Velocity of image

Case 1 : *Bird is the observer*

The apparent distance of fish from bird,

$$x = \frac{y}{\mu} + h$$



On differentiating w.r.t. time we get

$$\frac{dx}{dt} = \frac{1}{\mu}\left(\frac{dy}{dt}\right) + \left(\frac{dh}{dt}\right)$$

Here $\frac{dx}{dt} \rightarrow$ apparent velocity of fish w.r.t. bird

Take $\left(\frac{dy}{dt}\right)$ positive, if y increases (fish moves downwards) and $\left(\frac{dy}{dt}\right)$ positive when bird moves upwards.

Case 2 : *Fish is the observer*

The apparent distance of bird from fish

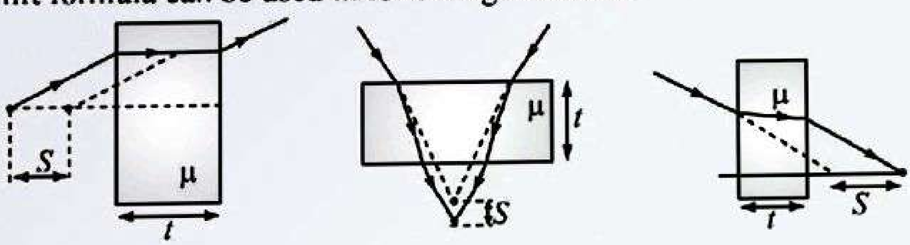
$$x = \mu h + y$$

On differentiating w.r.t. time, we get

$$\frac{dx}{dt} = \mu\left(\frac{dh}{dt}\right) + \left(\frac{dy}{dt}\right)$$

Here, $\frac{dx}{dt} \rightarrow$ apparent velocity of bird w.r.t. fish

11. Shift formula can be used in following situations



$$S = \left(\quad - \quad \right)$$

12. Thick mirror (silvered slab) forms infinite image, but second image is the brighest image.

Some Interesting facts about Refraction and Snell's Law

(i) When light enters from one to other medium at normal incident, it goes undeviated ($\angle i = \angle r = 0$). But speed of light changes, the phenomenon known as refraction (general definition of refraction).

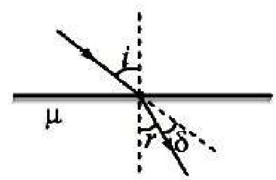
(ii) RI of the material is determined by observing angle of incidence i and angle of refraction r (But angle i and r should not equal to zero), so $\mu = \frac{\sin i}{\sin r}$. μ is the material property which does not depend on angle i and r.

(iii) We can use Snell's law even for $\angle i = 0$, which gives $\sin r = \frac{\sin i}{\mu} = \frac{\sin 0}{\mu} = 0$ or $\angle r = 0$.

Illustration 14

A ray deviates by 30° (which is one third of the angle of incidence) when it gets refracted from vacuum to a medium. Find the RI of the medium.

Short-cut solution :



We know, $\delta = i - r$

or $\frac{i}{3} = i - r = 30°$

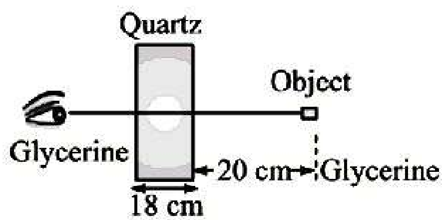
$\Rightarrow$ $i = 90°$ and $r = 60°$

So $\mu = \frac{\sin i}{\sin r} = \frac{\sin 90°}{\sin 60°} = \frac{1}{\sqrt{\frac{3}{2}}}$

$= \frac{2}{\sqrt{3}}$. *Ans.*

Illustration 15

Given that, velocity of light in quartz = 1.5 × 10^8 m/s and velocity of light in glycerine = $\frac{9}{4} \times 10^8$ m/s. Now a slab made of quartz is placed in glycerine as shown. Find shift produced by the slab.



Short-cut solution :

The RI of quartz, $\mu_1 = \frac{3 \times 10^8}{1.5 \times 10^8} = 2$

RI of glycerine, $\mu_2 = \frac{3 \times 10^8}{\left(\frac{9}{4}\right) \times 10^8} = \frac{4}{3}$

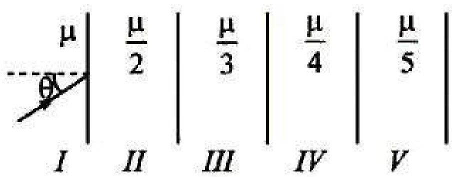
Shift produced,
$$S = t\left(1-\frac{1}{\left(\frac{\mu_1}{\mu_2}\right)}\right)$$
$$= t\left[1-\frac{\mu_2}{\mu_1}\right]$$
$$= 18\left[1-\frac{\left(\frac{4}{3}\right)}{2}\right]$$
$$= 6 \text{ cm.}$$ **Ans.**

Illustration 16

A light beam is travelling from region I to V as shown. The RI of the regions are also shown in figure. Find angle of incidence θ for which the beam just misses entering the region V.

Short-cut solution :

$$\frac{\sin 90°}{\sin} = \frac{\mu}{\left(\frac{\mu}{5}\right)}$$

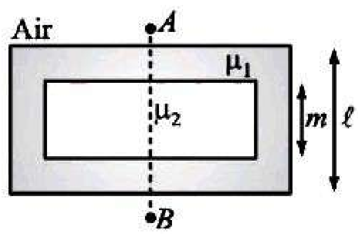


or $$\sin\theta = \frac{1}{5} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{5}\right).$$ **Ans.**

Illustration 17

In a thick slab of thickness 'ℓ' and RI μ_1 a cubical cavity of thickness 'm' is carved as shown in figure and is filled with a liquid of RI μ_2 ($\mu_1 > \mu_2$). Find the ratio of ℓ/m, so that shift produced by this slab is zero when observer A observes an object B with paraxial rays.

Short-cut solution :



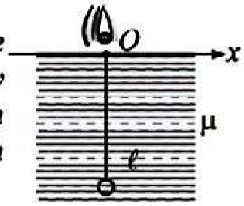
$$S_1 + S_2 = 0$$

or $$(\ell - m)\left(1-\frac{1}{\mu_1}\right)+m\left(1-\frac{1}{\mu_2}\right) = 0$$

$$\Rightarrow \frac{\ell}{m} = \frac{(\mu_1-\mu_2)}{\mu_2(\mu_1-1)}.$$ **Ans.**

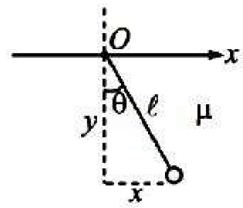
Illustration 18

A pendulum of length ℓ is free to oscillate in vertical plane above 'O'. An observer is viewing the bob of the pendulum directly from above. The pendulum is performing small oscillations in water (RI, μ) about its equilibrium position. Find the equation of trajectory of bob as seen by observer 'O'.



Short-cut solution :

At any instant the pendulum is as shown. The x-coordinate of the bob as seen by eye, $x = \ell \sin\theta$,



or $\quad \sin\theta = \dfrac{x}{\ell}$ ...(i)

y-coordinate, $\quad y = \dfrac{-\ell\cos\theta}{\mu}$

or $\quad \cos\theta = \dfrac{-y}{\left(\dfrac{\ell}{\mu}\right)}$ ...(ii)

Now squaring equations (i) and (ii), we have

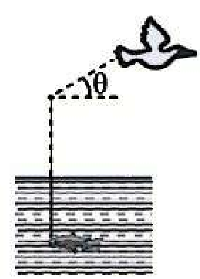
$$\frac{x^2}{\ell^2} + \frac{y^2}{\left(\dfrac{\ell}{\mu}\right)^2} = \sin^2\theta + \cos^2\theta$$

$$= 1.$$ ***Ans.***

Illustration 19

A bird is flying up with velocity 10 m/s at an angle $\sin^{-1}\left(\dfrac{3}{5}\right)$ ***with the horizontal. A fish in a pond looks at that bird when it is vertically above the fish. Find apparent velocity of bird as seen by fish.*** $\left(a\mu_w = \dfrac{4}{3}\right)$

Short-cut solution :



$$v_x(\text{apparent}) = v_x(\text{real})$$
$$= 10\cos\theta = 10 \times \frac{4}{5} = 8 \text{ m/s}$$

and
$$v_y(\text{apparent}) = [v_y(\text{real})]\mu = \mu v \sin\theta$$
$$= \frac{4}{3} \times 10 \times \frac{3}{5} = 8 \text{ m/s}.$$

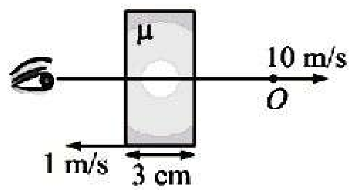
Therefore velocity of bird as seen by fish

$$\vec{v} = \left(8\hat{i} + 8\hat{j}\right) \text{m/s}.$$

Apparent angle, $\theta = \tan^{-1}\left(\frac{8}{8}\right) = \tan^{-1}(1) = 45°.$ ***Ans.***

Illustration 20

In the figure shown a slab of thickness 3 cm RI $\frac{3}{2}$ is moved towards a stationary observer with velocity 1 m/s. A point object 'O' moves towards right with a velocity of 10 m/s. Find apparent velocity of image as seen by observer, if both observer and object are in air.



Short-cut solution :

As object and observer both are in air, so there is no apparent change in position of image of '*O*' due to the motion of slab. Therefore velocity of image will appear 10 m/s. ***Ans.***

Vector form of Snell's law

$$\frac{\mu_2}{\mu_1} = \frac{\sin i°}{\sin r}$$

or $\mu_1 \sin i = \mu_2 \sin r$ ...(i)

$$|\hat{I} \times \hat{n}| = 1 \times 1 \times \sin(\pi - i) = \sin i$$

and $|\hat{R} \times \hat{n}| = 1 \times 1 \times \sin(\pi - r) = \sin r$

From equation (i), we have

$$\mu_1(\hat{I} \times \hat{n}) = \mu_2(\hat{R} \times \hat{n}).$$

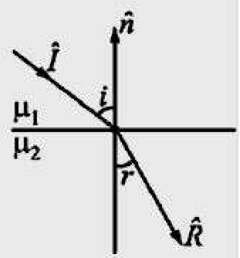
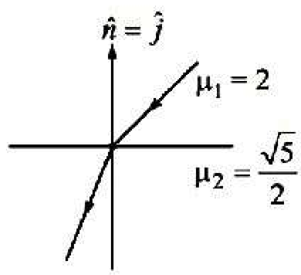


Illustration 21

A ray of light moving along the unit vector $(-\hat{i} - 2\hat{j})$ undergoes refraction at an interface of two media, which is xz-plane. The RI for y > 0 is 2 while for y < 0, it is $\frac{\sqrt{5}}{2}$. Find unit vector along the refractive ray.

Short-cut solution :



Using, $\mu_1(\hat{I} \times \hat{n}) = \mu_2(\hat{R} \times \hat{n})$

or $$2\left[\frac{(-\hat{i} - 2\hat{j})}{\sqrt{5}} \times (\hat{j})\right] = \frac{\sqrt{5}}{2}\left[\frac{(-\hat{i} - y\hat{j})}{\sqrt{1+y^2}} \times (\hat{j})\right]$$

Here *y* is the length of unit vector perpendicular to the refracting surface.

or $$\frac{2}{\sqrt{5}} = \frac{\sqrt{5}}{2}\left(\frac{1}{\sqrt{1+y^2}}\right)$$

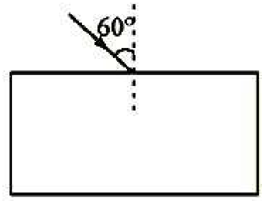
or $$\sqrt{1+y^2} = \frac{5}{4} \quad \Rightarrow \quad y = \frac{3}{4}$$

Therefore, $$\hat{R} = \frac{-\hat{i} - y\hat{j}}{\sqrt{1+y^2}} = \frac{-\hat{i} - \frac{3}{4}\hat{j}}{\sqrt{1+\left(\frac{3}{4}\right)^2}}$$

$$= \left(\frac{-4\hat{i} - 3\hat{j}}{5}\right).$$ *Ans.*

Illustration 22

A ray of light enters into a thick glass slab from air as shown in figure. The refractive index of the glass slab varies with depth x from the topmost surface of the slab as $\mu = \left(\sqrt{3} - \frac{x}{\sqrt{3}}\right)$, where x is in meter. Slab is of sufficiently large thickness. Find the maximum depth reached by the ray inside the slab :



***(a)** 1.5 m* ***(b)** 2.5 m* ***(c)** 1.25 m* ***(d)** 1.75 m*

Short-cut solution :

If r is the angle of refraction, then

$$\frac{\sin 60°}{\sin r} = \mu$$

For maximum depth, $r \to 90°$ and so

$$\frac{\sin 60°}{\sin 90°} = \left(\sqrt{3} - \frac{x}{\sqrt{3}}\right)$$

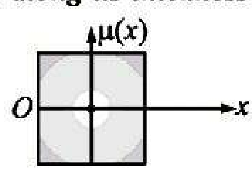
or $$\frac{\sqrt{3}}{2} = \sqrt{3} - \frac{x}{\sqrt{3}}$$

or $$x = 1.5 \text{ m.}$$ *Ans. (a)*

Illustration 23

A composite glass slab is manufactured so that its RI varies along its thickness according to the relation.

$$\mu(x) = 1 + \frac{\alpha x}{\ell},$$



where t is the thickness of the slab.

The optical path introduced by the slab when it is placed in the path of light passing normally through it, is given by

(a) $(1+\alpha)t$ *(b)* $\left(1+\frac{\alpha}{2}\right)t$ *(c)* $\frac{t}{1+\alpha}$ *(d)* $\frac{t}{\alpha}\ell n(t+\alpha)$

Short-cut solution :

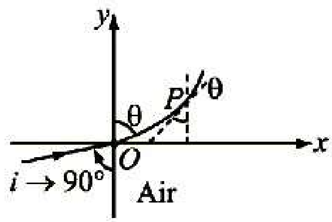
$$\text{Optical path} = \int_0^t \mu(dx) = \int_0^t \left(1+\frac{\alpha x}{\ell}\right)dx$$

$$= \left|x+\frac{\alpha x^2}{2t}\right|_0^t = t\left(1+\frac{\alpha}{2}\right).$$ ***Ans. (b)***

Illustration 24

A ray of light is incident on a glass slab at grazing incidence. The RI of the material of the slab is given by $\mu = \sqrt{1+y}$***. If the thickness of the slab is 'd', determine the equation of the trajectory of the ray inside the slab and the coordinates of the point where the ray exists from the slab. Take the origin to the point of entry of the ray.***

Short-cut solution :

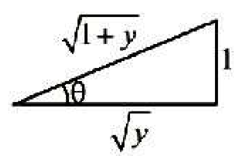


At origin, $$\frac{\sin 90°}{\sin\theta} = \mu$$

or $$\sin\theta = \frac{1}{\mu}$$

$$= \frac{1}{\sqrt{1+y}} \quad \text{...(i)}$$

Take a point P close to O, the slope



$$\frac{dy}{dx} = \tan(90° - \theta)$$

or $$\frac{dy}{dx} = \cot\theta \quad \text{...(ii)}$$

From above equations, we get

$$\frac{dy}{dx} = \sqrt{y}$$

or $$\int_0^y y^{-1/2}dy = \int_0^x dx$$

or $$2y^{1/2} = x$$

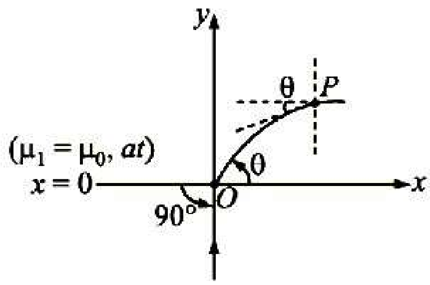
or $$y = \frac{x^2}{4}.$$

The ray will exit where $y = d \Rightarrow x = 2\sqrt{d}$. ***Ans.***

Illustration 25

The refractive index of an anisotropic medium varies as $\mu = \mu_0\sqrt{1+x}$, where $0 \le x \le a$. A ray of light is incident at the origin just along y-axis (shown in figure). Find the equation of ray in the medium.

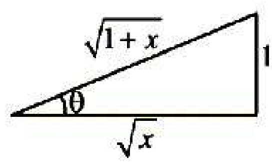
Short-cut solution :



At origin
$$\frac{\sin 90^\circ}{\sin\theta} = \frac{\mu}{\mu_1}$$

or
$$\sin\theta = \frac{\mu_1}{\mu}$$
$$= \frac{\mu_0}{\mu_0\sqrt{1+x}}$$
$$= \frac{1}{\sqrt{1+x}}.$$

Take a point '*P*' close to '*O*', the slope



$$\frac{dy}{dx} = \tan\theta$$
$$= \frac{1}{\sqrt{x}}$$

or
$$\int_0^x x^{-1/2}dx = \int_0^y dy$$

or
$$2x^{1/2} = y$$

$\therefore$
$$y = 2\sqrt{x}.$$
Ans.

***TOPIC 20.4**: Total Internal Reflection, Dispersion Produced by a Prism, Minimum Deviation Condition.*

Review of Formulae

1. Total internal reflection (TIR)

(i) TIR occurs when light ray passes from denser to rarer medium.

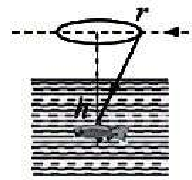
(ii) Angle of incidence in denser medium must be greater than critical angle. Critical angle is given by

$$\sin C = \frac{1}{\mu}.$$

Critical angle for water to air is 49° and for glass to air is 42°.

(iii) A fish inside water at a depth h can see outside world in horizontal circle of radius r, where

$$r = \frac{h}{\sqrt{\mu^2 - 1}}.$$



2. Dispersion produced by prism

The refractive index of material of a prism depends on wavelength of light. It approximately is given by Cauchy's equation as :

$$\mu = A + \frac{B}{\lambda^2}.$$

Here A and B are constants.

Dispersive power, $$\omega = \frac{\text{angular dispersion}}{\text{mean deviation}} = \frac{\delta_v - \delta_R}{\delta_y}$$

or $$\omega = \frac{\mu_v - \mu_R}{\mu_y - 1}$$

3. Combination of prisms

(i) **Dispersion without deviation :**

For two small angled prisms, we have

$$A' = -\frac{(\mu_y - 1)}{(\mu'_y - 1)} A$$

The total dispersion

$$D = (\mu_v - \mu_R)A + (\mu'_v - \mu'_R)A'$$

(ii) **Deviation with dispersion :**

For two small angled prisms, we have

$$A' = -\frac{(\mu_v - \mu_R)}{(\mu'_v - \mu'_R)} A$$

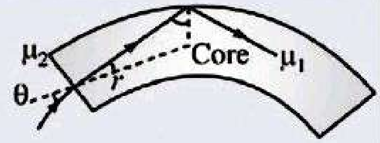
The total deviation

$$\delta = (\mu_y - 1)A + (\mu'_y - 1)A'$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. As $\sin C = \frac{1}{{}_r\mu_d} = \frac{\mu_r}{\mu_d}$, so greater is μ_d, smaller is critical angle and more chances of TIR.

2. An optical fibre of core material μ_1 and surrounded all over by medium μ_2 ($\mu_2 < \mu_1$). The maximum angle of incidence at one end of the fibre:



$$\frac{\sin\theta}{\sin r} = \frac{\mu_1}{\mu_2}$$

or $$\sin\theta = \frac{\mu_1}{\mu_2}\sin r$$

$$= \frac{\mu_1}{\mu_2}\sqrt{1-\cos^2 r}$$

At the boundary of the fibre

$$(90° - r) > C$$

or $$\sin(90° - r) > \sin C$$

or $$\cos r > \frac{1}{\left(\frac{\mu_1}{\mu_2}\right)}$$

or $$(\cos r)_{min} = \frac{\mu_2}{\mu_1}$$

$\therefore$ $$\sin\theta_{max} = \frac{\mu_1}{\mu_2}\sqrt{1-(\cos^2 r)_{min}}$$

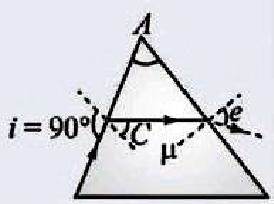
$$= \frac{\mu_1}{\mu_2}\sqrt{1-\left(\frac{\mu_2}{\mu_1}\right)^2}$$

$$= \sqrt{\left(\frac{\mu_1}{\mu_2}\right)^2 - 1}.$$

If light enters from air into the fibre, then $\mu_2 = 1$

$\therefore$ $$\sin\theta_{max} = \sqrt{\mu_1^2 - \mu_2^2}.$$

3. Condition of maximum deviation in prism



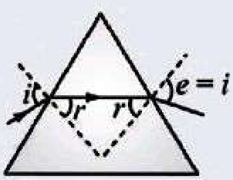
$$\delta_{max} = (90° + e) - A$$

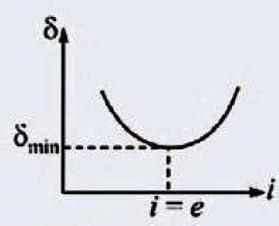
where, $$e = \sin^{-1}[\mu\sin(A - C)]$$

4. Condition of minimum deviation

$$i = e$$

and $$\delta_{min} = 2i - A$$





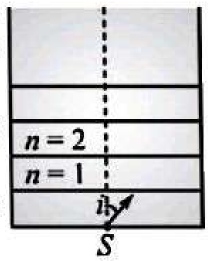
$$\mu = \frac{\sin\left(\frac{A+\delta_m}{2}\right)}{\sin\frac{A}{2}}$$

At minimum deviation, ray inside prism becomes parallel to the base of the prism.

5. For thin prism, $\delta = (\mu - 1)A$.
6. Condition of no emergence of light, $A > 2C$.

Illustration 26

A point source S is placed at the bottom of different layers as shown in figure. The RI of bottom most layer is μ_0. The RI of any other upper layer is $\mu = \mu_0 - \frac{\mu_0}{(4n-18)}$; n = 1, 2, ... A ray of light with angle i = 30° starts from the source S. Find the value of n so that TIR takes place at the upper surface of a layer.



Short-cut solution :

Using Snell's law

$$\frac{\mu}{\mu_0} = \frac{\sin 90°}{\sin 30°}$$

or $$\frac{\left[\mu_0 - \frac{\mu_0}{4n-18}\right]}{\mu_0} = \frac{1}{\left(\frac{1}{2}\right)}$$

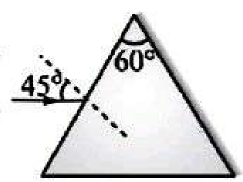
or $$1 - \frac{1}{4n-18} = 2$$

or $$n = \frac{17}{4} = 4.25$$

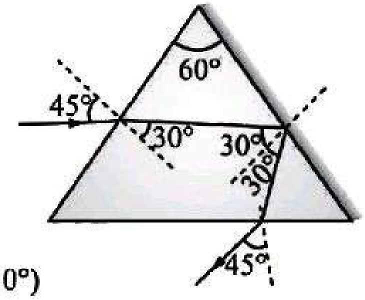
Therefore, TIR will take place in $n = 4$ layer.

Illustration 27

An equilateral prism is made of a transparent material of RI $\sqrt{2}$. A ray of light AB is incident at 45° as shown. Find the net deviation in the path of ray when it comes out of prism.



Short-cut solution :



$$\mu = \sqrt{2} = \frac{\sin 45°}{\sin r}$$

$$\Rightarrow \quad \sin r = \frac{1}{2} \text{ or } r = 30°$$

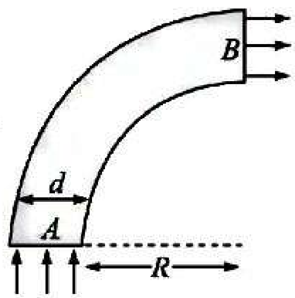
$$\text{Deviation, } \delta = (45° - 30°) + (180° - 60°) + (45° - 30°)$$

$$= 150° \text{ clockwise.}$$

Ans.

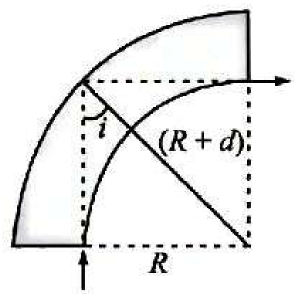
Illustration 28

A rod of glass ($\mu = 1.5$) and of square cross section is bent into the shape shown in the figure. A parallel beam of light falls on the plane flat surface A as shown in the figure. If d is the width of a side and R is the radius of circular arc then for what maximum value of $\frac{d}{R}$ light entering the glass slab through surface A emerges from the glass through B



(a) 1.5 *(b)* 0.5 *(c)* 1.3 *(d)* none of these

Short-cut solution :



For TIR, $i > C$

or $\sin i > \sin C$

or $\frac{R}{R+d} > \frac{1}{1.5}$

or $d < 0.5\,R$

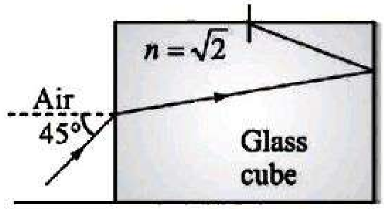
or $\frac{d}{R} < 0.5$

$$\therefore \quad \left(\frac{d}{R}\right)_{max} = 0.5.$$

Ans. (b)

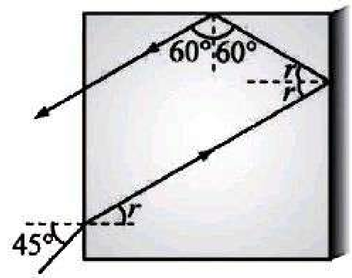
Illustration 29

Right face of a glass cube is silvered as shown. A ray of light is incident on left face of the cube as shown. The deviation of the ray when it comes out of the glass cube is



(a) 0° *(b)* 90° *(c)* 180° *(d)* 270°

Short-cut solution :



From Snell's law,

$$\frac{\sin 45°}{\sin r} = \sqrt{2}$$

or $\sin r = \frac{1}{2}$

$\therefore$ $r = 30°$

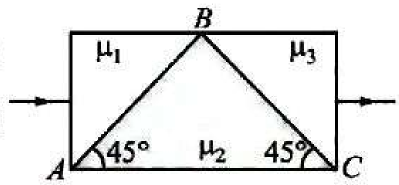
Critical angle, $\sin C = \frac{1}{\mu} = \frac{1}{\sqrt{2}}$

$\therefore$ $C = 45°$

So ray comes out antiparallel to the incident ray. ***Ans. (c)***

Illustration 30

A rectangular block is composed of three different glass prisms with refractive indices μ_1, μ_2 and μ_3 as shown in figure. A ray of light incident normal to the left face emerges normal to the right face. Find relation between μ_1, μ_2 and μ_3.



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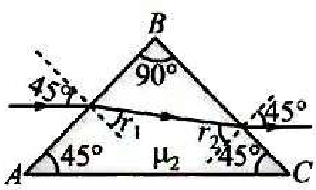
Short-cut solution :

For face AB, $\frac{\sin 45°}{\sin r_1} = \frac{\mu_2}{\mu_1}$

or $\sin r_1 = \frac{\mu_1}{\mu_2}\sin 45° = \frac{\mu_1}{\sqrt{2}\mu_2}$ ...(i)

On face BC,



$$\frac{\sin 45°}{\sin r_2} = \frac{\sin 45°}{\sin(90° - r_1)} = \frac{\mu_2}{\mu_3}$$

or $\cos r_1 = \frac{\mu_3}{\mu_2}\sin 45° = \frac{\mu_3}{\sqrt{2}\mu_2}$ ...(ii)

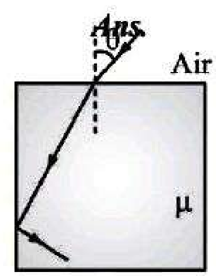
Now squaring (i) and (ii), we get

$$\frac{\mu_1^2}{2\mu_2^2} + \frac{\mu_3^2}{2\mu_2^2} = 1$$

or $\mu_1^2 + \mu_3^2 = 2\mu_2^2$. **Ans.**

Illustration 31

A ray of light is incident at the top surface of a transparent slab. Find minimum value of its RI, so that TIR will occur at its vertical face.



Short-cut solution :

Using, $\sin\theta_{max} = \sqrt{\mu_1^2 - \mu_2^2}$

Here, $\mu_2 = 1$

$\therefore$ $\sin\theta = \sqrt{\mu^2 - 1^2} \Rightarrow \mu = \sqrt{1+\sin^2\theta}$. ***Ans.***

TOPIC 20.5: *Refraction Through Spherical Surface, Magnification, Lens Formula, Lens Maker's Formula, Power of Lens, Combination of Lenses.*

Review of Formulae

1. **Refraction formula through single spherical surface**

$$\frac{{}_1\mu_2}{v} - \frac{1}{u} = \frac{{}_1\mu_2 - 1}{R}$$

or

$$\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}.$$

2. (i) **Lateral magnification**

$$m = \frac{I}{O} = \frac{\mu_1 v}{\mu_2 u}.$$

(ii) **Longitudinal magnification**

$$m_L = \frac{\mu_1}{\mu_2}\frac{v^2}{u^2}.$$

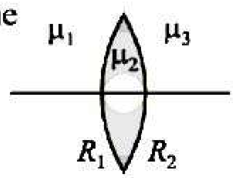
3. **Lens formula**

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}.$$

Lens maker's formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

4. If μ_1 and μ_3 are the refractive indexes on both sides of the lens of material of refractive index μ_2, then



$$\frac{\mu_3}{v} - \frac{\mu_1}{u} = \frac{(\mu_2 - \mu_1)}{R_1} + \frac{(\mu_3 - \mu_2)}{R_2}.$$

5. **Lateral magnification,**

$$m = \frac{I}{O} = \frac{v}{u} = \frac{f}{u - f}.$$

6. **Velocity of image** $v_i = \frac{v^2}{u^2} v_o$.

7. **Minimum distance between object and its real image in convex lens**

$$D_{min} = 4f$$

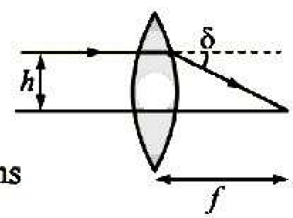
and $$D_{max} = \infty.$$

8. **Focal length of convex lens by displacement method**

$$f = \frac{D^2 - x^2}{4D}.$$

9. **Deviation produced by a lens**

$$\delta = \frac{h}{f}.$$



where h is the height of incident of ray on the lens of focal length f.

10. **Power of a lens** $P = \frac{1}{f}$.

11. **Combined focal length**

(i) When lenses are placed in contact

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} + ...,$$

and $$P = P_1 + P_2 + ...$$

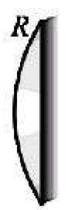
(ii) If two lenses of focal lengths f_1 and f_2 are placed at a separation of d, the equivalent focal length

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}.$$

and $$p = p_1 + p_2 - dp_1p_2.$$

12. When one face of a lens is silvered, it behaves as a concave mirror. If f_e is the effective focal length of the lens, then

$$\frac{1}{f_e} = \frac{2}{f_\ell} + \frac{1}{f_m}.$$



(i) Plano-convex lens silvered at plane surface, then

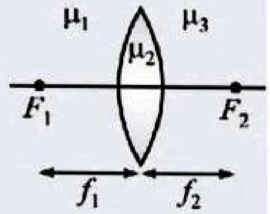
$$f_e = \frac{R}{2(\mu - 1)}.$$

(ii) Plano-convex lens silvered at convex surface

$$f = \frac{R}{2\mu}$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. First principal focus



$$\frac{1}{f_1} = -\frac{1}{\mu_1}\left(\frac{\mu_2-\mu_1}{R_1}+\frac{\mu_3-\mu_2}{R_2}\right)$$

Second principal focus

$$\frac{1}{f_2} = \frac{1}{\mu_3}\left(\frac{\mu_2-\mu_1}{R_1}+\frac{\mu_3-\mu_2}{R_2}\right)$$

$\therefore$
$$\frac{f_1}{f_2} = \frac{-\mu_1}{\mu_2}$$

2. Focal length ($\mu_1 = \mu_3 = 1$ and $\mu_2 = \mu$)

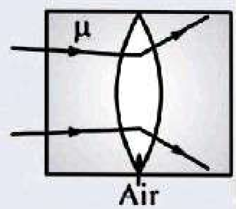
(i) Convex lens : $f = \dfrac{R}{2(\mu-1)}$

(ii) Concave lens : $f = -\dfrac{R}{2(\mu-1)}$.

3. If distance between object and real image is D and magnification is m, then

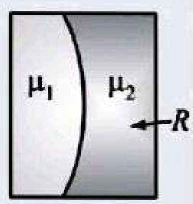
$$f = \frac{mD}{(1+m)^2}.$$

4. Focal length of air lens in glass slab



$$f = -\frac{\mu R}{2(\mu-1)}.$$

5. Focal length of a slab with two materials,

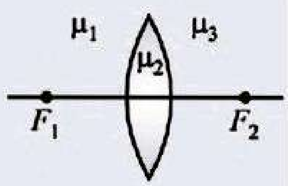


$$f = \frac{R}{\mu_1-\mu_2}.$$

If $\mu_1 > \mu_2, f \Rightarrow$ +ve

If $\mu_1 < \mu_2, f \Rightarrow$ –ve.

6. If f_0 is the focal length of the lens in air, and if μ_1 and μ_3 are the medium on its both sides, then



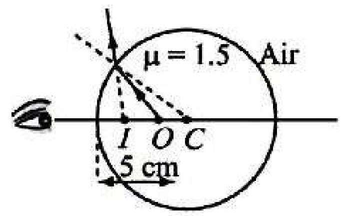
$$f_1 = f_0\left[\frac{2\mu_1(\mu_2-1)}{2\mu_2-\mu_1-\mu_3}\right].$$

and
$$f_2 = f_0\left[\frac{2\mu_3(\mu_2-1)}{2\mu_2-\mu_1-\mu_3}\right].$$

Illustration 32

There is a transparent sphere of radius 10 cm and RI 1.5. A point object is placed at a distance 5 cm from the centre of the sphere. Find the position of final image, when it seen from (i) left of the sphere (ii) right of the sphere.

Short-cut solution :



(i) Using, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$

Here $\mu_1 = 1.5$ and $\mu_2 = 1$

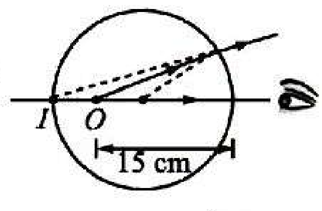
$u = -5$ cm, $R = -10$ cm

So $\frac{1}{v} - \frac{1.5}{-5} = \frac{1-1.5}{-10}$

or $v = -4$ cm. ***Ans.***

(ii) Here, $\mu_1 = 1.5$ and $\mu_2 = 1$

$\mu = -15$ cm, $R = -10$ cm

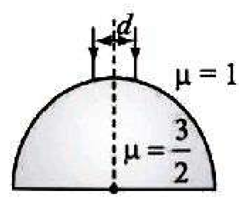


$\therefore \quad \frac{1}{-} \; \frac{1.5}{15} = \frac{1-1.5}{-10}$

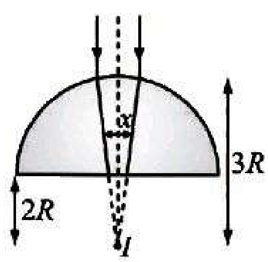
or $v = -20$ cm. ***Ans.***

Illustration 33

A beam of diameter d is incident on a glass hemisphere as shown. If the radius of curvature of the hemisphere is very large in comparision to d, then find the diameter of the beam at the base of the hemisphere.



Short-cut solution :



Using, $\frac{\mu_2}{v} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R}$

$$\frac{\frac{3}{2}}{v} - \frac{1}{\infty} = \frac{\frac{3}{2} - 1}{R}$$

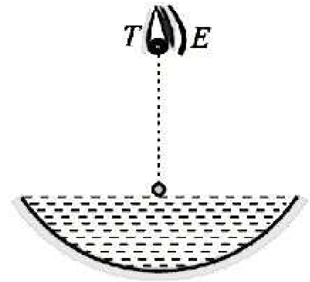
or $v = 3R$

From geometry,

$$\frac{x}{2R} = \frac{d}{3R} \Rightarrow x = \frac{2d}{3}.$$ ***Ans.***

Illustration 34

A concave mirror of radius 40 cm lies on a horizontal table and water is filled in it upto a height of 5.0 cm. A small dust particle P floats on the surface of water. Particle P lies vertically above the point of contact of the mirror with the table. Locate the image of the dust particle as seen from a point directly above it. The refractive index of water is 4/3.

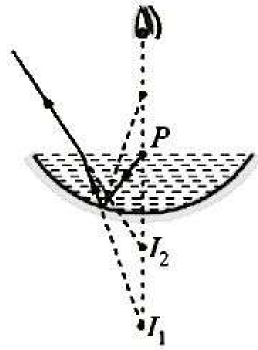


Short-cut solution :

For concave mirror, $u = -5$ cm, $f = -20$ m

By mirror formula, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$, we have

$$\frac{1}{v}+\frac{1}{-5} = \frac{1}{-20}$$



$$\therefore \qquad v = +\frac{20}{3} \text{ cm}$$

Thus $$OI' = \frac{20}{3}\text{cm}$$

$$= 11.67 \text{ cm}$$

The distance $$PI' = \frac{20}{3}+5$$

$$= 11.67 \text{ cm.}$$

Now for the refraction through water surface, we have

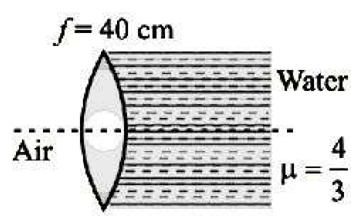
$$\mu = \frac{RD}{AD}$$

$$\therefore \qquad AD = \frac{RD}{\mu} = \frac{11.67}{4/3}$$

or $$PI_2 = 8.75 \text{ cm.}$$ ***Ans.***

Illustration 35

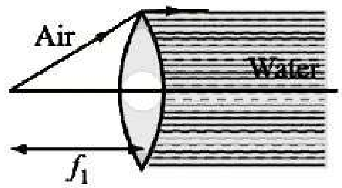
An equiconvex lens of glass ($\mu = 1.5$) of focal length 40 cm is placed such that on left side of it is air and that on the right side is water $\left(\mu = \frac{4}{3}\right)$. Determine the first and second focal length of the lens.



Short-cut solution :

Using, $f_1 = f_0\left[\dfrac{2\mu_1(\mu_2-1)}{2\mu_2-\mu_1-\mu_3}\right]$

Here, $\mu_1 = 1,\ \mu_2 = 1.5$ and $\mu_3 = \dfrac{4}{3}$

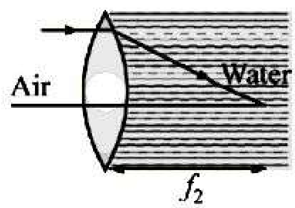


$\therefore$ $f_1 = 40\left[\dfrac{2\times1(1.5-1)}{2\times1.5-1-\dfrac{4}{3}}\right]$

$= 60$ cm. ***Ans.***

and $f_2 = f_0\left[\dfrac{2\mu_3(\mu_2-1)}{2\mu_2-\mu_1-\mu_3}\right]$

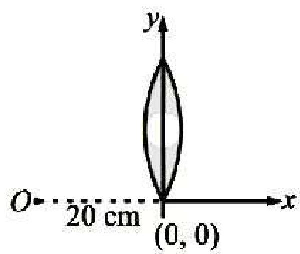
$= 40\left[\dfrac{2\times\dfrac{4}{3}(1.5-1)}{2\times1.5-1-\dfrac{4}{3}}\right]$



$= 80$ cm. ***Ans.***

Illustration 36

A point object 'O' is placed along the edge of the convex lens of focal length 10 cm, at a distance 20 cm. Find the coordinates of the image from origin of axes, if diameter of lens is 5 cm.



Short-cut solution :

As object is at distance $2f$ from the lens, so its image will also be at $2f = 2 \times 10 = 20$ cm. Therefore coordinates of image are (20 cm, 5 cm). ***Ans.***

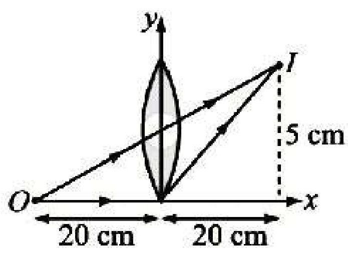
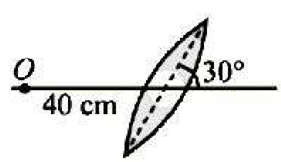


Illustration 37

A convex lens of focal length 10 cm is placed at an 30° with the horizontal axis. An object is placed at distance of 40 cm from the optical centre of the lens. Find distance of its image.



Short-cut solution :

The distance of object from principal axis of the lens $u = -40 \cos 60° = -20$ cm. So, image distance will also be 20 cm *i.e.*, $PI' = 20$ cm. The distance

$$PI = \frac{20}{\cos 60°} = 40 \text{ cm.} \qquad \textbf{Ans.}$$

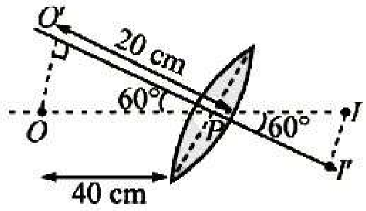
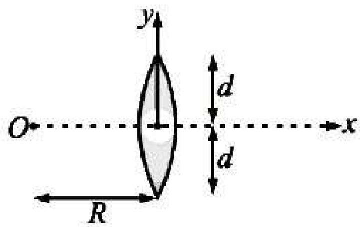


Illustration 38

A biconvex thin lens of radius of curvature R is made up of variable refractive index $\mu = 2\left(1+\frac{|y|}{d}\right)$. ***Assume 2d << R. There are infinite images of point object 'O' which is placed at a distance R on the principal axis from the lens as shown in figure. Find the distance of spread of images.***



Short-cut solution :

For $y = 0$, $\mu = 2\left(1+\frac{0}{d}\right) = 2$. Also $f = \frac{R}{2(\mu-1)}$.

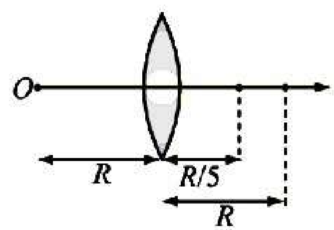
$$\therefore \quad \frac{1}{v} - \frac{1}{-R} = \frac{2(2-1)}{R}$$

or $\quad v = R.$

For $\quad y = d, \mu = 2\left(1+\frac{d}{d}\right) = 4$

$$\therefore \quad \frac{1}{v'} - \frac{1}{-R} = \frac{2(4-1)}{R}$$

or $\quad v' = \frac{R}{5}$



$$\therefore \quad \text{Spreading} = v - v' = R - \frac{R}{5} = \frac{4R}{5}. \qquad \textbf{Ans.}$$

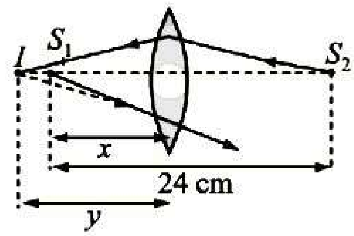
Illustration 39

Two point sources S_1 and S_2 are 24 cm apart. Where should a convex lens of focal length 9 cm can be placed in between them so that the images of both sources are formed at the same place?

Short-cut solution :

One of the source has real image then other will be virtual. So

$$\frac{1}{y}-\frac{1}{-x} = \frac{1}{9}$$



and
$$\frac{1}{-y}-\frac{1}{-x} = \frac{1}{9}.$$

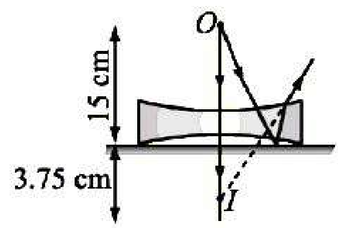
On solving above equations, we get

$$x = 6 \text{ cm.}$$ ***Ans.***

Illustration 40

A concave lens of focal length 10 cm is placed on a plane mirror. A point object is placed 15 cm above the lens. Find position of the final image.

Short-cut solution :



Using, $$\frac{1}{f_e} = \frac{2}{f}+\frac{1}{f_m}$$

$$= \frac{2}{-10}+\frac{1}{\infty}$$

or $$f_e = -5 \text{ cm}$$

This system like a concave mirror, so

$$f = -(-5) = 5 \text{ cm}$$

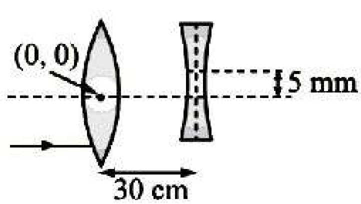
Using, $$\frac{1}{v}+\frac{1}{u} = \frac{1}{f}$$

or $$\frac{1}{v}+\frac{1}{-15} = \frac{1}{5}$$

∴ $$v = 3.75 \text{ cm.}$$ ***Ans.***

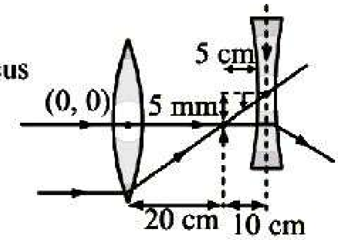
Illustration 41

If the optic axis of convex and concave lenses are separated by a distance 5 mm as shown in the figure. Find the coordinates of the final image formed by the combination if parallel beam of light is incident on lens. Origin is at the optical centre of convex lens.



Short-cut solution :

Parallel incident rays on convex lens will intersect at the focus of the lens. So for concave lens



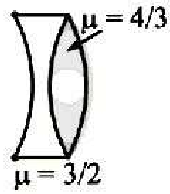
$$\frac{1}{v}-\frac{1}{-10} = \frac{1}{-10}$$

or $$v = -5 \text{ cm}$$

Also $$\frac{I}{O} = \frac{v}{u} \quad \text{or} \quad I = \frac{v}{u}\times O = \left(\frac{-5}{-10}\right)\times 5 = 2.5 \text{ mm.}$$

Coordinates of final image are (25 cm, 2.5 cm). ***Ans.***

Illustration 42



In the figure shown the radius of curvature of the left and right surface of the concave lens are 10 cm and 15 cm respectively. The radius of curvature of the mirror is 15 cm. Find focal length of the system.

Short-cut solution :

Using, $$\frac{1}{f} = (\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

For glass lens, $$\frac{1}{f_1} = \left(\frac{3}{2}-1\right)\left(\frac{1}{-10}-\frac{1}{15}\right) = -\frac{1}{12}$$

and for water lens, $$\frac{1}{f_2} = \left(\frac{4}{3}-1\right)\left(\frac{1}{15}-\frac{1}{-15}\right) = \frac{2}{45}$$

Now using, $$\frac{1}{f_e} = \frac{2}{f_1}+\frac{2}{f_2}+\frac{1}{f_m}$$

$$= 2\left(\frac{-1}{12}\right)+2\left(\frac{2}{45}\right)+\left(\frac{1}{+\frac{15}{2}}\right)$$

$\Rightarrow$ $$f_e = 18 \text{ cm.}$$ ***Ans.***

As $f = -f_e = -18$ cm, so system behaves like concave mirror.

TOPIC 20.6: *Simple and Compound Microscope and Astronomical Telescope.*

Review of Formulae

1. **Simple microscope**

   Angular magnification, $M = 1+\dfrac{D}{f_e}$

   Here $D = 25$ cm.

2. **Compound microscope**

   (i) When final image is formed at near point

   $$M = -\frac{v_o}{u_o}\left(1+\frac{D}{f_e}\right)$$

   Length of the microscope

   $$L = |v_o| + |u_e|$$

   (ii) When final image is formed at infinity

   $$M = -\frac{v_0}{u_0}\frac{D}{f_e}$$

   Length of the microscope $L = |v_0| + f_e$.

3. **Astronomical telescope**

   (i) When final image is formed at near point

   $$M = -\frac{f_o}{f_e}\left(1+\frac{f_e}{D}\right).$$

   Length of the telescope,

   $$L = f_o + |u_e|$$

   (ii) When final image is formed at infinity

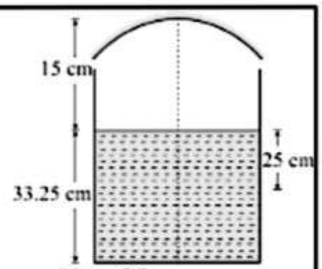
   $$M = -\frac{f_o}{f_e}$$

   Length of the telescope,

   $$L = f_o + f_e.$$

Video Solution

Q. *A container is filled with water ($\mu = 1.33$) upto a height of 33.25 cm. A concave mirror is placed 15 cm above the water level and the image of an object placed at the bottom is formed 25 cm below the water level. The focal length of the mirror is*



(a) *10 cm* *(b)* *15 cm* *(c)* *20 cm* *(d)* *25 cm*

To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=JoNcrSgUUX4

Illustration 43

A person uses + 1.5 D glasses to have normal vision from 25 cm onwards. He uses a 20 D lens as a simple microscope to see an object. Find the maximum magnification power if he uses the microscope (a) together with his glass (b) without the glass.

Short-cut solution :

The focal length of the glasses (lens) used

$$f = \frac{100}{1.5}\text{ cm}$$

If y is the distance of near point, then

$$\frac{1}{y} - \frac{1}{-25} = \frac{1.5}{100}$$

or $y = -40$ cm

(a) The focal length of the lens of the microscope

$$f_e = \frac{1}{P} = \frac{100}{20} = 5\text{ cm}$$

The magnifying power of the microscope together with the glass

$$M = 1 + \frac{D}{f_e} = 1 + \frac{25}{5}$$

$$= 6X \qquad \textbf{Ans.}$$

(b) Without the glass, $D' = y = 40$ cm

$$\therefore \quad M' = 1 + \frac{D'}{f_e}$$

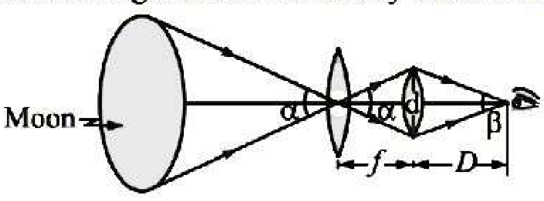
$$= 1 + \frac{40}{5} = 9X \qquad \textbf{Ans.}$$

Illustration 44

The image of the moon is focused by a converging lens of focal-length 50 cm on a plane screen. The image is seen by an unaided eye from a distance of 25 cm. Find the angular magnification achieved due to the converging lens.

Short-cut solution :

The ray diagram of the image of moon formed by a lens is shown in figure.



Suppose d is the diameter of the image of the moon. If α and β are the angle made by moon and its image respectively, then

$$\alpha = \frac{d}{f}$$

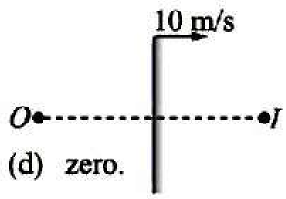
and $$\beta = \frac{d}{D}.$$

Angular magnification $$M = -\frac{\beta}{\alpha}$$

$$= -\left|\frac{f}{D}\right| = -\left|\frac{50}{25}\right| = -2. \qquad \textbf{\textit{Ans.}}$$

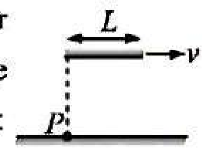
Concept Booster Exercise

1. An object is kept fixed in front of a plane mirror which is moved by 10 m/s away from the object, the velocity of the image



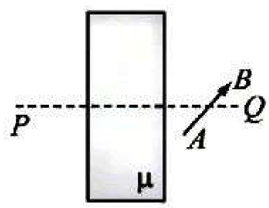
(a) $10\hat{i}$ m/s (b) $20\hat{i}$ m/s (c) $5\hat{i}$ m/s (d) zero.

2. A mirror of length L moves horizontally with a velocity v. The mirror is illuminated by a point source of light P placed on the ground. The rate at which the length of the light spot on the ground increases is :



(a) v (b) $2v$ (c) $3v$ (d) zero

3. In the figure shown an object AB makes small angle with the normal line PQ. The length of AB is ℓ. The RI of the slab is μ and surrounding medium is air. AB is seen with the help of paraxial rays, from the left side of the slab. The size of the image of AB is:



(a) ℓ (b) $\frac{\ell}{\mu}$ (c) $\mu\ell$ (d) $\ell\left(1-\frac{1}{\mu}\right)$

4. P is a point on the axis of a concave mirror. The image of P formed by the mirror coincides with P. A rectangular glass slab of thickness t and RI, μ is now introduced between P and the mirror. For the image of P coincide with P again, the mirror must be moved:

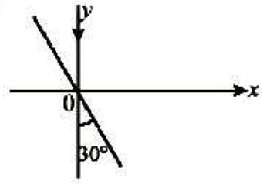
(a) towards P by $(\mu - 1)t$ (b) away from P by $(\mu - 1)t$

(c) towards P by $t\left(1-\frac{1}{\mu}\right)$ (d) away from P by $t\left(1-\frac{1}{\mu}\right)$

5. The width of man's face is 10 cm. The distance between the eyes of the man is 4 cm. Then the minimum width of plane mirror to see his full face is: **Numeric/Integer**

(a) 3 cm (b) 4 cm (c) 5 cm (d) 10 cm

6. A mirror is placed at an angle of 30° with respect to y-axis as shown in figure. A light rays travelling in the negative y-direction strikes the mirror. The direction of the reflected ray is given by the vector : *[KVPY-2017]*



(a) i (b) $i-\sqrt{3}j$

(c) $\sqrt{3}i-j$ (d) $i-2j$

7. Given magnetic field equation is $B=3\times10^{-8}\sin(\omega t+kx+\phi)\hat{j}$, then appropriate equation for electric field (E) will be: *[JEE Main 2020]*

(a) $20\times10^{-9}\sin(\omega t+kx+\phi)\hat{k}$ (b) $9\sin(\omega t+kx+\phi)\hat{k}$

(c) $16\times10^{-9}\sin(\omega t+kx+\phi)\hat{k}$ (d) $3\times10^{-9}\sin(\omega t+kx+\phi)\hat{k}$

8. A concave mirror is placed on a horizontal table, with its axis directly upwards. Let 'O' be the pole of the mirror and C its centre of curvature. A point object is placed at C. It has a real image also located at C. If the mirror is now filled with water, the image will be:

(a) real, and will remain at C
(b) real, located at a point between C and O
(c) virtual, and located at a point between C and O
(d) real, and located at a point between C and O

9. If relative permittivity and relative permeability of a medium are 3 and $\frac{4}{3}$ respectively. The critical angle for this medium is. *[JEE Main 2020]*

(a) 45° (b) 60° (c) 30° (d) 15°

10. Focal length of convex lens in air is 16 cm ($\mu_{glass}=1.5$). Now the lens is submerged in liquid of refractive index 1.42. Find the ratio of focal length in medium to focal length in air has closest value *[JEE Main 2020]*

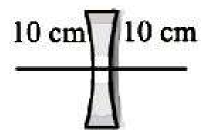
(a) 9 (b) 17 (c) 1 (d) 5

11. A thin concavo-convex lens has two surfaces of radii of curvature R and $2R$. The material of the lens has a RI, μ. When kept in air, the focal length of the lens:

(a) $\frac{R}{\mu}$ (b) $\frac{2R}{\mu-1}$ (c) $\frac{R}{\mu-1}$ (d) none of these

12. A plano-convex lens of radius of curvature 30 cm made of refractive index 1.5 is kept in air. Find its focal length (in cm). **Numeric/Integer** *[JEE Main 2020]*

13. A concave lens of ROC 10 cm each and made of material of RI 1.5 is silvered at its one face. Its focal length **Numeric/Integer**



(a) 2.5 cm (b) 5.0 cm
(c) 10.0 cm (d) 20.0 cm

14. An object is placed in front of a thin convex lens of focal length 30 cm and a plane mirror is placed 15 cm behind the lens. If the final image of the object coincides with the object, the distance of the object from the lens is: **Numeric/Integer**

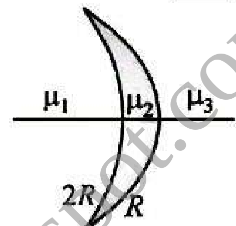
(a) 15 cm (b) 25 cm (c) 30 cm (d) 60 cm

15. Two lenses of focal lengths $f_1 = 10$ cm and $f_2 = 20$ cm are kept as shown. The power of combination will be **Numeric/Integer**

(a) −10 D (b) 5 D

(c) 0 (d) 10 D

16. Figure shows a concavo-convex lens of *RI* μ_2. What is the condition on the refractive indices so that the lens is diverging?



(a) $2\mu_3 < (\mu_1 + \mu_2)$

(b) $2\mu_3 > (\mu_1 + \mu_2)$

(c) $\mu_3 > 2(\mu_1 - \mu_2)$

(d) none of these

17. Two plane mirrors are combined to each other, one of them in *yz*-plane and other in *xz*-plane. A ray of light is incident along $\left(\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}\right)$ on the first mirror. Find the unit vector of reflected ray:

(a) $\frac{-\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$ (b) $\frac{-\hat{i}}{\sqrt{3}} - \frac{\hat{j}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$

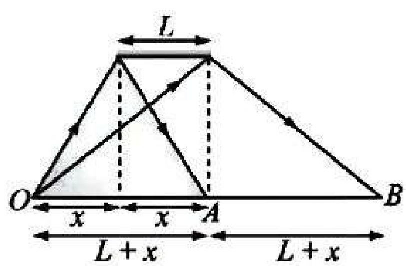
(c) $\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} - \frac{\hat{k}}{\sqrt{3}}$ (d) none of these

Solutions

1. **(b)** As
$$\vec{v}_{im} = -\vec{v}_{om}$$
or
$$\vec{v}_i - \vec{v}_m = (\vec{v}_o - \vec{v}_m)$$
∴
$$\vec{v}_i = 2\vec{v}_m - \vec{v}_o$$
$$= 2 \times 10\hat{i} - 0 = 20\hat{i} \text{ m/s.}$$
Ans.

2. **(d)** Length of spot, $AB = OB - OA = 2(L + x) - 2x = 2L$.

→ constant, so $\frac{dx}{dt} = 0$.

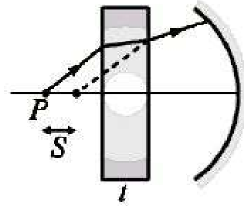


Ans.

3. **(a)** As object and observer both are in same medium so length *AB* remains same, only there is shift in its position. ***Ans.***

4. **(d)** Shift produced by slab $S = t\left(1 - \frac{1}{\mu}\right)$ towards right of P

so mirror must be moved away by distance S. ***Ans.***

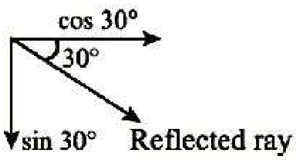


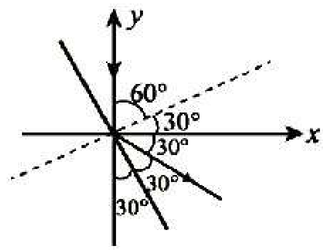
5. **(a)** $w = \frac{10-4}{2} = 3$ cm. ***Ans.***

6. **(c)** Incident ray $= -j$

Reflected ray vector

$\Rightarrow \frac{\sqrt{3}\,i}{2} - \frac{j}{2}$.





7. **(b)**

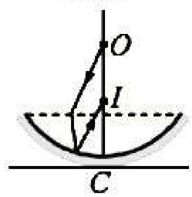
$$\frac{E_0}{B_0} = C(\text{speed of light in vacuum})$$

$$E_0 = B_0C = 3 \times 10^{-8} \times 3 \times 10^{8}$$

$$= 9 \text{ N/C}$$

So $E = 9\sin(\omega t + kx + \phi)$ ***Ans.***

8. **(d)** The image will be real and between C and O (see figure) ***Ans.***



9. **(c)**

$$V = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$n = \sqrt{\mu_r\varepsilon_r} = 2$$

$$\sin c = \frac{1}{2}$$

$$c = 30°.$$ ***Ans.***

10. **(a)**

$$\frac{1}{f_a} = \left(\frac{\mu_g}{\mu_a} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_m} = \left(\frac{\mu_0}{\mu_m} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \quad \frac{f_a}{f_m} = \frac{\left(\frac{\mu_g}{\mu_m} - 1\right)}{\left(\frac{\mu_g}{\mu_a} - 1\right)} = \frac{\left(\frac{1.50}{1.42} - 1\right)}{\left[\frac{1.50}{1} - 1\right]} = \frac{0.08}{(1.92)(0.5)}$$

$$\frac{f_m}{f_a} = \frac{(1.42)(0.5)}{0.08} = 8.875 \approx 9$$

11. **(b)** $\frac{1}{f} = (\mu-1)\left(\frac{1}{-2R}-\frac{1}{-R}\right) \Rightarrow f=\frac{2R}{\mu-1}$. ***Ans.***

12. **(60)** $\frac{1}{f} = (\mu-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$

$R_1 = \infty$

$R_2 = -30$ cm

$\frac{1}{f} = (1.5-1)\left(\frac{1}{\infty}-\frac{1}{-30}\right)$

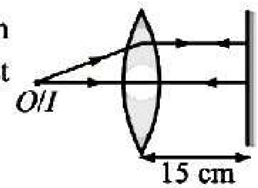
$\frac{1}{f} = \frac{0.5}{30}$

$f = 60$ cm. ***Ans.***

13. **(a)** $f_\ell = -\frac{R}{2(\mu-1)} = \frac{-10}{2(1.5-1)} = -10$ cm

Now $\frac{1}{f_e} = \frac{2}{f_\ell}+\frac{1}{f_m} = \frac{2}{-10}+\frac{1}{\left(\frac{-10}{2}\right)} \Rightarrow f_e = -2.5$ cm. ***Ans.***

14. **(c)** The rays after refraction from lens will reflect back from mirror, if they fall normally on the mirror, so object must be at the focal point of the lens. ***Ans.***



15. **(d)** $\frac{1}{f_e} = \frac{2}{f_1}+\frac{2}{f_2}+\frac{1}{f_m}$

or $P = \frac{2}{0.10}+\frac{2}{-0{,}20}+\frac{1}{\infty}$

$= 10$ D. ***Ans.***

16. **(b)** Using, $\frac{\mu_3}{v}-\frac{\mu_1}{u} = \frac{\mu_1-\mu_2}{2R}+\frac{\mu_2-\mu_3}{R}$

or $\frac{\mu_3}{(f)}-\frac{\mu_1}{\infty} = \left[\frac{\mu_1-\mu_2}{2R}+\frac{\mu_2-\mu_3}{R}\right]$

$\frac{1}{f} = \frac{1}{\mu_3}\left[\frac{\mu_1-\mu_2}{2R}+\frac{\mu_2-\mu_3}{R}\right]$

For $f < 0 \Rightarrow 2\mu_3 > (\mu_1+\mu_2)$. ***Ans.***

17. **(b)** When ray is reflected from *yz*-plane, only $\hat{i}$ reverse in sign, so

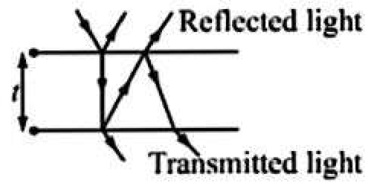
$\hat{R}_1 = \frac{-\hat{i}}{\sqrt{3}}+\frac{\hat{j}}{\sqrt{3}}+\frac{\hat{k}}{\sqrt{3}}$. Now when this ray is reflected from *xz*-plane, only $\hat{j}$ gets reverse, so $\hat{R}_2 = \frac{-\hat{i}}{\sqrt{3}}-\frac{\hat{j}}{\sqrt{3}}+\frac{\hat{k}}{\sqrt{3}}$. ***Ans.***

Wave Optics 21

TOPIC*: *Interference, Diffraction, Polarization, Malus Law, Brewster's Law and YDSE.

Review of Formulae

1. Interference in thin films :



(a) In reflected light :

$2\mu t \cos r = (2 \quad 1)-$, $n = 1, 2, ...$ for constructive interference and

$2\mu t \cos r = n\lambda$, $n = 0, 1, 2, ...$ for destructive interference

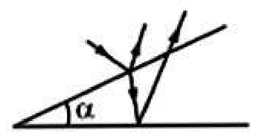
(b) In transmitted light :

$2\mu t \cos r = n\lambda$, $n = 0, 1, 2, ...$ for constructive interference

and $2\mu t \cos r = (2n-1)\dfrac{\lambda}{2}$, $n = 1, 2, ...$ for destructive interference

2. Interference in wedge shaped film :

Fringe width, $\beta = \dfrac{\lambda}{2\mu \tan\alpha}$.

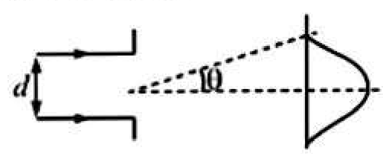


3. Fraunhoffer diffraction at single slit : Diffraction occurs due to super position between the wavelets originated from same wavefront. For diffraction, size of aperture is order of wavelength of wave.

$a \sin\theta = \lambda$, for first order minima

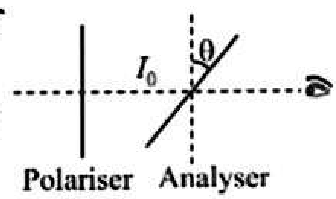
Width of principal maxima

$= 2\theta = 2\sin^{-1}(\lambda/a)$



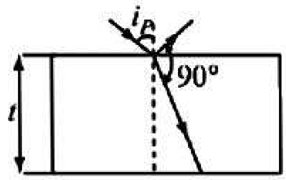
4. Polarization : Symmetry to asymmetry of vibrations of optic vector is called polarisation.

*When natural light falls on polariser, its intensity becomes half the intensity of incident light.



Malus Law : $I_\theta = I_0 \cos^2 \theta$

Brewster's law : $\mu = \tan i_p$.



TIPS & TRICKS Tips and Tricks for Shortcut Solutions

1. Brightness of the fringe is directly related to the intensity of source. When the slits of unequal width (area) are used, the intensity of fringe will not change but brightness of fringe will change. In this case we should take the power of source emerging from the slits. If P_1 and P_2 are the powers corresponding to area of slits A_1 and A_2, then

$$P_1 = IA_1 \text{ and } P_2 = IA_2$$

Therefore, $$\frac{P_{max}}{P_{min}} = \frac{\left(\sqrt{P_1}+\sqrt{P_2}\right)^2}{\left(\sqrt{P_1}-\sqrt{P_2}\right)^2}.$$

2. For sources of intensities I_1 and I_2 originated from slits of equal width,

$$\frac{I_{max}}{I_{min}} = \frac{(a_1+a_2)^2}{(a_1-a_2)^2} = \frac{\left(\sqrt{I_1}+\sqrt{I_2}\right)^2}{\left(\sqrt{I_1}-\sqrt{I_2}\right)^2}.$$

3. Path difference, $\Delta x = \frac{2\pi}{\lambda}\phi$.

The resultant intensity in interference

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\cos\phi$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\frac{\Delta x \lambda}{2\pi}\right).$$

→ For two identical coherent sources, $I_R = I_{max}\cos^2\frac{\phi}{2} = 4I\cos^2\frac{\phi}{2}$.

→ Intensity at a point y from the centre of screen $I = 4I_0\cos^2\frac{\pi y}{\beta}$.

4. For maxima, $\Delta x = \pm n\lambda$; $n = 0, 1, 2, \ldots$ and for minima, $\Delta x = (2n-1)\frac{\lambda}{2}$; $n = 1, 2, \ldots$

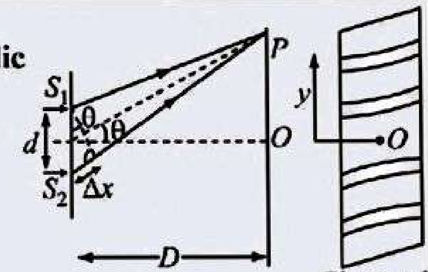
5. Path difference in different cases of YDSE

(i) *When screen is parallel to the plane of the slits*

Path difference in case $D >> d$:

$\Delta x = S_2P - S_1P = d \sin \theta$

The shape of the fringes are **hyperbolic** with straight central section.



Shape of fringes

If $\lambda << d$, then $\sin \theta \simeq \tan \theta = \dfrac{y}{D}$

$\therefore \quad \Delta x = d \sin \theta = d \tan \theta = \dfrac{dy}{D}.$

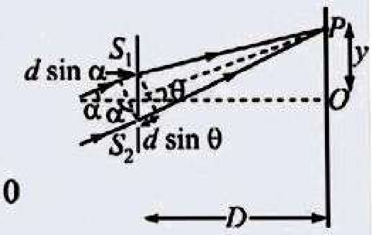
Fringe width, $\beta = \dfrac{D\lambda}{d}$

(ii) *Oblique incident of plane wavefront on the slits*

Path difference, $\Delta x = d \sin \theta - d \sin \alpha$

$= d \tan \theta - d \sin \alpha$

$= \dfrac{dy}{D} - d \sin \alpha.$



Position of zero order bright fringe, $\Delta x = 0$

or $\quad 0 = \dfrac{dy_0}{D} - d \sin \alpha$

$\therefore \quad y_0 = D \sin \alpha.$

(iii) *When transparent sheet is placed in front of S_1 or S_2.*

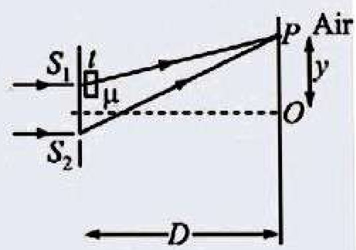
Optical path of S_1P

$= S_1P + (\mu - 1)t$

Path difference, $\Delta x = S_2P - [S_1P + (\mu - 1)t]$

$= (S_2P - S_1P) - (\mu - 1)t$

$= \dfrac{dy}{D} - (\mu - 1)t$

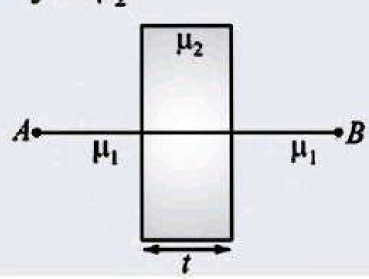


Position of zero order fringe, $\Delta x = 0$

or $\quad 0 = \dfrac{dy_0}{D} - (\mu - 1)t$

$\therefore \quad y_0 = \dfrac{D(\mu - 1)t}{d}.$

(iv) *Optical path of ray when passes through a medium of RI μ_1 into transparent sheet of RI μ_2.*



$= (AB - t)$ in medium of RI, μ_1 + t in medium of RI, μ_2.

$= (AB - t)\,\mu_1 + \mu_2 t$

$= (AB)\,\mu_1 + (\mu_2 - \mu_1)t.$

(v) *When transparent sheets are placed in front of both the slits*

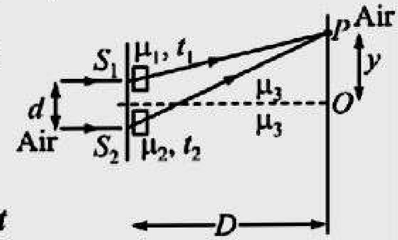
Path difference

$$\Delta x = \text{optical path of } S_2 - \text{optical path of } S_1$$

$$= [(S_2P)\mu_3 + (\mu_2 - \mu_3)t] - [(S_1P)\mu_3 + (\mu_1 - \mu_3)t]$$

$$= (S_2P - S_1P)\mu_3 + (\mu_2 - \mu_1)t$$

$$= \mu_3\left(\frac{dy}{D}\right) + (\mu_2 - \mu_1)t$$



For zero order fringe, $\Delta x = 0$

or $$0 = \mu_3\left[\frac{dy_0}{D}\right] + (\mu_2 - \mu_1)t$$

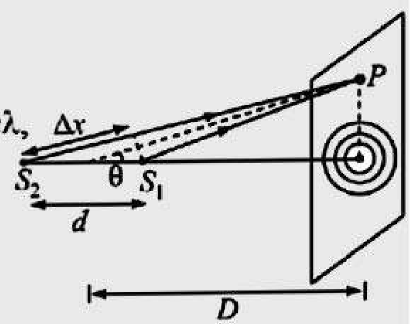
∴ $$y_0 = D\frac{(\mu_1 - \mu_2)t}{\mu_3 d}.$$

(vi) *When screen is placed perpendicular to the plane of the slits*

path difference, $\Delta x = d\cos\theta$ $(d << D)$

For $d = m\lambda$ and for bright fringe $\Delta x = n\lambda$,

$$n\lambda = (m\lambda)\cos\theta$$



or $$\cos\theta = \frac{n}{m}.$$

Here $n \le m$.

The shape of the fringes are circular and of different width.

(vii) *In YDSE, using two wavelengths, λ_1 and λ_2 and if bright fringes of both coincides, then*

$$n_1\lambda_1 = n_2\lambda_2 \quad \Rightarrow \quad \frac{n_1}{n_2} = \frac{\lambda_2}{\lambda_1}.$$

Illustration 1

In YDSE, $d = 1$ mm; $\lambda = \frac{1}{3}$ mm and $D = 1$ m.

(i) Find the distance between the first and central maximum on the screen.

(ii) Find number of maximum and minimum obtained on the screen.

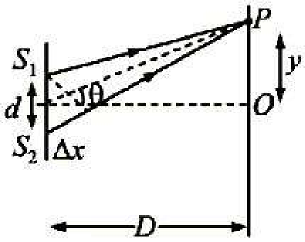
Solution :

(i) Here path difference between the waves on reaching at P

$$\Delta x = d \sin\theta$$

For maxima, $\Delta x = \pm n\lambda = \pm\frac{n}{3}$

$\therefore$ $\pm\frac{n}{3} = 1 \times \sin\theta \quad \Rightarrow \quad \sin\theta = \pm\frac{n}{3}$



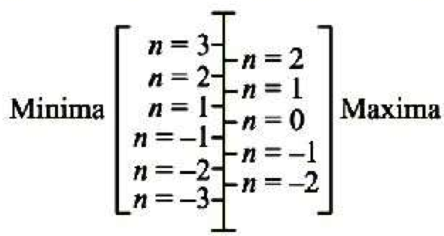
For I maxima, $n = 1$, $\therefore \sin\theta = \frac{1}{3}$ or $\tan\theta = \frac{1}{\sqrt{8}}$

Now $\frac{y}{D} = \tan\theta \quad \Rightarrow \quad y_1 = D\tan\theta = 1 \times \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{8}}$ m.

(ii) We have, $\sin\theta = \pm\frac{n}{3}$, here $n \le 3$.

For $n = 3$, $\sin\theta = 1 \quad \Rightarrow \quad \theta = 90°$. So maxima corresponding to $n = 3$ are not possible. Therefore total maximas are:

One for $n = 0$, and two for $n = 1$ and two for $n = 2$, so total are 5.



For minima, $\Delta x = \pm(2n-1)\frac{\lambda}{2} = \pm(2n-1)\frac{1/3}{2} = \pm\frac{(2n-1)}{6}$

and $\Delta x = d\sin\theta = 1\sin\theta$

or $\sin\theta = \pm\frac{2n-1}{6}$, here $n \le 5$.

Therefore possible minima are two for each $n = 1$, 2 and 3, so there are 6 minima. ***Ans.***

Illustration 2

In YDSE set-up d = 1 mm, λ = 600 nm and D = 1 m. The slits produce same intensity on the screen. Find the minimum distance between two points on the screen having 75% intensity of the maximum intensity.

Solution :

Using, $$I = 4I_0 \cos^2\frac{\phi}{2}$$

or $$\frac{3(4I_0)}{4} = 4I_0 \cos^2\frac{\phi}{2}$$

∴ $$\cos\frac{\phi}{2} = \frac{\sqrt{3}}{2} \Rightarrow \frac{\phi}{2} = \frac{\pi}{6} \quad \text{or} \quad \phi = \frac{\pi}{3}.$$

Now using $$\Delta x = \frac{\lambda}{2\pi}\phi$$

or $$\frac{dy_1}{D} = \frac{\lambda}{2\pi}\times\frac{\pi}{3}$$

∴ $$y = \frac{D\lambda}{6d}.$$

The minimum distance between two maximas

$$= 2y_1 = \frac{2D\lambda}{6d} = \frac{1\times600\times10^{-9}}{3\times10^{-3}}$$

$$= 0.2 \text{ mm}$$ ***Ans.***

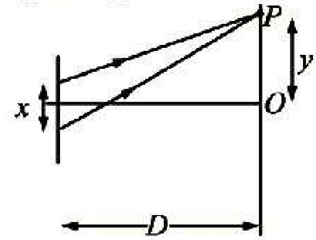
Illustration 3

If slits of the double slit were moved symmetrically apart with relative velocity v, calculate the rate at which fringes pass a point at a distance y from the centre of the fringe system formed on a screen D distance away from the double slits, if wavelength of light is λ. Assume λ >> d.

Solution :

Number of fringes passing through P

= number of maxima passing through P



For minima,

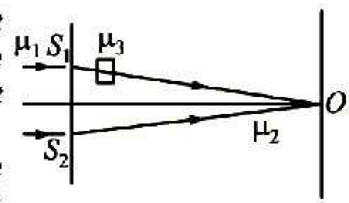
$$\Delta x = \frac{dy}{D} = \frac{xy}{D}$$

or $$n\lambda = \frac{xy}{D}$$

or $$\left(\frac{dn}{dt}\right)\lambda = \left(\frac{dx}{dt}\right)\frac{y}{D} \Rightarrow \frac{dn}{dt} = \frac{vy}{D\lambda}.$$ ***Ans.***

Illustration 4

In the figure shown in a YDSE, a parallel beam of light is incident on the slits from a medium of RI, μ_1. The wavelength of light in this medium is λ_1. A transparent slab of thickness t and RI, μ_3 is put infront of one slit. The medium between the screen and the plane of the slits is μ_2. Find the phase difference between the light waves reaching at O (symmetrically relative to the slits).



Short-cut solution :

Optical path of S_1O, $x_1 = \mu_2(S_1O) + (\mu_3 - \mu_2)t$

optical path of S_2O, $x_2 = \mu_2(S_1O)$

So path difference, $\Delta x = x_1 \sim x_2 = \mu_2(S_1O) + (\mu_3 - \mu_2)t - \mu_2(S_1O)$

As $S_1O = S_2O$

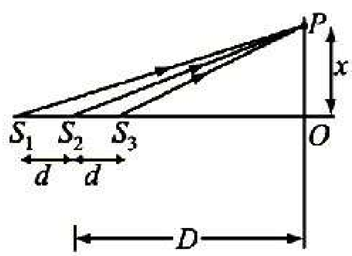
$\therefore$ $\Delta x = (\mu_3 - \mu_2)t$

and $\lambda_{air} = \mu_1\lambda_1$

Phase difference, $\phi = \dfrac{2\pi}{\lambda_{air}} \cdot \Delta x = \dfrac{2\pi}{\mu_1\lambda_1}(\mu_3 - \mu_2)t.$ ***Ans.***

Illustration 5

Three identical coherent point sources S_1, S_2 and S_3 are placed on a line perpendicular to the screen as shown in the figure. The wavelength of light emitted by source is λ, $d = 3\lambda$. The distance of S_2 from the screen is $D >> d$. Find the minimum distance x of the point P on the screen at which complete darkness is obtained.



Short-cut solution :

For complete darkness at P_1 the resultant amplitude of three sources will be zero.

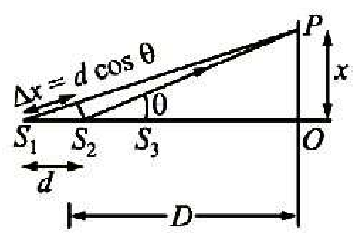
Therefore the angle between nearest two must be $120° = \dfrac{2\pi}{3}$.

Path difference, $\Delta x = \dfrac{\lambda}{2\pi} \times \phi = \dfrac{\lambda}{2\pi} \times \dfrac{2\pi}{3} = \dfrac{\lambda}{3}$.

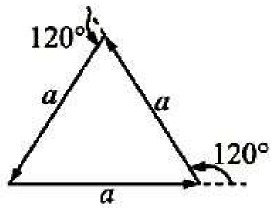
or $d\cos\theta = \dfrac{\lambda}{3}$

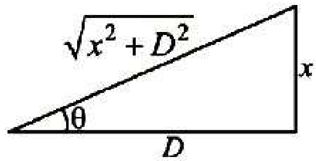
or $3\lambda\cos\theta = \dfrac{\lambda}{3}$

$\therefore$ $\cos\theta = \dfrac{1}{9}$



or $\dfrac{D}{\sqrt{x^2+D^2}} = \dfrac{1}{9} \Rightarrow 81D^2 = x^2 + D^2$

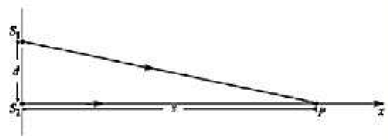




$\therefore$ $x = \sqrt{80}D = 4\sqrt{5}D.$ ***Ans.***

Video Solution

Q. ***Two coherent point sources S_1 and S_2 vibrating in phase emit light of wavelength l. The separation between the sources is 2l. Consider a line passing through S_2 and perpendicular to the line S_1S_2. What is the smallest distance S_2 where a minimum of intensity occurs?***

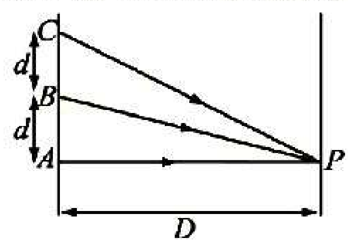


To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=XmP_cVphKFs

Illustration 6

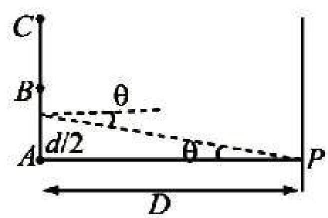
Figure shows three equidistant slits being illuminated by a monochromatic parallel beam of light. Let $BP - AP = \dfrac{\lambda}{3}$ and $D >> \lambda$.



(i) Show that $d = \sqrt{\dfrac{2\lambda D}{3}}$.

(ii) Find the resultant intensity at point P.

Solution :



(i) Given, $BP - AP = \dfrac{\lambda}{3}$

or $d\sin\theta = \dfrac{\lambda}{3}$

or $d\tan\theta = \dfrac{\lambda}{3}$

$$d \times \frac{\left(\frac{d}{2}\right)}{D} = \frac{\lambda}{3}$$

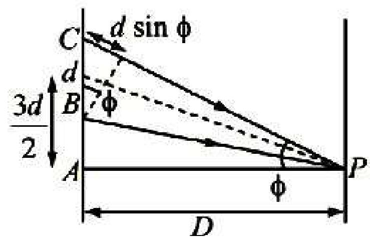
$$\therefore \qquad d = \sqrt{\frac{2\lambda D}{3}}.$$

(ii) Phase difference between A & B

$$\Delta x = BP - AP = \frac{\lambda}{3}$$

$$\therefore \qquad \phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{\lambda}\times\frac{\lambda}{3} = \frac{2\pi}{3}$$

Phase difference between B and C



$$\Delta x = CP - BP$$

$$= d\sin\phi$$

$$= d\times\left(\frac{\frac{3d}{2}}{D}\right)$$

$$= \frac{3d^2}{2D}$$

$$= \frac{3}{2D}\times\frac{2\lambda D}{3} = \lambda.$$

$$\therefore \qquad \phi = 2\pi.$$

Therefore B and C add together becomes $2a$ and with A,

$$R = \sqrt{a^2 + 2a^2 + 2a\times 2a\cos\frac{2\pi}{3}}$$

$$= \sqrt{3}a.$$

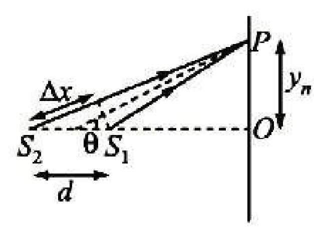
Intensity, $\quad I = R^2 = \left(\sqrt{3}a\right)^2 = 3a^2.$ **Ans.**

Illustration 7

In figure S_1 and S_2 are two coherent sources. A large screen is placed perpendicular to the plane of the slits. Find total number of bright fringes can be seen on the screen. Given $D >> d$, and $d = 3\lambda$. ***[JEE Main 2013]***

Solution :

Path difference between the waves on reaching P,



$$\Delta x = d\cos\theta.$$

For bright fringe, $\Delta x = n\lambda$

or $\qquad d\cos\theta = n\lambda$

or $\cos\theta = \frac{n\lambda}{d} = \frac{n\lambda}{3\lambda} = \frac{n}{3}.$

Here $n = 0, 1, 2, 3.$

For $n = 0$, $\cos\theta = 0$ or $\theta = 90°$, which practically not possible

$$n = 1, \cos\theta = \frac{1}{3} \text{ or } \theta = \cos^{-1}\frac{1}{3}$$

$$n = 2, \cos\theta = \frac{2}{3} \text{ or } \theta = \cos^{-1}\frac{2}{3}$$

$$n = 3, \cos\theta = 1 \text{ or } \theta = 0°.$$

Therefore there are total three circular bright fringes can be possible on the screen.

Ans.

Concept Booster Exercise

1. When light of different colours move through glass, they must have different

(a) frequencies (b) wavelengths
(c) amplitudes (d) speeds

2. A light of wavelength 6000Å and frequency 5×10^{14} Hz in air enters a medium of *RI* 1.5. Inside the medium, its wavelength and frequency **Numeric/Integer**

(a) 4000 Å, 5×10^{14} Hz (b) 6000 Å, 3.33×10^{14} Hz
(c) 4000 Å, 3.33×10^{14} Hz (d) 6000 Å, 5×10^{14} Hz

3. In a *YDSE*, let *A* and *B* be the two slits. A thin film of thickness *t* and *RI*, μ is placed in front of *A*. Let β is the fringe width. The central maximum will shift by

(a) μt (b) $\mu t\frac{\beta}{\lambda}$
(c) $\frac{\beta}{\lambda}(\mu-1)t$ (d) $\frac{\lambda}{\beta}(\mu-1)t$

4. In a *YDSE*, let β be the fringe width, and let I_0 be the intensity at the central bright fringe. At a distance *x* from the central bright fringe, the intensity will be :

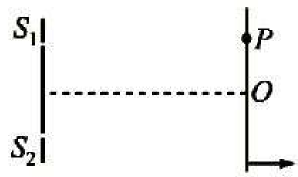
(a) $I_0 \cos\left(\frac{x}{\beta}\right)$ (b) $I_0 \cos^2\left(\frac{x}{\beta}\right)$
(c) $I_0 \cos^2\left(\frac{\pi x}{\beta}\right)$ (d) $I_0 \sin^2\left(\frac{\pi x}{\beta}\right)$

5. In a *YDSE*, *d* is the separation between the slits and *D* is distance of screen from slits. The distance of nearest point to the central maximum where the intensity is same as that due to a single slit : ($\lambda \rightarrow$ wavelength of light used)

(a) $\frac{D\lambda}{d}$ (b) $\frac{D\lambda}{2d}$

(c) $\frac{D\lambda}{3d}$ (d) $\frac{2D\lambda}{d}$

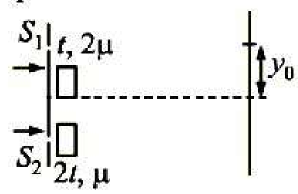
6. In a *YDSE*, first maxima is observed at a fixed point *P* on the screen. Now the screen is continuously moved away from the plane of the slits. The ratio of intensity at point *P* to the intensity at point *O* (centre of the screen)



(a) remains constant (b) keep on decreasing

(c) first decreases and then increases (d) first decreases then increases

7. In the *YDSE* shown the two slits are covered with thin sheets having thickness *t* and 2*t* and *RIs*, 2μ and μ. Find the position of central maximum (y_0) :



(a) $\frac{Dt}{d}$ (b) $-\frac{Dt}{d}$

(c) $D(\mu - 1)t$ (d) zero

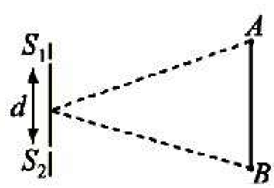
8. In *YDSE*, the wavelength of red light is 7.5×10^{-5} cm and that of blue light is 5.0×10^{-5} cm. The value of *n* for which $(n + 1)^{th}$ blue bright band coincides with n^{th} red band is:

Numeric/Integer

(a) 8 (b) 4

(c) 2 (d) 1

9. Figure shows two coherent sources S_1 and S_2 vibrating is same phase. *AB* is an irregular wire lying at a far distance from the sources. Let — = 10^{-3}. $\angle BOA = 0.12°$. How many bright spots will be seen on the wire, including *A* and *B*.

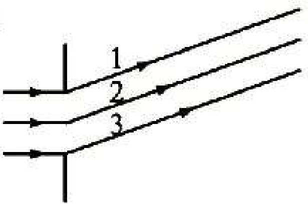


Numeric/Integer

(a) 2 (b) 3

(c) 4 (d) more than 4

10. The adjacent figure shows Fraunhoffer's diffraction due to single slit. If first minimum is obtained in the direction shown, then path difference between rays 1 and 3 is :



(a) 0 (b) $\frac{\lambda}{4}$

(c) $\frac{\lambda}{2}$ (d) λ

11. Unpolarised light of intensity I0 is incident on surface of a block of glass at Brewster's angle. In that case, which one of the following statements is true?

(a) Reflected light is completely polarised with intensity $\frac{I_0}{2}$.

(b) Transmitted light is completely polarised with intensity less than $\frac{I_0}{2}$.

(c) Transmitted light is partially polarised with intensity $\frac{I_0}{2}$.

(d) Reflected light is partially polarised with intensity $\frac{I_0}{2}$.

12. Three waves of same intensity (I_0) having initial phases $0, \frac{\pi}{4}, -\frac{\pi}{4}$ rad respectively interfere at a point. Find the resultant Intensity *[JEE Main 2020]*

(a) I_0 (b) 0 (c) $5.8\,I_0$ (d) $0.2\,I_0$

13. Assuming human pupil to have a radius of 0.25 cm and a comfortable viewing distance of 25 cm, the minimum separation between two objects that human eye can resolve at 500 nm wavelength is : **Numeric/Integer**

(a) 100 μm (b) 300 μm

(c) 1 μm (d) 30 μm

Solutions

1. **(a, b)** Different colours have different speeds, and wavelength. As $v = f\lambda$, so frequency will also be different. ***Ans.***

2. **(a)** $\lambda_m = \frac{\lambda}{\mu} = \frac{6000}{1.5} = 4000$ Å. ***Ans.***

3. **(c)** $\beta = \frac{D\lambda}{d} \Rightarrow \frac{D}{d} = \frac{\beta}{\lambda}$. Shift $\Delta = \frac{D}{d}(\mu-1)t = \frac{\beta}{\lambda}(\mu-1)t$. ***Ans.***

4. **(c)** $\Delta x = d\tan\theta = \frac{dx}{D}$, phase difference $\phi = \frac{2\pi}{\lambda}(\Delta x)$.

$$I = I_0\cos^2\frac{\phi}{2} = I_0\cos^2\left(\frac{2\pi}{\lambda}\times\frac{dx}{2D}\right) = I_0\cos^2\left(\frac{\pi x}{\beta}\right);\ \text{As } \beta = \frac{D\lambda}{d}. \quad \textbf{Ans.}$$

5. **(c)** Using,
$$I = 4I_0\cos^2\frac{\phi}{2}$$

or
$$I_0 = 4I_0\cos^2\frac{\phi}{2}$$

or
$$\cos\frac{\phi}{2} = \frac{1}{2} \Rightarrow \frac{\phi}{2} = \frac{\pi}{3} \text{ or } \phi = \frac{2\pi}{3} \text{ and } \Delta x = \frac{\lambda}{3}.$$

$\therefore$
$$\Delta x = d\tan\theta = \frac{dy}{D}$$

or
$$\frac{\lambda}{3} = \frac{dy}{D} \Rightarrow y = \frac{D\lambda}{3d}. \quad \textbf{Ans.}$$

6. **(c)** $I = 4I_0\cos^2\frac{\pi y}{\beta} = 4I_0\cos^2\left(\frac{\pi y d}{D\lambda}\right)$. ***Ans.***

7. **(a)** $\Delta x = [S_2P + (\mu-1)2t] - [S_1P + (2\mu-1)t]$

or
$$0 = (S_2P - S_1P) - t$$

or
$$0 = \frac{dy_0}{D} - t \Rightarrow y_0 = \frac{Dt}{d}. \quad \textbf{Ans.}$$

8. **(c)** $n(7.5\times10^{-5}) = (n+1)(5\times10^{-5}) \Rightarrow n = 2.$ ***Ans.***

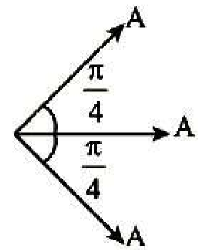
9. **(a)** Angular width,
$$\alpha = \frac{\lambda}{d} = 10^{-3}$$

$$n = \frac{\theta}{\alpha} = \frac{0.12\times2\pi}{360\times10^{-3}} \approx 2.09. \quad \textbf{Ans.}$$

10. **(d)** Path difference between 1 and 2 is $\frac{\lambda}{2}$ and between 2 and 3 is also $\frac{\lambda}{2}$. So path difference between 1 and 3 will be λ. ***Ans.***

11. **(a)** At Brewster's angle the reflected light gets completely polarised. ***Ans.***

12. **(c)** $A_{res} = \left(\sqrt{2}+1\right)A$

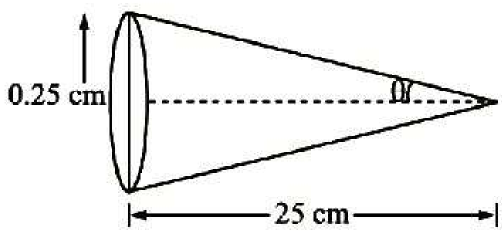


$I_{res} = \left(\sqrt{2}+1\right)^2 I_0$

$= \left(3+2\sqrt{2}\right)I_0 = 5.8I_0s$ ***Ans.***

13. **(d)** $\sin\theta = \frac{0.25}{25} = \frac{1}{100}$

Resolving power $= \frac{1.22\lambda}{2\mu\sin\theta} = 30\ \mu\text{m}.$ ***Ans.***



Photons, Atoms and Nuclei 22

***TOPIC 22.1**: Planck's Hypothesis, Photoelectric Effect, de-Broglie Waves, Rutherford α-Particle Scattering Experiment, Bohr's Model, Spectrum of Hydrogen Atom and X-rays.*

Review of Formulae

1. **Planck's hypothesis :** Light propagate in small packets of energy, called photon. Each photon having energy

$$E = hf = hc/\lambda.$$

*Rest mass of photon is zero.

Momentum of photon = $h\nu/c = h/\lambda$

Number of photon per second emitted by a source of power P is

$$N = \frac{P}{(hc/\lambda)}$$

2. **Photoelectric effect :** The emission of electron from metal surface by light is called photo electric effect (PEE).

Einstein's photo-electric equation

$$E = W_0 + \text{K.E}_{max}$$

or

$$hf = hf_0 + (1/2)mv^2_{max}$$

$$\frac{hc}{\lambda} = \frac{hc}{\lambda_0} + \frac{1}{2}mv^2_{max}$$

*With increase in intensity of incident light, photoelectric current increase but maximum K.E. of electron remains same.

*With increase in frequency of incident light, photo electric current will not change but maximum K.E. of photoelectrons increases.

3. **de Brogile Waves :** If a particle of m is moving with a speed v then corresponding wavelength

$$\lambda = \frac{h}{mv}$$

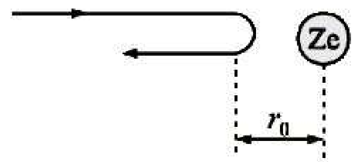
$$= \frac{h}{\sqrt{2m\,K.E.}}$$

4. Rutherford α - scattering formulae

Atomic Model

(i) **Size of nucleus :** Distance of closest approach

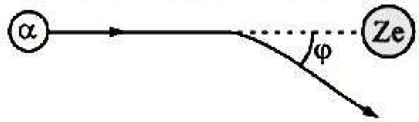
$$E_k = \frac{1}{4\pi \epsilon_0}\frac{(Ze)(2e)}{r_0}$$



$$\Rightarrow \quad r_0 = \frac{1}{4\pi \epsilon_0}\frac{2Ze^2}{E_k}.$$

(ii) The number of α-particles scattered at an angle φ by a nucleus is

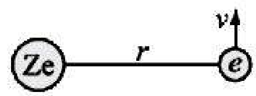
$$N_\varphi \propto \frac{1}{\sin^4 \frac{\varphi}{2}}.$$



5. Bohr's Model :

(i) Electrons revolve around the certain circular orbits without radiating energy, such that

$$\frac{1}{4\pi \epsilon_0}\frac{Ze^2}{r} = \frac{mv^2}{r}$$



(ii) Quantization of orbit

$$mvr = \frac{nh}{2\pi}, \; n = 1, 2, ...$$

(iii) Quantization of energy $hf = E_2 - E_1$.

(iv) Mass of revolving electron remains constant.

Deductions: $r_n = (0.53/Z)n^2 \mathring{A}$

$$V_m = \frac{c}{137}\frac{Z}{n}\text{ m/s}$$

$$-\text{T.E.} = \text{K.E.} = (-\text{P.E.}/2)$$

$$E_n(\text{T.E.}) = (-13.6\, Z^2/n^2)\text{eV}$$

6. Wave number : $\dfrac{1}{\lambda} = RZ^2\left[\dfrac{1}{n_1^2} - \dfrac{1}{n_2^2}\right].$

Spectrum of hydrogen atom :

1. Lymen (Ultraviolet) $n_1 = 1, n_2 \geq 2$
2. Balmer (Visible) $n_1 = 2, n_2 \geq 3$
3. Paschen (Infrared) $n_1 = 3, n_2 \geq 4$
4. Brackett (Infrared) $n_1 = 4, n_2 \geq 5$
5. Pfund (Infrared) $n_1 = 5, n_2 \geq 6$

7. Limitations of Bohr's theory

1. It can only explain the structure of hydrogen like atom.
2. It is unable to explain the fine spectral lines.

Phosphorescence and fluorescence : Certain substance emit visible light when high frequency radiation incident on it. If the emission continues after the source of excitation is removed then it is called phosphorescence. If not, is called fluorescence.

First excitation potential $V_1 = [(E_2 - E_1)/e]$

Ionisation potential $= [(E_\infty - E_1)/e]$

$= 13.6$ V for hydrogen

8. X-rays (1Å to 100 Å)

When fast moving electrons strike to a target of high melting point and atomic mass, X-rays are produced.

*Intensity of X-rays is proportional to filament current.

*Penetrating power is proportional to potential difference across filament & target.

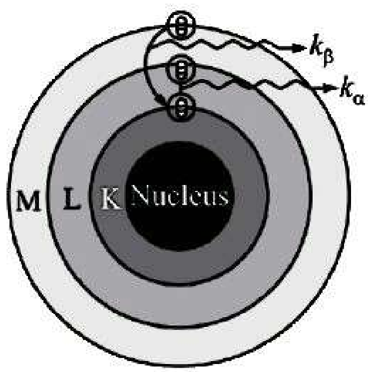
*X-rays of $\lambda > 4$Å are called soft and X-rays $\lambda \leq 4$Å are called hard X-rays.

9. X-rays Spectra :

1. Continuous spectrum are produced due to deceleration of electrons.
2. Characteristic spectrum are produced due to knock of bound electrons from the inner orbit and the vacancy so caused is filled by either free electron or electrons from higher orbits.

$E_L - E_k = h f_{k\alpha}$

$E_M - E_k = h f_{k\beta}$



Energy of X-ray $= hf = \dfrac{hc}{\lambda}$.

Dynamic mass of X-ray $= \dfrac{hf}{c^2} = \dfrac{h}{c\lambda}$.

Minimum wavelength of continuous X-rays

$$\lambda_{min} = \frac{hc}{eV} = \frac{12345}{V} \text{ Å}.$$

Bragg's law $2d \sin\theta = n\lambda$ ($n = 1, 2, ...$)

Intensity of X-rays after passing through a medium of thickness x is $I = I_0 e^{-kx}$.

TIPS & TRICKS! Tips and Tricks for Shortcut Solutions

1. **Intensity of photons :**

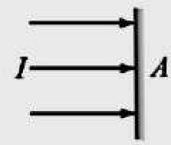
   (i) $I = \frac{E}{At} = \frac{\text{Power}}{A}$.

   (ii) Intensity due to a point source at a distance r, $I = \frac{\text{Power}}{4\pi r^2}$

   (iii) Intensity due to a line source at a distance r, $I = \frac{\text{Power}}{2\pi r}$.

2. **Radiation force (Normal incidence):**

   (i) If coefficient of reflection is r, then force exerted

   $F = \frac{(1+\text{r})}{c}$ power

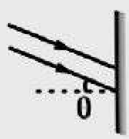
   (ii) For perfectly reflecting surface

   $F = \frac{2}{c}$ power $= \frac{2IA}{c}$.

   (iii) For perfectly absorbing surface

   $F = \frac{IA}{c}$.

   (iv) When radiation of intensity I incident at an angle θ with the normal of surface, then $F = \frac{IA(1+r)\cos^2\theta}{c}$.

3. **Photoelectric equation:** If V_0 is the stopping potential then

   $V_0 = -\frac{W_0}{e} + \left(\frac{h}{e}\right)f$.

   The slope of V_0 vs f is a constant and equal to $\left(\frac{h}{e}\right)$.

4. D-Broglie wavelength associated with an electron $\lambda = \frac{12.27}{\sqrt{V}}$ Å.

5. Shortest wavelength in continuous X-rays, $\lambda_{min} \simeq \frac{12345}{V}$ Å and longest wavelength will be infinite.

6. Remember $hc \approx 2 \times 10^{-25}$ J-m.

Illustration 1

How many photons are emitted per second by a 5 mW laser source operating at 632.8 nm?

Short-cut solution :

The energy of each photon

$$E = \frac{hc}{\lambda}$$

$$= \frac{(6.63\times10^{-34})\times(3\times10^{8})}{632.8\times10^{-9}}$$

$$= 3.14 \times 10^{-19}\ \text{J}.$$

The number of photons emitted per second is given by

$$n = \frac{\text{Power}}{E}$$

$$= \frac{5\times10^{-3}}{3.14\times10^{-19}}$$

$$= 1.6 \times 10^{16}.$$ ***Ans.***

Illustration 2

A beam of white light in incident normally on a plane surface absorbing 70% of the light and reflecting the rest. If the incident beam carries 10 W of power, find the force exerted by it on the surface.

Short-cut solution :

Using, $F = \frac{(1+r)\ \text{power}}{c}$, here

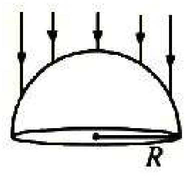
$r = 0.3.$

$\therefore$ $F = \frac{1.3\ \text{Power}}{c}$

$$= \frac{1.3\times10}{3\times10^{8}} = 4.3 \times 10^{-8}\ N.$$ ***Ans.***

Illustration 3

Radiation of intensity I is incident on a perfectly absorbing hemispherical surface of radius R. Find force exerted by radiation on the hemisphere.



Solution :

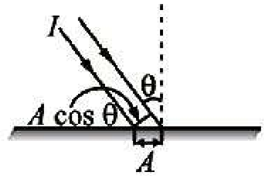
Force on perfectly absorbing surface

$$F = \frac{IA}{c} = \frac{I\times\text{projected area}}{c} = \frac{I(\pi R^2)}{c}.$$ ***Ans.***

Note: If surface is perfectly reflecting, then force remains same.

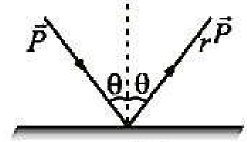
Illustration 4

The radiation of intensity I is incident at an angle θ with the normal of the surface. If coefficient of reflection is r than find force exerted by the radiation on area A of the surface.



Short-cut solution :

Momentum of incident photons $P = \frac{E}{c} = \frac{I(A\cos\theta)t}{c}$



Force, $F = \frac{P}{\Delta t} = \frac{P(1+r)\cos\theta}{t} = \left[\frac{IA\cos\theta t}{c}\right]\frac{(1+r)\cos\theta}{t}$

$$= \frac{IA(1+r)\cos^2\theta}{c}.$$

Ans.

Illustration 5

A hydrogen atom at rest emits a photon of wavelength 122 nm. Find the recoil speed of the hydrogen atom.

Solution :

$$0 = \vec{P}_{\text{photon}} + \vec{P}_H$$

or $$\vec{P}_H = -\vec{P}_{\text{photon}}$$

or $$P_H = P_{\text{photon}}$$

or $$mv = \frac{h}{\lambda}$$

$\therefore$ $$v = \frac{h}{m\lambda} = \frac{6.63\times10^{-34}}{1.67\times10^{-27}\times122\times10^{-9}} \simeq 3.25 \text{ m/s}$$ ***Ans.***

Illustration 6

Calculate the value of the retarding potential needed to stop the photo-electrons ejected from a metal surface of work function 1.2 eV with light of frequency 5.5 × 10¹⁴ Hz.

Short-cut solution :

If V_0 is the stopping potential, then by Einstein photoelectric equation, we have

$$eV_0 = hf - W_0$$

$\therefore$ $$V_0 = \frac{hf - W_0}{e}$$

$$= \frac{6.62\times10^{-34}\times5.5\times10^{14} - 1.2\times1.6\times10^{-19}}{1.6\times10^{-19}}$$

$$= 1.07 \text{ V}$$ ***Ans.***

Illustration 7

What is the angular momentum of an electron in Bohr's hydrogen atom whose energy is – 3.4 eV ?

Short-cut solution :

We know that,

$$E = -\frac{13.6}{n^2} eV$$

$$\therefore \quad -3.4 = -\frac{13.6}{n^2}$$

or $$n = 2.$$

The angular momentum L of an electron is given by $\frac{nh}{2\pi}$.

For $$n = 2, L = \frac{h}{\pi}.$$ ***Ans.***

Illustration 8

Find the shortest wavelength of the X-rays emitted by an X-ray tube operating at 30 kV.

Short-cut solution :

Using, $$\lambda_{min} = \frac{12345}{V} Å$$

$$= \frac{12345}{30 \times 10^3} = 0.4125 Å.$$ ***Ans.***

Illustration 9

The wavelength of the characteristic X-ray K_α line emitted by a hydrogen-like element is 0.32 Å. Calculate the wavelength of K_β line emitted by the same element.

Short-cut solution :

For hydrogen like atom

$$\frac{1}{\lambda} = Z^2 R\left[\frac{1}{n_f^2} - \frac{1}{n_i^2}\right].$$

For K_α – line, $$\frac{1}{\lambda_{K_\alpha}} = Z^2 R\left[\frac{1}{1^2} - \frac{1}{2^2}\right]$$

$$= \frac{3Z^2R}{4}. \quad ...(i)$$

For K_β – line,

$$\frac{1}{\lambda_{K_\beta}} = Z^2R\left[\frac{1}{1^2} - \frac{1}{3^2}\right]$$

$$= \frac{8Z^2R}{9}. \quad ...(ii)$$

From equations (i) and (ii), we have

$$\frac{\lambda_{K_\beta}}{\lambda_{K_\alpha}} = \frac{3/4}{8/9} = \frac{3\times9}{4\times8} = \frac{27}{32}$$

$$\therefore \quad \lambda_{K_\beta} = \frac{27}{32}\lambda_{K_\alpha} = \frac{27}{32}\times 0.32\ \text{Å} = 0.27\ \text{Å}. \quad \textbf{\textit{Ans.}}$$

Illustration 10

Calculate the ratio of minimum to maximum wavelengths of radiation that an electron causes in a Bohr's hydrogen atom (if the electron is de-exciting to n = 1).

Short-cut solution :

$$\frac{\left(\frac{1}{\lambda_{max}}\right)}{\left(\frac{1}{\lambda_{min}}\right)} = \frac{R\left(\frac{1}{1^2} - \frac{1}{2^2}\right)}{R\left(\frac{1}{1^2} - \frac{1}{\infty^2}\right)}$$

or

$$\frac{\lambda_{min}}{\lambda_{max}} = \frac{1 - \frac{1}{4}}{1} = \frac{3}{4}. \quad \textbf{\textit{Ans.}}$$

Illustration 11

Two identical non-relativistic particles move at right angles to each other, possessing De-Broglie wavelengths λ_1 and λ_2. Find the De-Broglie wavelength of each particle in the frame of their centre of mass.

Solution :

If $\vec{P}_1$ and $\vec{P}_2$ are the momentum corresponding to wavelengths λ_1 and λ_2. The velocity of CM

$$\vec{v}_{cm} = \frac{\vec{P}_1 + \vec{P}_2}{m + m}$$

or

$$2mv_{cm} = \sqrt{P_1^2 + P_2^2}$$

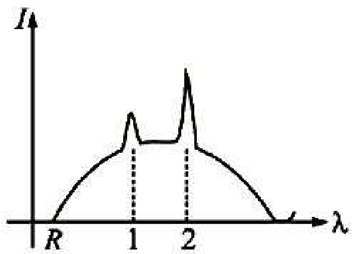
or $$2P_{cm} = \sqrt{P_1^2 + P_2^2}$$

or $$2\frac{h}{\lambda} = \sqrt{\left(\frac{h}{\lambda_1}\right)^2 + \left(\frac{h}{\lambda_2}\right)^2}$$

∴ $$\lambda = \frac{2\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_1^2}}.$$ ***Ans.***

Illustration 12

Characteristic-spectra of X-ray is shown in figure. Which is K_α and K_β?



Solution :

$$\lambda = \frac{hc}{\Delta E}$$

Since energy difference of K_α is less than K_β

$$\Delta E_{K\alpha} < \Delta E_{K\beta}$$

⇒ $$\lambda_{K\beta} < \lambda_{K\alpha}$$

Therefore 1 is K_β and 2 is K_α ***Ans.***

Illustration 13

An energy of 24.6 eV is required to remove one of the electrons from a neutral helium atom. The energy (in eV) required to remove both the electrons from a neutral helium atom is

(a) *79.0* ***(b)*** *51.8* ***(c)*** *49.2* ***(d)*** *38.2*

Short-cut solution :

The ionisation energy of helium atom

$= 13.6\ Z^2 = 13.6 \times 2^2 = 54.4$ eV

So total energy needed to remove both the electrons from helium atom = 24.6 + 54.4 = 79 eV. ***Ans. (a)***

***TOPIC 22.2**: Radius and Density of Nucleus, Mass-Defect, Packing Fraction, Binding Energy and Radioactivity.*

Review of Formulae

1. Radius of nucleus $R = R_0A^{1/3}$
   * Size of nucleus is of the order of 10^{-15} m.
   * Size of atom is of the order of 10^{-10} m.
   * Nuclear density is of the order of 10^{17} kg/ m^3
2. Mass defects (ΔM) = Mass of nucleons (M) – Mass of nucleus (A).
3. 1 amu = 1.66×10^{-27} kg
   = 931 MeV.
4. Packing fraction $f = \frac{\text{Mass defect}}{\text{Mass number}} = \frac{M-A}{A}$
5. B.E. = Δmc^2
6. **Radioactivity**
   (i) Spontaneous disintegration of nucleus is called radioactivity.
   * It is matter of chance, for any atom to disintegrate first. The life time of radioactive atoms will therefore be from zero to infinite.

   (ii) $N = N_n e^{-\lambda t}$.

   Also $N = N_0(1/2)^n,\ n = \frac{t}{t_{1/2}}$.

   (iii) Mean life $T = (1/\lambda) = 1.44t_{1/2.}$

   *β-particle is not initially present in nucleus, but is produced due to disintegration of neutron

   $_0n^1 \rightarrow {}_1H^1 + {}_{-1}\beta^0 + \bar{v}$.

   (iv) **Activity (A) :** $A = \left|\frac{dN}{dt}\right| = \lambda N$

   Also $A = A_0e^{-\lambda t}$.

   S.I. unit of radioactivity is becquerel.

   1 becquerel = 1 disintegration /s.

   * 1 rutherford = 10^6 disintegration /s.
   * 1 curie = 3.7×10^{10} disintegration /s.

Illustration 14

The binding energies per nucleons for deuteron $_1H^2$ and helium $_2He^4$ are 1.1 MeV and 7.0 MeV respectively. Calculate the energy released when two deuterons fuse to form a helium nucleus ($_2He^4$).

Short-cut solution :

The fusion of deutrons can be written as :

$$_1H^2 + {}_1H^2 \rightarrow {}_2He^4 + \text{Energy}$$

The binding energy per nucleon of deutrons is 1.1 MeV,

∴ net binding energy of deutrons will $4 \times 1.1 = 4.4$ MeV.

The binding energy per nucleon of $_2He^4$ is 7.0 MeV. The binding energy of helium nucleus is $4 \times 7.0 = 28.0$ MeV.

The energy released = 28.0 MeV – 4.4 MeV = 23.6 MeV ***Ans.***

Illustration 15

What is the power output of a $_{92}U^{235}$ reactor if it takes 30 days to use up 2 kg of fuel, and if each fission gives 185 MeV of usable energy?

Solution :

The number of $_{92}U^{235}$ in 2 kg,

$$N = \frac{(2 \times 1000)}{235} \times 6.02 \times 10^{23}$$

$$= 5.12 \times 10^{24}$$

The fission of each $_{92}U^{235}$ produces energy

$$= 185 \text{ MeV}$$

∴ Total energy produces,

$$E = (5.12 \times 10)^{24} \times (185 \times 1.6 \times 10^{-13})\, J$$

$$= 1572.3 \times 10^{11}\, J$$

The power output

$$P = \frac{E}{t} = \frac{1572.3 \times 10^{11}}{30 \times 24 \times 3600}$$

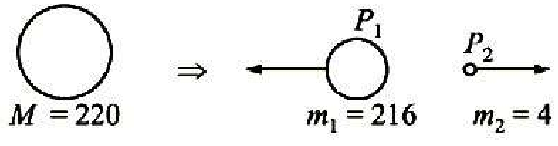
$$= 6.06 \times 10^{7} \text{ W}$$ ***Ans.***

Illustration 16

A nucleus with mass number 220 initially at rest emits an α-particle. If the Q value of the reaction is 5.5 MeV, calculate the kinetic energy of the α-particle.

***(a)* 4.4 MeV *(b)* 5.4 MeV *(c)* 5.6 MeV *(d)* 6.5 MeV**

Short-cut solution :



$$K_1 + K_2 = 5.5\ MeV \quad \text{...(i)}$$

and

$$P_1 = P_2 = \sqrt{2 \times 216 K_1} = \sqrt{2 \times 4 K_2} \quad \text{...(ii)}$$

On solving equation (i) and (ii), we get

$$K_2 = 5.4 \text{ MeV.}$$ ***Ans. (b)***

Illustration 17

Two radioactive materials X_1 and X_2 have decay constants 10λ and λ respectively. If initially they have the same number of nuclei, then the ratio of the number of nuclei of X_1 to that of X_2 will be 1/e after a time

(a) $1/(10\lambda)$ *(b)* $1/(11\lambda)$ *(c)* $11/(10\lambda)$ *(d)* $1/(9\lambda)$

Short-cut solution :

$N_1 = N_0e^{-10\lambda}$ and $N_2 = N_0e^{-\lambda t}$

$$\therefore \quad \frac{N_1}{N_2} = e^{(-10\lambda + \lambda t)} = e^{-9\lambda t}$$

$$\text{So} \quad - = e^{-9\lambda t}$$

$$\text{or} \quad t = \frac{1}{9\lambda}.$$

Ans. (d)

Illustration 18

Nuclei of radioactive element A are being produced at a constant rate α. The element has a decay constant λ. At time $t = 0$, there are N_0 nuclei of the element. Calculate the number N of nuclei of A at time t.

Solution :

The rate of formation of nuclei is $= \alpha$.

The rate of decay of nuclei $= \lambda N$.

Thus net rate of formation of nuclei, $\frac{dN}{dt} = (\alpha - \lambda N)$

$$\therefore \quad \frac{dN}{(\alpha - \lambda N)} = dt.$$

On integrating, we have

$$\int_{N_0}^{N} \frac{dN}{(\alpha - \lambda N)} = \int_0^t dt$$

$$\text{or} \quad \left|\frac{\ell n(\alpha - \lambda N)}{-\lambda}\right|_{N_0}^{N} = t$$

$$\ell n(\alpha - \lambda N) - \ell n(\alpha - \lambda N_0) = -\lambda t$$

$$\text{or} \quad \ell n\left(\frac{\alpha - \lambda N}{\alpha - \lambda N_0}\right) = -\lambda t$$

or $$\frac{\alpha-\lambda N}{\alpha-\lambda N_0} = e^{-\lambda t}$$

$\therefore$ $$N = \frac{\alpha}{\lambda}(1-e^{-\lambda t})+N_0e^{-\lambda t}.$$ ***Ans.***

Illustration 19

The activity of a sample of radioactive material is A_1 at time t_1 and A_2 at time t_2 ($t_2 > t_1$). If mean life is T, then find A_2.

Short-cut solution :

$$A_1 = A_0e^{-\lambda t_1}$$

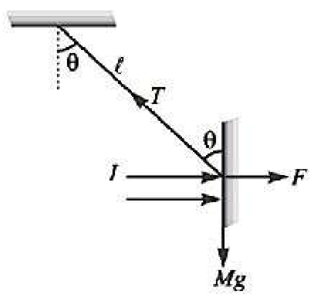
and $$A_2 = A_0e^{-\lambda t_2}, \qquad \text{also } \lambda = \frac{1}{T}$$

$\therefore$ $$\frac{A_2}{A_1} = \frac{e^{-\lambda t_2}}{e^{-\lambda t_1}} = e^{(-\lambda t_2+\lambda t_1)}$$

or $$A_2 = A_1e^{\lambda(t_1-t_2)} = A_1e^{(t_1-t_2)/T}.$$ ***Ans.***

Video Solution

Q. ***A small plane perfectly reflecting mirror of area A and mass M is hanging vertically with the help of a massless string of length l. Light of intensity I is incident normally on it. Find the angle made by the string with the vertical.***



To see the video solution, scan the QR code:

OR Visit https://www.youtube.com/watch?v=PSbcFnJNpSQ

Illustration 20

Suppose, the daughter nucleus in a nuclear decay is itself radioactive. Let λ_p and λ_d be the decay constants of the parent and the daughter nuclei. Also, let N_p and N_d be the number of parent and daughter nuclei at time t. Find the condition for which the number of daughter nuclei becomes constant.

Short-cut solution :

Activity, $A_P = A_d$

or $\lambda_P N_P = \lambda_d N_d$. **Ans.**

Illustration 21

A factory produces a radioactive substance A at a constant rate R which decays with a decay constant λ to form a stable substance. Find (i) the number of nuclei of A and (ii) number of nuclei of B at any time t assuming the production of A starts at t = 0. (iii) Also find out the maximum number of nuclei of A present at any time t during the formation.

Solution :

Factory $\xrightarrow{R} A \xrightarrow{\lambda} B$

(i) Let N be the number of nuclei of A at any time t

$$\therefore \quad \frac{dN}{dt} = R - \lambda N$$

or
$$\int_0^N \frac{dN}{R-\lambda N} = \int_0^t dt$$

or
$$\left|\frac{\ell n(R-\lambda N)}{(-\lambda)}\right|_0^N = t$$

or
$$\ell n(R-\lambda N) - \ell n R = -\lambda t$$

or
$$N = \frac{R}{\lambda}\left(1-e^{-\lambda t}\right).$$

(ii) Number of nuclei of B at any time t

$$N_B = Rt - N_A$$
$$= Rt - \frac{R}{\lambda}\left(1-e^{-\lambda t}\right)$$
$$= \frac{R}{\lambda}\left(\lambda t - 1 + e^{-\lambda t}\right).$$

(iii) Maximum number of nuclei of 'A' present at any time during its formation

$= \frac{R}{\lambda}$. **Ans.**

Illustration 22

A radioactive sample decays with an average life of 20 ms. A capacitor of capacitance 100 μF is charged to some potential and then the plates are connected through a resistance R. What should be the value of R so that the ratio of the charge on the capacitor to the activity of the radioactive sample remains constant in time?

Solution :

The activity of the sample at any time t is given by

$$A = A_0e^{-\lambda t}.$$

The charge on the capacitor $Q = Q_0e^{-t/CR}$

Thus $$\frac{Q}{A} = \frac{Q_0e^{-t/CR}}{A_0e^{-\lambda t}}.$$

It is independent of t if $$\lambda = \frac{1}{CR} \qquad \left[\frac{1}{\lambda} = T\right]$$

or $$R = \frac{1}{\lambda C} = \frac{T}{C} = \frac{20\times10^{-3}}{100\times10^{-6}} = 200\,\Omega \quad \textit{Ans.}$$

Concept Booster Exercise

1. An electron and a photon have same energy E. Find the ratio of de-Broglie wavelength of electron to wavelength of photon. Given mass of electron is m and speed of light is C. ***[JEE Main 2020]***

(a) $\frac{1}{C}\left(\frac{E}{2m}\right)^{1/2}$ (b) $\left(\frac{E}{m}\right)^{1/2}C$ (c) $\frac{\sqrt{2mE}}{C}$ (d) $\left(\frac{E}{2m}\right)^{1/2}$

2. Let T be the mean life of a radioactive sample, 75% of the active nuclei present in the sample initially will decay in time

(a) $2T$ (b) $\frac{T\ell n2}{2}$ (c) $4T$ (d) $2T\ell n2$

3. The radiation force experience by a disc of radius R exposed to radiation of constant intensity I is F. The maximum value of F is : (c speed of light)

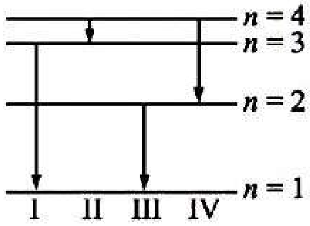
(a) $\frac{\pi R^2 I}{c}$ (b) $\frac{\pi R^2 I}{2c}$ (c) $\frac{2\pi R^2 I}{c}$ (d) $\pi^2 R^2 I$

4. Energy required for the electron excitation in Li^{++} from the first to the third Bohr orbit is :

Numeric/Integer

(a) 36.3 eV (b) 108.8 eV (c) 122.4 eV (d) 12.1 eV

5. The diagram shows the energy levels for an electron in a certain atom. Which transition shown represents the emission of a photon with the most energy ***[JEE Main 2005]***



(a) I (b) II (c) III (d) IV

6. An electron of mass 'm' and charge 'e' initially at rest gets accelerated by a constant electric field E. The rate of change of de-Broglie wavelength of this electron at time t, ignoring relativistic effects is :

(a) $\frac{-h}{eEt^2}$ (b) $\frac{-eht}{E}$ (c) $\frac{-mh}{eEt^2}$ (d) $\frac{-h}{eE}$

7. The potential energy of a particle of mass m is given by $U(x)=\begin{cases}E_0; 0 \le x \le 1\\ 0; \quad x>1\end{cases}$, λ_1 and λ_2 are the de-Broglie wavelengths of the particle, when $0 \le x \le 1$ and $x > 1$ respectively. If the total energy of particle is $2E_0$, the ratio $\frac{\lambda_1}{\lambda_2}$ will be **[JEE Adv. 2005]**

Numeric/Integer

(a) 2 (b) 1 (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

8. Consider identical particles A and B moving along x and y-axis respectively and having respective De-Broglie wavelengths 3λ and 4λ. De-Broglie wavelength of one of the particles w.r.t. the other is: [consider no relativistic variation of mass]

Numeric/Integer

(a) 1.2λ (b) 2λ (c) 2.4λ (d) 3.6λ

9. An electron is in an excited state in a hydrogen-like atom. It has a total energy of –3.4 eV. The kE of the electron is E and its De-Broglie wavelength is λ

Numeric/Integer

(a) $E = 6.8$ eV, $\lambda \sim 6.6 \times 10^{-10}$ m (b) $E = 3.4$ eV, $\lambda \sim 6.6 \times 10^{-10}$ m

(c) $E = 3.4$ eV, $\lambda \sim 6.6 \times 10^{-10}$ m (d) $E = 6.8$ eV, $\lambda = 6.6 \times 10^{-11}$ m

10. The count rate from 100 cm^3 of a radioactive liquid is c. Some of the liquid is now discarded. The count rate of the remaining liquid is found to be $\frac{c}{10}$ after three half-lives. The volume of the remaining liquid in cm^3 is:

Numeric/Integer

(a) 20 (b) 40 (c) 60 (d) 80

11. In the Bohr model of the hydrogen atom, let R, V and E represent the radius of orbit, speed of election and the total energy of the electron respectively. Which of the following quantities are proportional of the quantum number n?

(a) VR (b) RE (c) $\frac{V}{E}$ (d) $\frac{R}{E}$

12. An electron of a stationary hydrogen atom passes from 4th energy level to the ground state. Recoil velocity of the atom as a result of photon emission is:
(h = Plank's constant, m = mass of atom, R = Rydberg constant)

(a) $\frac{15hR}{16m}$ (b) $\frac{hR}{m}$ (c) $\frac{13hR}{15m}$ (d) $\frac{16hR}{15m}$.

13. Activity of a substance changes from 700 s^{-1} to 500 s^{-1} in 30 minute. Find its half-life in minutes. ***[JEE Main 2020]***

(a) 66 (b) 62 (c) 56 (d) 50

14. Two radioactive elements A and B have half lives T and $2T$ respectively. In the beginning the number of atoms in both samples is same. After $4T$ time the ratio of remaining atom of A and B will be:

Numeric/Integer

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

15. Let λ_α, λ_β and λ'_α denote the wavelengths of the X-rays of the K_α, K_β and L_α lines in the characteristic X-rays for a metal

(a) $\lambda'_\alpha > \lambda_\alpha > \lambda_\beta$ (b) $\lambda'_\alpha > \lambda_\beta > \lambda_\alpha$

(c) $\frac{1}{\lambda_\beta} = \frac{1}{\lambda_\alpha} + \frac{1}{\lambda'_\alpha}$ (d) $\frac{1}{\lambda_\alpha} + \frac{1}{\lambda_\beta} = \frac{1}{\lambda'_\alpha}$

16. 90% of the active nuclei present in a radioactive sample are found to remain undecayed after one day. The percentage of undecayed nuclei left after two days will be

Numeric/Integer

(a) 85% (b) 81% (c) 80% (d) 79%

Solutions

1. **(a)**

$$\lambda_d \text{ for electron} = \frac{h}{\sqrt{2mE}}$$

$$\lambda \text{ for photon} = \frac{hC}{E}$$

$$\text{Ratio} = \frac{h}{\sqrt{2mE}}\frac{E}{hC} = \frac{1}{C}\sqrt{\frac{E}{2m}}$$ ***Ans.***

2. **(d)** When 75% decays, 25% is left undecayed. This requires a time $t = 2t_{1/2}$ $= 2\frac{\ell n2}{\lambda}$

As $\frac{1}{\lambda} = T,$

$\therefore$ $t = 2T\ln 2.$ ***Ans.***

3. **(c)** $F = \frac{2IA}{c} = \frac{2I \times \pi R^2}{c}$ ***Ans.***

4. **(b)** Energy of excitation,

$$\Delta E = 13.6\, Z^2\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) eV$$

$$\Rightarrow \quad \Delta E = 13.6\,(3)^2\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = 108.8\ eV$$ ***Ans.***

5. **(a)**

6. **(a)** Acceleration, $a = \frac{Ee}{m}$

Velocity of electron, $v = at = \frac{Eet}{m}$.

Wavelength, $\lambda = \frac{h}{mv} = \frac{h}{Eet}$.

Now $\frac{d\lambda}{dt} = -\frac{h}{eEt^2}$. ***Ans.***

7. **(c)** For $0 \le x \le 1$, $K_1 = E - U = 2E_0 - E_0 = E_0$

For $x > 1$, $K_2 = E - U = 2E_0 - 0 = 2E_0$

Now $\frac{\lambda_1}{\lambda_2} = \frac{h/\sqrt{2mK_1}}{n/\sqrt{2mK_2}} = \sqrt{\frac{K_2}{K_1}}$

$= \sqrt{2}$. ***Ans.***

8. **(c)** $\lambda' = \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} = \frac{(3\lambda)(4\lambda)}{\sqrt{(3\lambda)^2 + (4\lambda)^2}} = 2.4\,\lambda$ ***Ans.***

9. **(b)** The $PE = -2(KE) = -2E$

$\therefore$ Total energy $= -2E + E = -E = -3.4$ eV

$$\lambda = \frac{h}{\sqrt{2mK}} = 6.6 \times 10^{-10}\ m.$$ ***Ans.***

10. **(d)** Initial count rate (CR) for 1 cm^3 of liquid $= \frac{c}{100}$

After 3 half-lives, CR for 1 cm^3 of liquid $= \frac{1}{8} \times \frac{c}{100}$

If V is the volume of remaining liquid, then

$$V \times \frac{c}{800} = \frac{c}{10}$$

$\therefore$ $V = 80.$ ***Ans.***

11. **(a)** $R \propto n^2$, $V \propto \frac{1}{n}$ and $E \propto n^2$. ***Ans.***

12. **(a)** $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{4^2}\right) = \frac{15R}{16}$

$$P = mv = \frac{h}{\lambda} = \left(\frac{15R}{16}\right)h$$

$\therefore$ $v = \frac{15hR}{16m}$. ***Ans.***

13. **(b)** $\ell n\left[\frac{A_0}{A_t}\right] = \lambda t$

$\Rightarrow$ $\ell n2 = \lambda t_{1/2}$ ...(i)

$\Rightarrow$ $\ell n\left[\frac{700}{500}\right] = \lambda(30 \text{ min})$ ...(ii)

(i)/(ii)

$\Rightarrow$ $\frac{\ell n2}{\ell n(7/5)} = \frac{t_{1/2}}{(30 \text{ min})}$

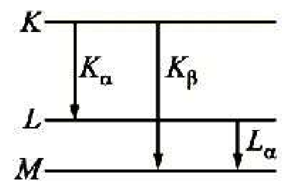
$\Rightarrow$ $(2.06004)\ 30 = t_{1/2} = 61.8$ min. ***Ans.***

14. **(c)** $\frac{N_1}{N_2} = \frac{N_0 e^{-\lambda_1 t}}{N_0 e^{-\lambda_2 t}} = e^{-(\lambda_1 - \lambda_2)t} = e^{-\left[0.693\left(\frac{1}{T} - \frac{1}{2T}\right) \times 4T\right]}$

$= \frac{1}{4}$. ***Ans.***

15. **(a, c)** $E_K - E_L = \dfrac{hc}{\lambda_\alpha}$

$$E_K - E_M = \frac{hc}{\lambda_\beta}$$



$$E_L - E_M = \frac{hc}{\lambda'_\alpha} = \frac{hc}{\lambda_\beta} - \frac{hc}{\lambda_\alpha}$$

or $$\frac{1}{\lambda_\beta} = \frac{1}{\lambda_\alpha} + \frac{1}{\lambda'_\alpha}$$

Also $(E_K - E_M) > (E_K - E_L) > (E_L - E_M)$

or $$\frac{hc}{\lambda_\beta} > \frac{hc}{\lambda_\alpha} > \frac{hc}{\lambda'_\alpha}.$$ ***Ans.***

16. **(b)** As equal fraction decay in equal times, so if a fraction of 0.9 remains undecayed after one day, a fraction of $(0.9)^2 = 0.81$ will remain undecayed after two days. ***Ans.***

Semiconductor Electronics: Materials, Devices and Simple Circuits 23

TOPIC 23.1: *Band Theory, Forbidden Gap, Conductivity of a Semiconductor, P-type and N-type Semiconductor, PN-junction Diode, Zener Diode, LED Half-wave and Full-wave Rectification.*

Review of Formulae

1. According to band theory, the substance is a semi conductor if forbidden gap is order of 1 eV.

   *For Ge, $E_g = 0.7\ eV$ and for Si, $E_g = 1.1\ eV$.

2. Conductivity of a semiconductor

$$\sigma = e(n_e\mu_e + n_h\mu_h)$$

   for intrinsic semiconductor $n_e = n_h$

   for extrinsic semiconductor

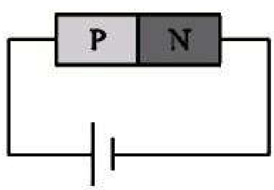
   (a) N-type, $n_e > n_h$

   (b) P-type $n_h > n_e$

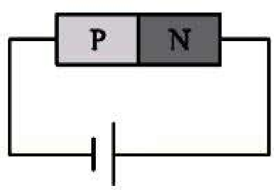
   *In N-type semiconductor, impurity added is pentavalent, (Antimony, Arsenic etc.)

   *In P-type semiconductor, impurity added is trivalent, (Aluminium, Boron, Indium etc)

3. Forward bias



   Reverse bias



   *For an ideal junction diode, forward resistance $R_f = 0$ and reverse resistance $R_r = \infty$.

4. Current in p-n junction

$$i = i_0 e^{(ev/\eta^{kt})} - 1;\ \eta = 1 \text{ for Ge and } \eta = 2 \text{ for Si.}$$

5. **Rectification (A.C. → D.C.) by diode :**

(a) **Half wave :** $i_{dc} = i_0/\pi$, $i_{rms} = i_0/\sqrt{2}$

$$P_{DC} = i_{dc}^2 R_L.$$

$$P_{AC} = i^2_{rms}(R_L + R_f).$$

$$\eta_{rec} = \frac{P_{dc}}{P_{ac}} \times 100\% = \frac{40.6}{1 + \frac{R_f}{R_L}}$$

(b) **Full wave :** $i_{dc} = 2i_0/\pi$, $i_{rms} = i_0/2$

$$P_{DC} = i_{dc}^2 R_L$$

$$P_{AC} = i^2_{rms}(R_L + R_f)$$

$$\eta_{rec} = \frac{P_{dc}}{P_{ac}} \times 100\% = \frac{81.2}{1 + \frac{R_f}{R_L}}.$$

TIPS! & TRICKS Tips and Tricks for Shortcut Solutions

1. **Diffusion current** in the diode is due to difference in concentration of charge carriers and from *p*-side to *n*-side of the crystal.
2. **Drift current** is due to movement of bond electrons into conduction band, in depletion region. It is from *n*-side to *p*-side of the crystal.
3. In forward bias, the current is due to majority carriers and *mA* order, while in reverse bias the current is due to minority carriers and of *μA* order.
4. Zener diode is highly doped diode and always uses in reverse bias. Zener diode is used for voltage stabilization.
5. LED is usually made of GaAs or indium phosphide.
6. PN-junction is a non-ohmic resistance.

Illustration 1

An intrinsic sample of germanium crystal has a hole of 10^{13}cm^{-3} at the room temperature. When doped with antimony the hole density is decreased to 10^{11}cm^{-3} at the same temperature. Find the number density of majority charge carriers.

Short-cut solution :

We know that

$$n_e n_h = n_i^2$$

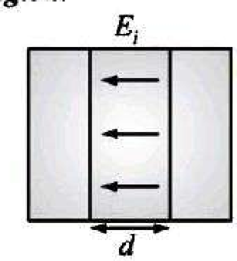
$$\therefore \quad n_e = \frac{n_i^2}{n_h}$$

$$= \frac{(10^{13})^2}{10^{11}} = 10^{15}\ \text{cm}^{-3}. \qquad \textbf{\textit{Ans.}}$$

Illustration 2

A potential barrier of 0.50 V exists across a P-N junction. If the depletion region is 5.0 × 10^{-7} m wide. Find the intensity of electric field in the region.

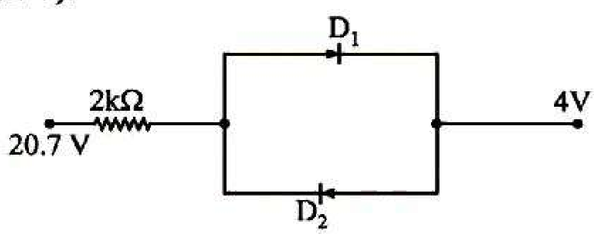
Solution :



$$E = \frac{V}{d} = \frac{0.5}{5 \times 10^{-7}} = 10^6\ \text{V/m}. \qquad \textbf{\textit{Ans.}}$$

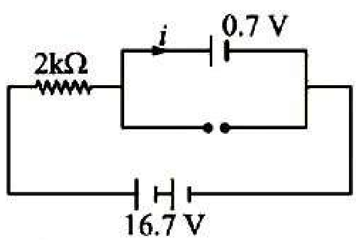
Illustration 3

Determine the current i in the circuit shown in figure. Assume diodes are made of silicon (V_0 = 0.7 V).



Short-cut solution :

In the given circuit, D_1 is in FB and D_2 is in RB and so current will pass through D_1. The equivalent circuit is :

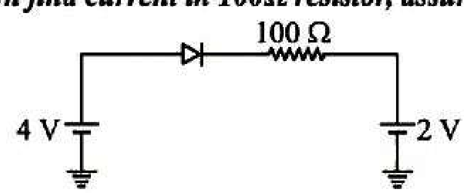


Current

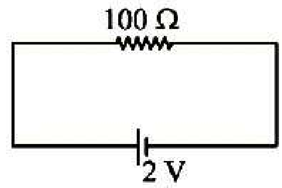
$$i = \frac{16.7 - 0.7}{2 \times 10^3} = 8 \times 10^{-3}\ \text{A} = 8\ \text{mA} \qquad \textbf{\textit{Ans.}}$$

Illustration 4

In the circuit shown find current in 100Ω resistor, assuming diode is ideal.



Short-cut solution :

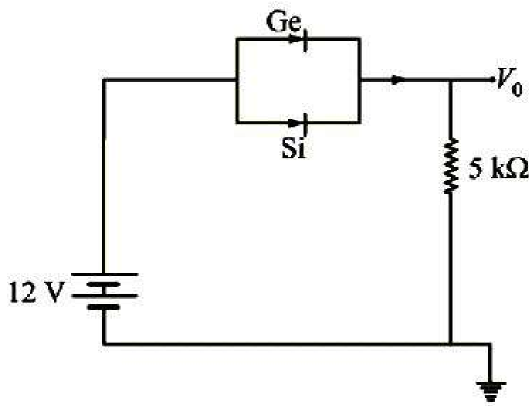


The effective circuit is shown in figure.

$$i = \frac{4-2}{100} = 0.02 \text{ A.} \quad \textit{Ans.}$$

Illustration 5

Calculate the value of V_0 and i if the Si diode and the Ge diode conduct at 0.7V and 0.3V respectively, in the circuit given below:



If now the Ge diode connection are reversed, what will be the new values of V_0 and i?

Short-cut solution :

The potential barrier of Ge is 0.3 V, which is less than the potential barrier of Si (0.7 V). So Ge diode will conduct, therefore

$$i = \frac{V_{net}}{R} = \frac{12-0.3}{5\times10^3} = 2.34 \; mA$$

$$V_0 = iR = 2.34 \times 10^{-3} \times 5 \times 10^3$$

$$= 11.7 \text{ V.} \qquad \textit{Ans.}$$

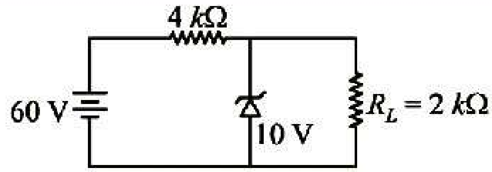
When Ge diode is reversed, it offers infinite resistance, and now Si diode will conduct. Thus

$$V_0 = 12 - 0.7 = 11.3 \text{ V}$$

$$i = \frac{11.3}{5\times10^3} = 2.26 \; mA \qquad \textit{Ans.}$$

Illustration 6

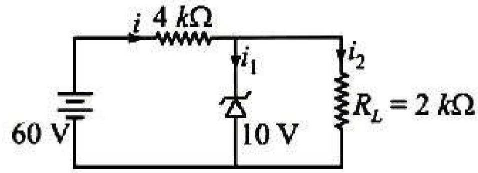
A zener diode is connected to a battery and a load as shown below. Find current in diode and load resistor.



Short-cut solution :

The current in load resistor R_L

$$i_2 = \frac{V}{R_L} = \frac{10}{2 \times 10^3} = 5 \text{ mA}.$$



The current $$i = \frac{V - V_z}{R}$$

$$= \frac{60 - 10}{4 \times 10^3} = 12.5 \text{ mA}.$$

The current $$i_1 = i - i_2 = 12.5 - 5 = 7.5 \text{ mA}.$$ ***Ans.***

TOPIC 23.2*: *Transistor-current Gain, Voltage Gain, Power Gain and Transistor as an Oscillator.

Review of Formulae

1. **Transistor :** (Willium Shockley)

   Function : *Amplification :* To convert weak signal to strong signal.

   Oscillation : To convert AC into DC.

   For any transistor, $i_e = i_b + i_c$.

2. **Current gain :** $\alpha = \frac{\Delta I_c}{\Delta I_e}$ in common base

   $\beta = \frac{\Delta I_c}{\Delta I_b}$ in common emitter

   $\alpha = \frac{\beta}{1+\beta}$ and

   $\beta = \frac{\alpha}{1-\alpha}$.

3. **Voltage gain :** $A_v = \alpha \frac{R_2}{R_1}$ in common base

$A_v = \beta \frac{R_2}{R_1}$ in common emitter

4. **Power gain :** $A_p = \alpha^2 \frac{R_2}{R_1}$ in common base

$A_p = \beta^2 \frac{R_2}{R_1}$ in common emitter

5. **Transistor as Oscillator :** It consists of :
   (i) LC Circuit
   (ii) Transistor Amplifier
   (iii) Positive feed back circuit, $f = \frac{1}{2\pi\sqrt{LC}}$.

   *Negative feed back is used in amplification to reduce distortions.

   *Positive feedback is used in oscillators.

Illustration 7

In a common base transistor amplifier, the input and output resistance are 500 Ω and 40 kΩ, and the emitter current is 1.0 mA. Find the input and the output voltages. Given α = 0.95.

Solution :

$$V_{in} = i_e \times R_{in} = (1.0 \times 10^{-3}) \times 500 = 0.5 \text{ V}$$

Similarly, the output voltage

$$= i_c \times R_{out} = (\alpha i_e) R_{out}$$
$$= (0.95 \times 1.0 \times 10^{-3}) \times 40 \times 10^3$$
$$= 38 \text{ V.}$$

Ans.

Illustration 8

In a NPN transistor 10^{10} electrons enter the emitter in 10^{-6} s. 2% of the electrons are lost in the base. Calculate the current transfer ratio and current amplification factor.

Solution :

$$i_e = \frac{N_e}{t} = 10^{10} \times 1.6 \times 10^{-19} = 1.6 \text{ mA}$$

and

$$i_b = \frac{2}{100} \times i_e = \frac{2}{100} \times 1.6 = 0.032 \text{ mA}$$

∴

$$i_c = i_e - i_b = 1.6 - 0.032 = 1.57 \text{ mA}$$

Current transfer ratio

$$\alpha = \frac{i_c}{i_e} = \frac{1.57}{1.6} = 0.98$$

Current amplification factor, $\beta = \frac{i_c}{i_b} = \frac{1.57}{0.032} = 49.$ **Ans.**

Illustration 9

A transistor is connected is common-emitter configuration. The collector-supply is 8V and the voltage drop across a resistor of 800Ω in the collector circuit is 0.5 V. If the current gain factor (α) is 0.96, find the base current.

Solution :

Current gain, $\beta = \frac{\alpha}{1-\alpha} = \frac{0.96}{1-0.96} = 24$

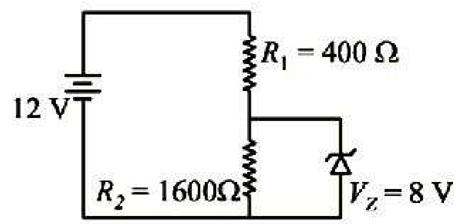
The collector-current is $= \frac{\text{voltage drop across collector resistor}}{\text{resistance}}$

$$= \frac{0.5}{800} \times 10^{-3} = 0.625 \times 10^{-3} \text{ A}$$

As $\beta = \frac{i_c}{i_b}$, $\therefore$ $i_b = \frac{i_c}{\beta} = \frac{0.625 \times 10^{-3}}{24} = 26\ \mu\text{A}.$ **Ans.**

Illustration 10

In the circuit shown in figure. Zener diode is properly biased. Find power dissipated in diode.

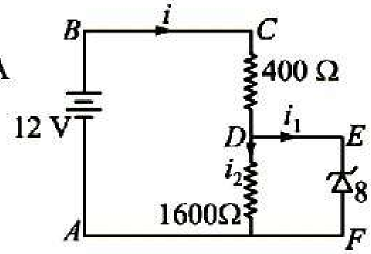


Solution :

By Kirchhoff's law in loop *ABCDEFA*

$$12 - 400\,i - 8 = 0$$

or $i = \frac{4}{400} = 10$ mA



Also $i_2 = \frac{V_z}{R} = \frac{8}{1600}$

$= 5$ mA.

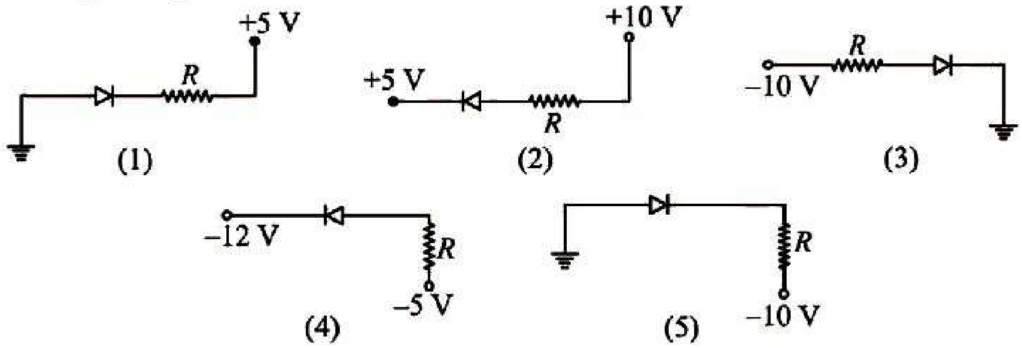
$\therefore$ $i_1 = i - i_2 = 10 - 5 = 5$ mA.

Power dissipated $= V_i = 8 \times 5 = 40$ mW. **Ans.**

Concept Booster Exercise

1. In a P-N junction with open ends
   (a) there is no systematic motion of charge carriers
   (b) holes and conductor electrons systematically go from the *P*-side and from the *n*-side to the *p*-side respectively.
   (c) there is no net charge transfer between the two sides.
   (d) there is a constant electric field near the junction.

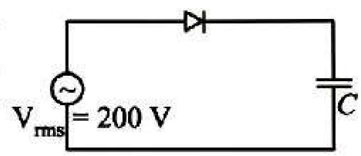
2. In the given figure, which of the diodes are in forward biased?



   (a) 1, 2, 3 (b) 2, 4, 5 (c) 1, 3, 4 (d) 2, 3, 4

3. Potential barrier developed in a junction diode opposes
   (a) minority carriers in both regions only
   (b) majority carriers
   (c) electrons in N-region
   (d) holes in P-region

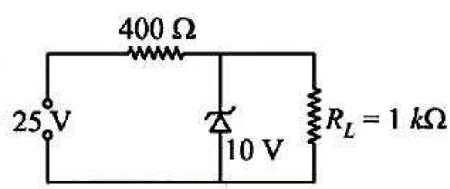
4. In the figure, an AC of rms voltage 200 V is applied to the circuit containing diode and the capacitor and it is being rectified. The potential across the capacitor *C* in volt will be **Numeric/Integer**



   (a) 500 V (b) 283 V (c) 200 V (d) 141 V

5. On increasing the temperature of a semiconductor material
   (a) density of charge carriers as well as their mobility both increases
   (b) density of charge carriers increases, but their mobility decreases
   (c) density of charge-carriers decreases, but their mobility increases
   (d) none of these

6. A zener diode of breakdown voltage, $V_z = 10$ V is used in the given circuit. The current through the load resistor (R_L) is **Numeric/Integer**



   (a) 10 μA (b) 5 μA
   (c) 10 mA (d) 5 mA

7. In a photo-diode the current increases if it is exposed to light of wavelength 621 nm or lesser. The band gap of the material of photo diode is: **Numeric/Integer**

(a) 0.7 eV (b) 1.2 eV (c) 2 eV (d) 2.3 eV

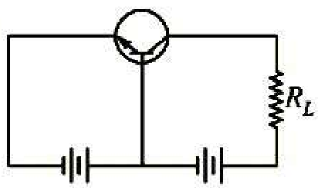
8. In a transistor used in common-emitter mode, current amplification factor (β) is 100 and base current (i_b) is 50 μA. The emitter current is: **Numeric/Integer**

(a) 2 mA (b) 2.05 mA (c) 5.05 mA (d) 7 mA

9. In a common-base transistor circuit, emitter current is 1 mA and base current is 40 μA. The value of current amplification factor(α) is: **Numeric/Integer**

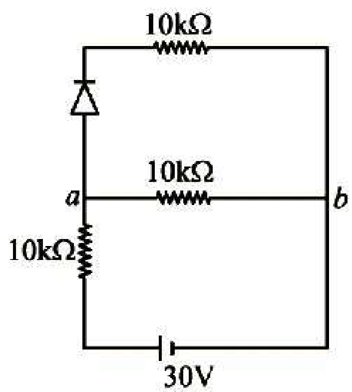
(a) 0.95 (b) 0.96 (c) 0.98 (d) 0.99

10. In a common base configuration of transistor shown in figure, $\alpha = 0.98$, $i_b = 0.02$ mA, $R_L = 5\text{k}\Omega$. Output voltage across load is: **Numeric/Integer**



(a) 3.2 V (b) 4.9 V
(c) 5.2 V (d) 6.2 V

11. There is a electric circuit as shown in the figure. Find potential difference between points *a* and *b*. *[JEE Main 2020]*



(a) 0 V (b) 15 V (c) 10 V (d) 5 V

Solutions

1. **(c, d)** Due to the accumulation of holes and electrons on both sides of the junction.
2. **(b)** In 2, 4 and 5 P-crystals are move positive than N-crystals.
3. **(b)**
4. **(b)** $V = \sqrt{2}V_{rms} = \sqrt{2} \times 200 = 283$ V.
5. **(b)** $\mu = \frac{v_d}{E}$.
6. **(c)** $i = \frac{V_z}{R} = \frac{10}{1000} = 10$ mA.

7. **(c)** $E_g = \frac{hc}{\lambda}$.

8. **(c)** $\beta = \frac{i_c}{i_b}$ $\Rightarrow$ $i_c = \beta i_b = 100 \times 50 \times 10^{-6} = 5$ mA.

Now $i_e = i_b + i_c = 50\ \mu A + 5\ mA = 5.05$ mA.

9. **(b)** $i_c = i_e - i_b = 1\ mA - 40\ \mu A = 0.001 - 0.0004 = 0.00096$

Now $\alpha = \frac{i_c}{i_e} = \frac{0.00096}{0.001} = 0.96.$

10. **(b)** $\alpha = \frac{i_c}{i_e} = \frac{i_c}{i_c + i_b}$

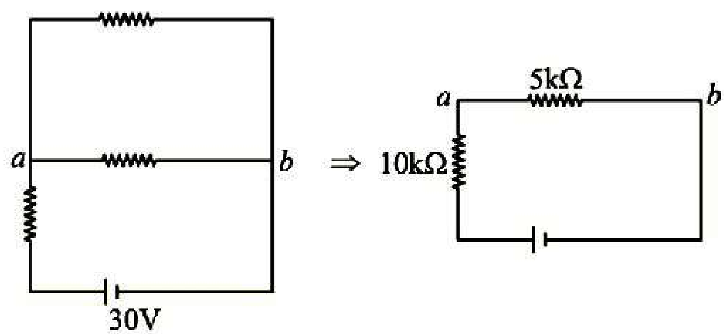
$$0.98 = \frac{i_c}{i_c + 0.02}$$

$\therefore$ $i_c = 0.98$ mA

Now $V_0 = i_e R_L = 0.98 \times 10^{-3} \times 5 \times 10^3$

$= 4.9$ V

11. **(c)** Diode is in forward bias, so it will behave as simple wire so,



So, $V_{ab} = \frac{30}{5+10} \times 5 = 10$ V ***Ans.***

Logic Gates and Communication Systems 24

***TOPIC**: Communication System-Modulation, Modulation Index, LOS Distance and Logic Gates.*

Review of Formulae

1. Low frequencies cannot be transmitted to long distances. Therefore, they are superimposed on a high frequency carrier signal by a process known as modulation.
2. In modulation, some characteristic of the carrier signal like amplitude, frequency or phase varies in accordance with the modulating or message signal. Correspondingly, they are called Amplitude Modulated (AM), Frequency Modulated (FM) or Phase Modulated (PM) waves.
3. For transmission over long distances, signals are radiated into space using devices called antennas. The radiated signals propagate as electromagnetic waves and the mode of propagation is influenced by the presence of the earth and its atmosphere. Near the surface of the earth, electromagnetic waves propagate as surface waves. **Surface waves** propagation is useful up to a **few MHz** frequencies.
4. Long distance communication between two points on the earth is achieved through reflection of electromagnetic waves by ionosphere. Such waves are called sky waves. **Sky wave** propagation takes place up to frequency of about **30 MHz**. Above this frequency, electromagnetic waves essentially propagate as space waves. **Space waves** are used for line-of-sight (LOS) communication and satellite communication.
5. If an antenna radiates electromagnetic waves from a height h_T, then the range d_T is given by $\sqrt{2Rh_T}$ where R is the radius of the earth. Line-of-sight distance, $d_m = \sqrt{2Rh_T} + \sqrt{2Rh_R}$.
6. Amplitude modulated signal contains frequencies $(\omega_c - \omega_m)$, ω_c and $(\omega_c + \omega_m)$.
7. Amplitude modulated waves can be produced by application of the message signal and the carrier wave to a non-linear device, followed by a band pass filter.

   Amplitude modulation index, $\mu = \dfrac{A_m}{A_c}$; $\mu \leq 1$.

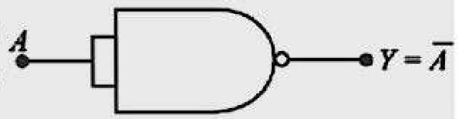
Tips and Tricks for Shortcut Solutions

Building blocks

The NAND gate is known as universal building block; with the help of it, we can get basic gates. The other universal building block is NOR gate.

NOT from NAND gate

When both the inputs A and B of the NAND gate are joined together then it works as the NOT gate. Thus we have;



$$\overline{A\cdot A} = \overline{A}$$

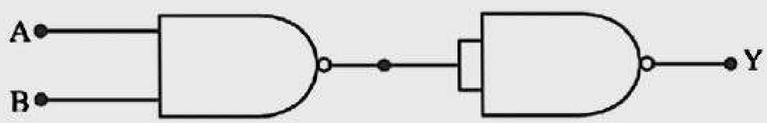
Truth table

Input	Output
$A = B$	Y
0	1
1	0

AND from NAND gate

When two NAND gates are connected in series, the output is equivalent to AND gate. That is;

$$\text{NAND} + \text{NAND} = (\text{AND} + \text{NOT}) + (\text{AND} + \text{NOT}) \Rightarrow \text{AND}$$

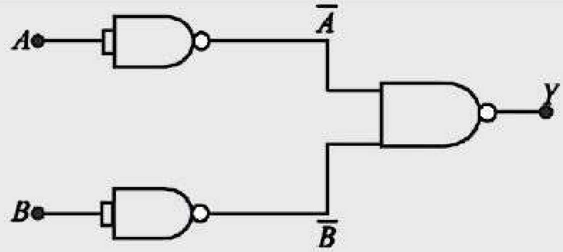


Truth table

A	B	$\overline{A\cdot B}$	$(\overline{\overline{A\cdot B}}) = Y$	
0	0	1	0	→ NOT
0	1	1	0	
1	0	1	0	
1	1	0	1	

OR from NAND gate

When the outputs of two NOT gates (which are obtained from NAND gates) is given to the input of the another NAND gate, the resultant gate works as the OR gate.



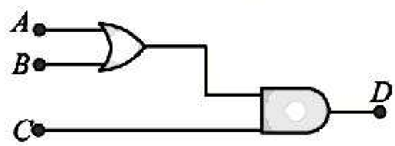
Truth table

A	B	$\overline{A}$	$\overline{B}$	$(\overline{\overline{A}\,\overline{B}}) = Y$	
0	0	1	1	0	
0	1	1	0	1	→ OR
1	0	0	1	1	
1	1	0	0	1	

Illustration 1

To get an output 1 from the circuit shown in the figure, the input must be

(a) $A = 0, B = 1, C = 0$
(b) $A = 1, B = 0, C = 0$
(c) $A = 1, B = 0, C = 1$
(d) $A = 1, B = 1, C = 0$

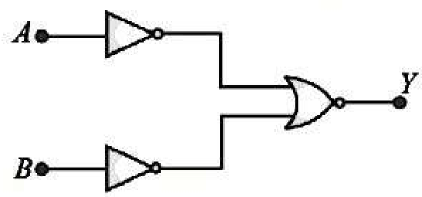


Short-cut solution :

$(A + B) \cdot C$. For $A = 1$, $B = 0$ and $C = 1$ will give 1. ***Ans. (c)***

Illustration 2

Which logic gate is represented by the following combination of logic gates



(a) *OR* *(b)* *NAND* *(c)* *AND* *(d)* *NOR*

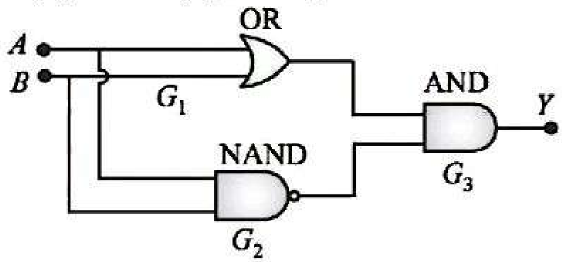
Short-cut solution :

A	B	$\overline{A}$	$\overline{B}$	$\overline{Y} = \overline{A} + \overline{B}$	$\overline{\overline{Y}}$
0	0	1	1	1	0
0	1	1	0	1	0
1	0	0	1	1	0
1	1	0	0	0	1

This table is of AND gate. ***Ans. (c)***

Illustration 3

The following configuration of gate is equivalent to



(a) NAND (b) XOR (c) OR (d) None of these

Short-cut solution :

Ans. (b)

Illustration 4

An audio signal of amplitude 0.1 V is used in amplitude modulation of a carrier wave of amplitude 0.2 V. Calculate the modulation index.

Short-cut solution :

$$\mu = \frac{A_m}{A_c} = \frac{0.1}{0.2} = 0.5$$ ***Ans.***

Illustration 5

A transmitting antenna at the top of a tower has a height 32 m and the height of the receiving antenna is 50 m. What is the maximum distance between them for satisfactory communication in LOS mode ? Given radius of earth 6.4 × 106 m.

Short-cut solution :

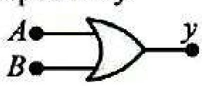
$$d_m = \sqrt{2Rh_T} + \sqrt{2Rh_R}$$

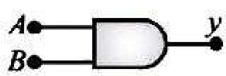
$$= \sqrt{2\times64\times10^5\times32} + \sqrt{2\times64\times10^5\times50}$$

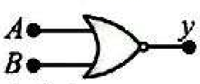
$$= 45.5 \times 10^3 \text{ m} = 45.5 \text{ km}$$ ***Ans.***

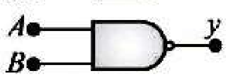
Concept Booster Exercise

1. Given below are four logic gate symbol (figure). Those for OR, NOR and NAND are respectively

(1) 

(2) 

(3) 

(4) 

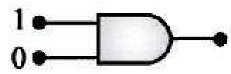
(a) 1, 4, 3 (b) 4, 1, 2 (c) 1, 3, 4 (d) 4, 2, 1

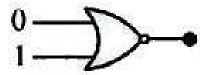
2. A truth table is given below. Which of the following has this type of truth table

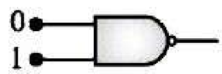
A	0	1	0	1
B	0	0	1	1
y	1	0	0	0

(a) XOR gate (b) NOR gate (c) AND gate (d) OR gate

3. Which of the following gates will have an output of 1

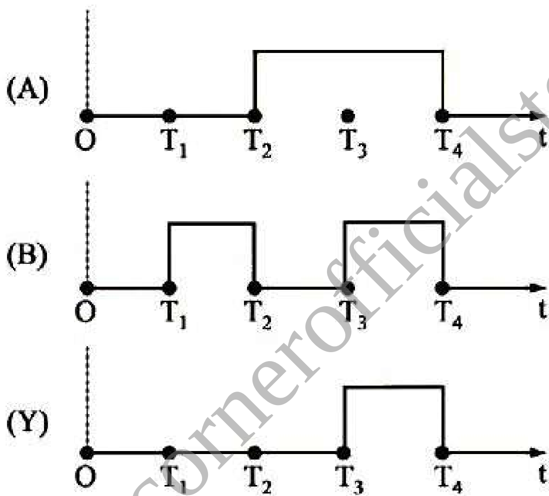
(a) 

(b) 

(c) 

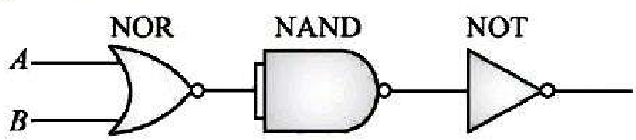
(d) 

4. The given figure shows the wave forms for two inputs A and B and that for the output Y of a logic circuit. The logic circuit is



(a) an AND gate (b) an OR gate
(c) a NAND gate (d) an NOT gate

5. The circuit is equivalent to



(a) NOR gate (b) OR gate (c) AND gate (d) NAND gate

6. A carrier wave of peak voltage 12 V is used to transmit a message signal. The modulation index is 75%. The peak value of voltage of modulation index is:

Numeric/Integer

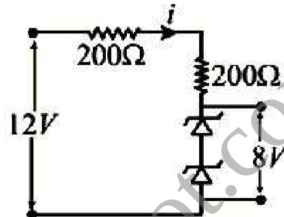
(a) 3 V (b) 6 V (c) 9 V (d) 12 V

7. For an amplitude modulated wave, the maximum amplitude is found to be 10 V while the minimum is found to be 2V. The modulation index is: **Numeric/Integer**

(a) 1/3 (b) 2/3 (c) 3/2 (d) 1/4

8. For sky wave propagation, the radio waves must have a frequency range in between:
   (a) 1 MHz to 2 MHz (b) 5 MHz to 25 MHz
   (c) 35 MHz to 40 MHz (d) 45 MHz to 50 MHz

9. Find the power loss in each diode (in mW), if potential drop across the zener diode is 8V.
   Numeric/Integer
   [JEE Main 2020]



10. A signal of 5 kHz frequency is amplitude modulated on a carrier wave of frequency 2 MHz. The frequencies of the resultant signal is/are :
   (a) 2005 kHz, 2000 kHz and 1995 kHz (b) 2000 kHz and 1995 kHz
   (c) 2 MHz only (d) 2005 kHz and 1995 kHz

Solutions

1. (c) 2. (b) 3. (c)

4. (a) The truth table of the wave forms are as follows :

A	*B*	*Y*
0	0	0
0	1	0
1	0	0
1	1	1
0		0

The truth table is corresponding to AND gate. ***Ans.***

5. (a) NOR + (NAND = NOT) + NOT
   $\Rightarrow$ NOR. ***Ans.***

6. (c) $\dfrac{A_m}{A_c} = 0.75$

   $\therefore \quad A_m = 0.75\,A_c = 0.75 \cdot 12 = 9$ V. ***Ans.***

7. (b) $M = \dfrac{A_m}{A_c} = \dfrac{M_1 - M_2}{M_1 + M_2} = \dfrac{10-2}{10+2} = \dfrac{2}{3}$ ***Ans.***

8. (b) Sky wave propagation is suitable for frequency range 5 MHz to 25 MHz. ***Ans.***

9. (40) $i = \dfrac{(12-8)}{(200+200)}A = \dfrac{4}{400} = 10^{-2}A$

   Power loss in each diode = $(4)(10^{-2})$ W = 40 mW ***Ans.***

10. (a) Amplitude modulated wave consists of three frequencies are
   $\omega_c + \omega_m, \omega, \omega_c - \omega_m$
   i.e., 2005 KHz, 2000 KHz, 1995 KHz ***Ans.***